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CONTENTS

RÉSUMÉ

Sur une variété complexe kählérienne X, on introduit les notions de courbure scalaire à poids et de métrique kählérienne à courbure scalaire à poids constante, dépendant d'un tore réel fixé $\mathbb{T}$ dans le groupe réduit des automorphismes de X, et de deux fonctions lisses (poids) $\mathrm{v} > 0$ et w, définies sur le polytope moment de X (par rapport à une classe de Kähler fixée sur X) dans l'algèbre de Lie du tore $\mathbb{T}$. Pour des choix spécifiques des fonctions poids v et w, la recherche de métriques kählériennes à courbure scalaire pondérée constante dans une classe de Kähler α, correspond à des problèmes bien connues de recherche de métriques spéciales en géométrie kählérienne, tels que l'existence de métriques kählériennes extrémales, des solitons de Kähler-Ricci, des métriques kählériennes conformes à une métrique hermitienne d'Einstein-Maxwell, ou la prescription de la courbure scalaire sur une variété torique.

On montre que la plupart des résultats connues sur l'obstruction à l'existence des métriques kählériennes à courbure scalaire constante (cscK) peuvent s'étendre au cadre pondéré. En pariculier, en introduisant une fonctionelle $\mathcal{M}_{\mathrm{v,w}}$ sur l'espace des métriques kählériennes $\mathbb{T}$-invariantes dans α, qui généralise la fonctionnelle de Mabuchi dans le cas cscK, on montre que lorsque α est une class de Hodge, les métriques kählériennes à courbure scalaire pondérée dans α minimisent $\mathcal{M}_{\mathrm{v,w}}$. Nous définissons un invariant de Futaki (v,w)-pondéré pour des configurations test lisses $\mathbb{T}$-compatibles associées à $(X, \mathbb{T}, \alpha)$, et on montre que si l'énergie de Mabuchi pondérée $\mathcal{M}_{\mathrm{v,w}}$ est bornée inférieurement, alors ceci impliquera une notion de K-semistabilité (v,w)-pondérée.

Nous illustrons cette théorie sur des variétés toriques et sur des fibrations toriques principales. Comme application, nous obtenons une correspondance de Yau-Tian-Donaldson pour les métriques extrémales (v,w)-pondérées sur des $\mathbb{P}^1$-fibrations au dessus d'un produit de variétés de Hodge cscK, et une description des classes de Kähler sur les surfaces complexes réglées de genre 2, qui admettent une métrique kählérienne conforme à une métrique hermitienne d'Einstein-Maxwell.

Mots clés : Métriques kählériennes extrémales, Courbure scalaire à poids, K-semistabilité, Variétés toriques, Fibrations toriques principales, Métriques d'Einstein-Maxwell, Solitons de Kähler-Ricci.

ABSTRACT

We introduce a notion of a Kähler metric with constant weighted scalar curvature on a compact Kähler manifold X, depending on a fixed real torus $\mathbb{T}$ in the reduced group of automorphisms of X, and two smooth (weight) functions $\mathrm{v} > 0$ and w, defined on the momentum image with respect to a given Kähler class α on X in the dual Lie algebra of $\mathbb{T}$. A number of natural problems in Kähler geometry, such as the existence of extremal Kähler metrics and conformally Kähler Einstein–Maxwell metrics, Kähler-Ricci solitons, or prescribing the scalar curvature on a compact toric manifold reduce to the search of Kähler metrics with constant weighted scalar curvature in a given Kähler class α, for special choices of the weight functions v and w.

We prove that a number of known results obstructing the existence of constant scalar curvature Kähler (cscK) metrics can be extended to the weighted setting. In particular, we introduce a functional $\mathcal{M}_{\mathrm{v,w}}$ on the space of $\mathbb{T}$-invariant Kähler metrics in α, extending the Mabuchi energy in the cscK case, and show that if α is Hodge, then constant weighted scalar curvature metrics in α are minima of $\mathcal{M}_{\mathrm{v,w}}$. We define a (v, w)-weighted Futaki invariant of a $\mathbb{T}$-compatible smooth Kähler test configuration associated to $(X, \alpha, \mathbb{T})$, and show that the boundedness from below of the (v, w)-weighted Mabuchi functional $\mathcal{M}_{\mathrm{v,w}}$ implies a suitable notion of a (v, w)-weighted K-semistability.

We illustrate our theory with specific computations on smooth toric varieties and on the toric fibre bundles. As an application, we obtain a Yau–Tian–Donaldson type correspondence for (v, w)-extremal Kähler classes on $\mathbb{P}^1$-bundles over products of compact Hodge cscK manifolds, and a description of the Kähler classes on geometrically ruled complex surfaces of genus greater than 2, which admit Kähler metrics conformally equivalent to Einstein-Maxwell metrics.

Keywords : Extremal Kähler metrics, Weighted scalar curvature, K-semistability, Toric varieties, Toric fibre bundles, Einstein-Maxwell metrics, Kähler-Ricci solitons.

CHAPTER I

INTRODUCTION

Recently, research activities in Kähler geometry were primarily concerned with so-called Yau-Tian-Donaldson (YTD) conjecture which relates the existence of *constant scalar curvature Kähler metrics* (cscK) on a projective manifold to *K-stability*, an algebro-geometric condition in the sense of geometric invariant theory (GIT) of the underlying projective manifold. The efforts of many mathematicians culminated in the resolution of the YTD conjecture in the case of Fano manifolds, where the cscK property of the metric is equivalent to being *Kähler-Einstein* (KE). The cscK metrics can be also viewed as a higher dimensional generalization in Riemannian signature of Einstein's equations describing the space-time in 4 dimensions.

On a 4-dimensional Riemanian manifold $(X, \tilde{g})$, another natural generalization of the *Einstein equation* $\mathrm{Ric}_0^{\tilde{g}} = 0$ is given by the *Einstein-Maxwell equations*

$$\begin{cases} d\Phi = 0, \ \star_{\tilde{g}}\Phi = \Phi, \\ d\Psi = 0, \ \star_{\tilde{g}}\Psi = -\Psi, \\ \mathrm{Ric}_0^{\tilde{g}} = \Phi^\sharp \circ \Psi^\sharp. \end{cases} \tag{1.1}$$

where $\mathrm{Ric}_0^{\tilde{g}}$ is the trace free part of the Ricci endomorphism, $\Phi, \Psi \in A^2(X)$ is a pair of 2-forms on X, $\star_{\tilde{g}}$ is the Hodge star operator of $\tilde{g}$, and $\Phi^\sharp, \Psi^\sharp$ are the skew-symmetric endomorphisms associated to Φ, Ψ by $\tilde{g}$.

Apostolov–Calderbank–Gauduchon [4] and LeBrun [65, 66] observed that on a Kähler surface (X, J, g, ω) with complex structure J, Kähler metric g and Kähler form ω, a

Hermitian metric $\tilde{g} := f^2 g$ conformaly equivalent to a Kähler metric g with positive conformal factor $f > 0$, is a solution to the Einstein-Maxwell equations with $\Phi = \omega$ if and only if

$$\begin{cases} \xi := J\mathrm{grad}_g(f) \text{ is a Killing field for } g, \\ \mathrm{Scal}_{\tilde{g}} = \mathrm{const}, \end{cases} \tag{1.2}$$

where $\mathrm{Scal}_{\tilde{g}}$ is the scalar curvature of $\tilde{g}$. The Kähler metrics g satisfying the condition (1.2) are called *conformally Einstein-Maxwell Kähler metrics* (cKEM for short). The condition (1.2) provides a natural generalization of cscK metrics (corresponding to the case when $f = 1$), and allows one to define an extension of the Einstein-Maxwell equations to higher dimensionsional Kähler manifolds.

The initial motivation of this thesis was the systematic study of conformally Einstein-Maxwell metrics in line with the YTD conjecture alluded to above. To this end, we propose to study the more general notion of *Kähler metrics with weighted constant scalar curvature* (weighted cscK for short), which contains the Kähler metrics conformal to Einstein-Maxwell metrics (1.2) as a special case.

To define a weighted cscK metric we first introduce a "weighted" version of the scalar curvature. More precisely, on a compact Kähler manifold (X, α) of complex dimension $m \geq 1$ with a Kähler class α, and a Hamiltonian torus action $\mathbb{T}$ with momentum polytope $\mathrm{P}_\alpha \subset \mathrm{Lie}(\mathbb{T})^*$, for any positive smooth $\mathrm{v} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R})$ (called *weight function*), we define the v-*scalar curvature* $\mathrm{Scal}_\mathrm{v} : \mathcal{K}(X,\alpha)^\mathbb{T} \to \mathbb{R}$ on the space $\mathcal{K}(X,\alpha)^\mathbb{T}$ of $\mathbb{T}$-invariant Kähler metrics in the Kähler class α, by

$$\mathrm{Scal}_\mathrm{v}(\omega) := \mathrm{v}(m_\omega)\mathrm{Scal}(\omega) + 2\Delta_\omega(\mathrm{v}(m_\omega)) + \mathrm{tr}(\mathrm{G}_\omega \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)). \tag{1.3}$$

Here $\mathrm{Scal}(\omega)$ is the usual scalar curvature, $m_\omega : X \to \mathrm{Lie}(\mathbb{T})^*$ is the ω-moment map of the $\mathbb{T}$-action normalized by $m_\omega(X) = \mathrm{P}_\alpha$, Δ_ω is the Riemannian Laplacian of the Kähler metric g_ω and $\mathrm{Hess}(\mathrm{v})$ is the hessian of v, viewed as a bilinear form on $\mathrm{Lie}(\mathbb{T})^*$ whereas G_ω is the bilinear form with smooth coefficients on $\mathrm{Lie}(\mathbb{T})$, given by the restriction of the Kähler metric g_ω on fundamental vector fields. We say that a $\mathbb{T}$-invariant Kähler metric $\omega \in \alpha$ is a *weighted cscK metric* if

$$\mathrm{Scal}_{\mathrm{v}}(\omega) = c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\omega), \tag{1.4}$$

for a couple (v, w) of weight functions on the polytope P_α (with $\mathrm{v} > 0$), where $c_{\mathrm{v},\mathrm{w}}(\alpha)$ is a suitable constant. The above definition may appear rather tedious at first glimpse, but it turns out that for suitable choices of the weight functions v, w on P_α, the problem of finding a (v, w)-weighted cscK metric in the Kähler class α corresponds to some well studied problems in Kähler geometry. There is a list of such examples:

1. Letting $\mathrm{v} \equiv 1$, and $\mathrm{w} \equiv \mathrm{const}$ we obtain the Calabi problem of finding cscK metrics in α;

2. Letting $\mathbb{T}$ be a maximal torus in the group $\mathrm{Aut}_{\mathrm{red}}(X)$ of reduced automorphisms of X, $\mathrm{v} \equiv 1$ and w be the affine-linear function on P_α given by the extremal vector field of α, the solutions of (1.4) are the extremal Kähler metrics in the sense of Calabi [22] in α;

3. If X is a Fano manifold equipped with the Kähler class $\alpha = 2\pi c_1(X)$, $\mathrm{v}(p) = e^{\langle \xi, p \rangle}$ and $\mathrm{w}(p) = 2e^{\langle \xi, p \rangle}(\langle \xi, p \rangle + a)$ for $\xi \in \mathfrak{t}$, $a \in \mathbb{R}$, then solutions of (1.4) are Kähler-Ricci solitons on X (see [58, 59]);

4. Letting $\mathrm{v}(p) = \left(\langle \xi, p \rangle + a\right)^{-2m+1}$, $\mathrm{w}(p) = \left(\langle \xi, p \rangle + a\right)^{-2m-1}$ for $\xi \in \mathfrak{t}$ and $a \in \mathbb{R}$ such that $\langle \xi, p \rangle + a > 0$ over P_α, (1.4) describes the Kähler metrics in α, which are conformal to Einstein–Maxwell metrics in the sense of [9, 64–66];

5. If $\alpha = 2\pi c_1(L)$ is the Kähler class associated to an ample holomorphic line bundle L over X, $\mathrm{v}(p) = \left(\langle \xi, p \rangle + a\right)^{-m-1}$, $\mathrm{w}(p) = \left(\langle \xi, p \rangle + a\right)^{-m-3}\mathrm{w}_{\mathrm{ext}}(p)$ for $\xi \in \mathfrak{t}$, $\mathrm{w}_{\mathrm{ext}}(p)$ is a suitable affine linear function on P_α and $a \in \mathbb{R}$ such that $\langle \xi, p \rangle + a > 0$ over P_α, then (1.4) describes Kähler metrics on X giving rise to extremal Sasaki metrics on the unit circle bundle associated to L^{-1}, see [5];

6. The search for extremal Kähler metrics, or more generally, prescribing the scalar curvature of a class of Kähler metrics on toric fibre-bundles given by the generalized

Calabi ansatz [8] or on manifolds with free multiplicity [45] reduces to finding solutions of (1.4) on the (toric) fibre. In this toric setting (1.4) is known as the *generalized Abreu equation*, see [70, 71].

Instead of (1.4), one can more generally consider the condition

$$\mathrm{Scal}_{\mathrm{v}}(\omega) = \mathrm{w}(m_\omega)(m_\omega^\xi + c) \tag{1.5}$$

for a $\mathbb{T}$-invariant Kähler metric ω in α, where $\xi \in \mathfrak{t}$, $c \in \mathbb{R}$ and $m_\omega^\xi := \langle m_\omega, \xi \rangle$ is the Killing potential associated to ξ. A $\mathbb{T}$-invariant Kähler metric satisfying (1.5) generalizes the notion of an extremal Kähler metric (see 2 above), and will be referred to as a (v, w)-*weighted extremal* Kähler metric. As it is apparent from the example 2 above, and as we establish more generally in Section 2.2, when $\mathrm{w} > 0$ the smooth function $(m_\omega^\xi + c)$ in the RHS of (1.5) must be of the form $\mathrm{w}_{\mathrm{ext}}(m_\omega)$ for an affine-linear function $\mathrm{w}_{\mathrm{ext}}(p) = \langle \xi, p \rangle + c$ on $\mathfrak{t}^*$ defined in terms of $(\mathbb{T}, \alpha, \mathrm{P}, \mathrm{v}, \mathrm{w})$. Thus, the problem (1.5) of finding (v, w)-extremal Kähler metrics in α reduces to the problem (1.4) of finding $(\mathrm{v}, \mathrm{w}\mathrm{w}_{\mathrm{ext}})$-cscK metrics.

Besides the above mentioned list of examples, our intrinsic motivation for defining the v-scalar curvature is twofold. On the one hand, in the cases 1 and 2 above, there is a well known interpretation, due to Donaldson [40] and Fujiki [48], of the scalar curvature as a formal moment map

$$\begin{aligned}\mathrm{Scal} : \mathcal{AC}(X, \omega) &\to \mathrm{Lie}(\mathrm{Ham}(X, \omega))^*, \\ \langle \mathrm{Scal}(J), f \rangle &= \int_X \mathrm{Scal}(g_J) f \omega^{[n]},\end{aligned}$$

for the action of the group of Hamiltonian transformations $\mathrm{Ham}(X, \omega)$, on the space of all ω-compatible almost complex structures $\mathcal{AC}(X, \omega)$, where $\mathrm{Scal}(g_J)$ is the scalar curvature of the Kähler metric $g_J := \omega(\cdot, J\cdot)$, and the identification of $\mathrm{Lie}(\mathrm{Ham}(X, \omega))$ with the space of smooth functions of zero average is obtained by using the global L^2 inner product with respect to $\omega^{[n]} := \frac{\omega^n}{n!}$.

Following an idea due to Apostolov–Mashler in [9], on a Kähler manifold (X, ω) with a Hamiltonian torus action $\mathbb{T}$, one can use two positive weight functions v, w on the momentum polytope $\mathrm{P} := m_\omega(X)$ of the $\mathbb{T}$-action on X to modify the formal symplectic structure on $\mathcal{AC}(X,\omega)^{\mathbb{T}}$ on the subspace of $\mathbb{T}$-invariant almost complex structures, and the L^2 inner product on the Lie algebra of $\mathbb{T}$-equivariant Hamiltonian transformations $\mathrm{Lie}(\mathrm{Ham}(X,\omega)^{\mathbb{T}})$. This modification yields a modified formal momentum map for the action of $\mathrm{Ham}(X,\omega)^{\mathbb{T}}$ on $\mathcal{AC}(X,\omega)^{\mathbb{T}}$, given in terms of the v-scalar curvature and the function w

$$\frac{\mathrm{Scal}_{\mathrm{v}}}{\mathrm{w}(m_\omega)} : \mathcal{AC}(X,\omega)^{\mathbb{T}} \to \mathrm{Lie}(\mathrm{Ham}(X,\omega))^*.$$

This formal momentum map picture for the v-scalar curvature suggests the existence and uniqueness for solutions $g_J := \omega(\cdot, J\cdot)$ of the problem (1.4) in each 'complexified' orbit for the action of $\mathrm{Ham}(X,\omega)^{\mathbb{T}}$, under suitable stability conditions. However, it is well known that such a complexification of the group $\mathrm{Ham}(X,\omega)^{\mathbb{T}}$ does not exist, but it is possible to identify its orbits with the space $\mathcal{K}(X,\alpha)^{\mathbb{T}}$ of $\mathbb{T}$-invariant Kähler metrics in the class $\alpha := [\omega]$. Following the analogy with the case of cscK metrics, a direct consequence of the momentum map picture provides the definition of a (v,w)-*Futaki invariant* defined on the space $\mathfrak{h}^{\mathbb{T}}_{\mathrm{red}}$ of real holomorphic vector fields with zeroes commuting with $\mathrm{Lie}(\mathbb{T})$,

$$\mathcal{F}^{\alpha}_{\mathrm{v},\mathrm{w}} : \mathfrak{h}^{\mathbb{T}}_{\mathrm{red}} \to \mathbb{R},$$

giving a natural obstruction to the existence of weighted cscK metrics in a Kähler class α, similar to the famous Futaki invariant. Also, the momentum map interpretation allows us to introduce a weighted (v,w)-Mabuchi functional $\mathcal{M}_{\mathrm{v},\mathrm{w}}$, whose critical points are the (v,w)-cscK metrics, extending the well known Mabuchi functional [76].

On the other hand, our second motivation for introducing the v-scalar curvature comes from the Donaldson quantization of cscK metrics, based on the Catlin [24], Ruan [84], Tian [91] and Zelditch [98] asymptotic expansion of the Bergman Kernel $B_k(h)$ of a hermitian metric h on a prequantization line bundle $L \to X$:

$$(2\pi)^n B_k(h) = 1 + \frac{\mathrm{Scal}(\omega_h)}{4k} + \mathcal{O}(\frac{1}{k^2}),$$

where ω_h is the curvature 2-form of h. In the case where the polarized manifold (X, L) carries a torus action $\mathbb{T} \subset \mathrm{Aut}(X, L)$ in the automorphism group of the pair (X, L) with momentum polytope P (in this case P is determined by the lifted action of $\mathbb{T}$ on L), one can associate to each smooth strictly positive weight function v on P, a v-*equivariant Bergman kernel* $B_k(\mathrm{v}, h)$ defined by the restriction to the diagonal of $X \times X$ of the Schwartz kernel of the operator

$$\mathrm{v}(k^{-1}A^{(k)}_{\xi_1}, \cdots, k^{-1}A^{(k)}_{\xi_\ell}) \circ \Pi^{k\phi}_{\mathrm{v}},$$

where $A^{(k)}_{\xi_1}, \cdots, A^{(k)}_{\xi_\ell}$ are the infinitisimal actions on the space of global holomorphic sections $\mathcal{H}_k$ of L^k, induced from a basis $(\xi_1, \cdots, \xi_\ell)$ of $\mathrm{Lie}(\mathbb{T})$, and $\Pi^{k\phi}_{\mathrm{v}}$ is a v-weighted orthogonal projection on $\mathcal{H}_k$. We show, using the theory of functional calculus of Toeplitz operators developed by Charles in [25], that the v-equivariant Bergman kernel admits an asymptotic expansion given by

$$(2\pi)^m B_k(\mathrm{v}, h) = \mathrm{v}(m_{\omega_h}) + \frac{\mathrm{Scal}_{\mathrm{v}}(\omega_h)}{4k} + \mathcal{O}(\frac{1}{k^2}).$$

The above asymptotic expansion will be used to extend the Donaldson quantization scheme via approximations by balanced metrics. Also, it provides an asymptotic expansion for the trace $\mathrm{tr}(\mathrm{v}(A^{(k)}_{\xi_1}, \cdots, A^{(k)}_{\xi_\ell}))$, which allows us to give a quantized version for the (v, w)-Futaki invariant on a smooth polarized variety (X, L), and leads to a notion of (v, w)-*weighted K-stability* extending the usual K-stability obstruction to the existence of cscK metrics on (X, L) [89].

Thus motivated, the main achievement of this thesis is the proposition of a suitable generalization of the YTD correspondence for the problem of finding weighted cscK metrics, by extending the corresponding notion of K-stability in the cscK and the extremal cases, introduced by Donaldson [42], Tian [92, 94], and Székelyhidi [88]. We shall also establish one direction in this correspondence, by showing that a (v, w)-cscK metric is a minimum of the (v, w)-Mabuchi energy, and that the boundedness of the (v, w)-Mabuchi energy implies (v, w)-K-semistability.

In light of the recent generalization of the YTD conjecture to arbitrary Kähler manifolds by Dervan-Ross [36, 37] and Dyrfelt [46, 47], a definition of K-stability can be obtained from the intersection theoretic formula for the Donaldson–Futaki invariant due to Odaka

[80] and Wang [97]. However, it is not clear how to generalize directly the approaches of Odaka and Wang to define a (v,w)-Futaki invariant for $\mathbb{T}$-equivariant test configurations. We overcome this problem by defining the (v,w)-Futaki invariant as a global differential geometric quantity of the test configuration, given by the slope of the weighted Mabuchi energy on a family of Kähler potentials associated to the test-configuration.

A natural question that arises in the case when the test-configuration is a polarized projective variety is the interpretation of the (v,w)-Futaki invariant in terms of a purely algebraic invariant defined on the central fibre. This was in fact the initial approach of Tian [94] and Donaldson [44] in the cscK case for defining an invariant of a test configuration, and a similar definition of a (v,w)-Donaldson-Futaki invariant on the central fiber has been proposed in [9] (regarding the cases 4 and 5). We review this approach in Section 4.2. At this point, it is not clear to us whether or not such an algebraic definition of a (v,w)-Donaldson–Futaki invariant can be given for any central fibre, nor that it would agree with the differential geometric definition on the total space of a smooth test configuration we propose in this thesis. In fact, when v,w are not polynomials, the proposed algebraic definition of a (v,w)-Donaldson-Futaki invariant of X_0 involves transcendental quantities leading to difficulties reminiscent to the ones involved in the definition of the L^p-norm of a test configuration for positive real values of p, see the discussion at the end of [43].

Now, we give an outline of the principal results of this thesis.

In Chapter 2, we introduce the notion of weighted cscK metrics and describe the relevant examples. We recast the problem of finding weighted cscK metrics within the framework of moment maps, extending the momentum map picture of Donaldson [40] and Fujiki [48] in the cscK case. We also define a first obstruction to the existence of a weighed cscK metric in a Kähler class, in terms of a differential geometric Futaki invariant, and set a variational formulation for the problem of finding weighted cscK metrics in a Kähler class, in terms of minimizing a modified Mabuchi energy. The main results of this chapter are extensions of two fundamental results in the theory of extremal Kähler metrics to the more general (v,w)-cscK context. The first result is a generalization of Calabi's Theorem [22, 78] on the structure of the group of holomorphic automorphisms of a compact extremal manifold.

Theorem 1. *Let (X,ω,g) a compact Kähler manifold and g a (v,w)-extremal metric with v,w positive. Then the group $\mathrm{Isom}_0^{\mathbb{T}}(X,g)$ of $\mathbb{T}$-equivariant isometries of X is a maximal compact connected subgroup of the identity component of the $\mathbb{T}$-equivariant automorphisms $\mathrm{Aut}_0^{\mathbb{T}}(X)$ of X. In particular, if the metric g is a (v,w)-cscK metric (with $\mathrm{w}>0$), then $\mathrm{Aut}_0^{\mathbb{T}}(X)$ is a reductive complex Lie group.*

This result was independently proved in [51] and [61] in the case 4 of cKEM metrics. The case 3 is originally established by Tian-Zhu in [95].

Our second result is a suitable modification of the stability of the existence of extremal Kähler metrics under deformations of the Kähler class, proved by LeBrun–Simanca in [67, 68]. In the weighted setting, we show that if a compact Kähler manifold admits a (v,w)-extremal Kähler metric in a Kähler class α, then a small deformation of α admits a $(\tilde{\mathrm{v}},\tilde{\mathrm{w}})$-extremal metric with weights $(\tilde{\mathrm{v}},\tilde{\mathrm{w}})$ close to (v,w).

Theorem 2. *Suppose that $\omega\in\alpha$ is a (v,w)-extremal Kähler metric invariant with respect to a maximal torus $\mathbb{T}_{\max}\subset\mathrm{Aut}_{\mathrm{red}}(X)$ with momentum polytope P_α and $\mathrm{v},\mathrm{w}>0$ smooth functions over an open set $U\subset\mathfrak{t}^*$ such that $\mathrm{P}_\alpha\subset U$. Then for $\tilde{\mathrm{v}},\tilde{\mathrm{w}}\in C^\infty(U,\mathbb{R}_{>0})$, there exist $\varepsilon>0$, such that for any $|s|<\varepsilon,|t|<\varepsilon$, $|r|<\varepsilon$, there exists a $(\mathrm{v}+t\tilde{\mathrm{v}},\mathrm{w}+s\tilde{\mathrm{w}})$-extremal Kähler metric in the Kähler class $\alpha+r\beta$, associated to $(\mathrm{v}+t\tilde{\mathrm{v}},\mathrm{w}+s\tilde{\mathrm{w}})$ and momentum polytope $\mathrm{P}_{\alpha+r\beta}$.*

In Chapter 3, we extend Donaldson's quantization scheme of cscK metrics. The basic tool is the use of the asymptotic expansion of equivariant weighted Bergman kernels on the finite dimensional spaces $\mathcal{H}_k$ of holomorphic sections of a prequantization line bundle line bundle $L^{\otimes k}\to X$, $k\gg 1$. Our main result gives an obstruction to the existence of weighted cscK metrics in an integral Kähler class $\alpha=2\pi c_1(L)$ in terms of the boundedness of the (v,w)-Mabuchi functional introduced in Chapter 1.

Theorem 3. *Let (X,L) be a compact smooth polarized projective variety, $\mathbb{T}\subset\mathrm{Aut}_{\mathrm{red}}(X)$*

a real torus, and suppose that X admits a (v,w)*-cscK metric* ω *in* $\alpha = 2\pi c_1(L)$ *for some smooth functions* $\mathrm{v} > 0$ *and* w *on the momentum image* $\mathrm{P}_L \subset \mathfrak{t}^*$. *Then,* ω *is a global minima of the* (v,w)*-Mabuchi energy* $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ *of* $(\alpha, \mathbb{T}, \mathrm{P}_L, \mathrm{v}, \mathrm{w})$.

In Chapter 4, we introduce the notion of (v,w)-*K-stability* associated to $(X, \alpha, \mathbb{T}, \mathrm{P}_\alpha, \mathrm{v}, \mathrm{w})$, extending the corresponding notions in the cscK and the extremal cases, introduced by Donaldson [42], Tian [92,94], and Székelyhidi [88]. We define the (v,w)-Futaki invariant $\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}, \mathcal{A})$ of a smooth Kähler test configuration $(\mathcal{X}, \mathcal{A}, \hat{\mathbb{T}})$ with reduced central fibre, compatible with $(X, \alpha, \mathbb{T})$, as a global differential geometric quantity on the total space $\mathcal{X}$ and show that it must be non-negative should the (v,w)-Mabuchi energy associated to $(\alpha, \mathbb{T}, \mathrm{P}, \mathrm{v}, \mathrm{w})$ be bounded from below. This, combined with Theorem 3 yields one direction of a YTD type correspondence for the existence of (v,w)-cscK metrics.

Theorem 4. *Let* (X, L) *be a compact smooth polarized projective variety,* $\mathbb{T} \subset \mathrm{Aut}(X, L)/\mathbb{C}^*$ *a real torus, and suppose that X admits a* (v,w)*-cscK metric in* $\alpha = 2\pi c_1(L)$. *Then* X *is* (v,w)*-K-semistable on smooth,* $\mathbb{T}$*-compatible Kähler test configuration with reduced central fibre associated to* (X, α), *i.e. the* (v,w)*-Futaki invariant of any such test configuration is non-negative.*

In Chapter 5, we give specific applications of the results of the previous chapters to the problem of the existence and uniqueness of cKEM metrics.

Theorem 1 combined with the results in [9] and [65], where conformally-Kähler, Einstein–Maxwell metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are constructed, leads to

Corollary 1. *Any conformally-Kähler, Einstein–Maxwell metric on* $\mathbb{CP}^1 \times \mathbb{CP}^1$, *must be toric, and if it is not a product of Fubini-Study metrics on each factor, it must be homothetically isometric to one of the metrics constructed by LeBrun in* [65].

We can consider the case when $(X, \alpha, \mathbb{T})$ is a $\mathbb{P}^1$-bundle over the product of cscK smooth projective manifolds, given by the Calabi construction of [8]. We compute the (v,w)-Futaki invariant of certain test configurations of $(X, \alpha, \mathbb{T})$, which together with Theorem 2 and 3 yields to the following classification result.

Corollary 2. *Let* $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to C$ *be a geometrically ruled complex surface over a* *compact complex curve C of genus $\mathbf{g} \geq 2$, where $\mathcal{L}$ is a holomorphic line bundle over C of positive degree, and $\alpha_\kappa = 2\pi \left(c_1(\mathcal{O}(2)_{\mathbb{P}(\mathcal{O}\oplus\mathcal{L})}) + (1+\kappa) \cdot c_1(\mathcal{L}) \right)$, $\kappa > 1$ is the effective parametrization of the Kähler cone of X, up to positive scales, see e.g.* [11, 49]. *Then, there exists a real constant $\kappa_0(X) > 1$, such that for each $\kappa > \kappa_0(X)$, α_κ admits a cKEM metric (see* [60]*), whereas for any $\kappa \in (1, \kappa_0(X)]$, α_κ does not admit a cKEM metric.*

CHAPTER II

AUTOMORPHISMS AND DEFORMATIONS OF KÄHLER METRICS WITH CONSTANT WEIGHTED SCALAR CURVATURE.

2.1 The v-scalar curvature

Let X be a compact Kähler manifold of complex dimension $n \geq 2$. We denote by $\mathrm{Aut}_{\mathrm{red}}(X)$ the reduced automorphism group of X whose Lie algebra $\mathfrak{h}_{\mathrm{red}}$ is the ideal of the real holomorphic vector fields with zeros on X (see [53]). Let $\mathbb{T}$ be an ℓ-dimentional real torus in $\mathrm{Aut}_{\mathrm{red}}(X)$ with Lie algebra $\mathfrak{t}$, and ω a $\mathbb{T}$-invariant Kähler form on X. We denote by $\mathcal{K}_{\omega}^{\mathbb{T}}$ the space of $\mathbb{T}$-invariant Kähler potentials with respect to ω, and for any $\phi \in \mathcal{K}_{\omega}^{\mathbb{T}}$, we let $\omega_{\phi} = \omega + dd^c\phi$ be the corresponding Kähler form in the Kähler class α. It is well known that for any $\phi \in \mathcal{K}_{\omega}^{\mathbb{T}}$ the $\mathbb{T}$-action on X is ω_{ϕ}-Hamiltonian (see [53]) and we choose $m_{\phi} : X \to \mathfrak{t}^*$ to be a ω_{ϕ}-momentum map of $\mathbb{T}$. It is also known [13, 56] that $\mathrm{P}_{\phi} := m_{\phi}(X)$ is a convex polytope in $\mathfrak{t}^*$. Furthermore, the following is true.

Definition 1. Let θ be a $\mathbb{T}$-invariant closed $(1,1)$-form on X. A *θ-momentum map* for the action of $\mathbb{T}$ on X is a smooth $\mathbb{T}$-invariant function $m_{\theta} : X \to \mathfrak{t}^*$ with the property $\theta(\xi, \cdot) = -dm_{\theta}^{\xi}$ for all $\xi \in \mathfrak{t}$.

Lemma 1. *The following facts are equivalent:*

1. *For any $\phi \in \mathcal{K}_{\omega}^{\mathbb{T}}$ we have $\mathrm{P}_{\phi} = \mathrm{P}_{\omega}$.*

2. *For any $\phi \in \mathcal{K}_{\omega}^{\mathbb{T}}$ we have $\int_X m_{\phi}\omega_{\phi}^{[n]} = \int_X m_{\omega}\omega^{[n]}$, where $\omega_{\phi}^{[n]} := \frac{\omega_{\phi}^n}{n!}$ is the volume form.*

3. For any $\xi \in \mathfrak{t}$ and $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$ we have $m_\phi^\xi = m_\omega^\xi + d^c\phi(\xi)$, where $m_\phi^\xi := \langle m_\phi, \xi \rangle$.

Proof. Presumably, Lemma 1 is well-known, see e.g. [12, Section 4] and [90, Section 3.1] for the case of a single hamiltonian. We include here an argument covering the general case for the sake of completeness. We start by proving that 2 is equivalent with 3. By the very definition of the momentum map, Cartan's formula and the fact that ξ is a real holomorphic vector field we have

$$d(m_\omega^\xi - m_\phi^\xi) = -d(d^c\phi(\xi)). \tag{2.1}$$

Thus, there exist a $\lambda_\phi \in \mathfrak{t}^*$ such that

$$m_\phi^\xi = m_\omega^\xi + d^c\phi(\xi) + \lambda_\phi(\xi). \tag{2.2}$$

Suppose that 2 holds. Then λ_ϕ is given by

$$\lambda_\phi(\xi) = \frac{1}{\mathrm{Vol}(X,\alpha)}\left(\int_X m_\omega^\xi \omega^{[n]} - \int_X (m_\omega^\xi + d^c\phi(\xi))\omega_\phi^{[n]}\right).$$

For a variation $\dot\phi$ of ϕ in $\mathcal{K}_\omega^{\mathbb{T}}$, the corresponding variation of λ_ϕ is given by

$$\begin{aligned}
-\mathrm{Vol}(X,\alpha)\dot\lambda_\phi(\xi) &= \int_X m_\omega^\xi dd^c\dot\phi \wedge \omega_\phi^{[n-1]} + \int_X d^c\dot\phi(\xi)\omega_\phi^{[n]} + \int_X d^c\phi(\xi) dd^c\dot\phi \wedge \omega_\phi^{[n-1]} \\
&= \int_X d^c\phi(\xi) dd^c\dot\phi \wedge \omega_\phi^{[n-1]} + \int_X dm_\phi^\xi \wedge d^c\dot\phi \wedge \omega_\phi^{[n-1]} \\
&\quad + \int_X (-dm_\phi^\xi + d(d^c\phi(\xi))) \wedge d^c\dot\phi \wedge \omega_\phi^{[n-1]} \\
&= \int_X d(d^c\phi(\xi)) \wedge d^c\dot\phi \wedge \omega_\phi^{[n-1]} + \int_X d^c\phi(\xi) dd^c\dot\phi \wedge \omega_\phi^{[n-1]} = 0,
\end{aligned}$$

where we have used (2.1), the fact that $d^c\dot\phi(\xi)\omega_\phi^{[n]} = dm_\phi^\xi \wedge d^c\dot\phi \wedge \omega_\phi^{[n-1]}$, and integration by parts. It follows that $\lambda_\phi = \lambda_\omega = 0$ which gives the implication "2⇒3". Conversely if we suppose that 3 holds, then for any variation $\dot\phi$ of ϕ in $\mathcal{K}_\omega^{\mathbb{T}}$, we get

$$\frac{d}{dt}\int_X m_{\phi_t}^\xi \omega_{\phi_t}^{[n]} = \int_X m_{\phi_t}^\xi dd^c\dot\phi \wedge \omega_{\phi_t}^{[n-1]} + d^c\dot\phi(\xi)\omega_{\phi_t}^{[n]} = 0.$$

It follows that $\int_X m_\phi^\xi \omega_\phi^{[n]} = \int_X m_\omega^\xi \omega^{[n]}$ for any $\xi \in \mathfrak{t}$, which yields 2 .

Now we prove the equivalence between 1 and 3. Suppose that 1 is true and let $x \in X$ be a fixed point for the $\mathbb{T}$-action on X. Then we have

$$m_\phi(x) - m_\omega(x) = (d^c\phi)_x + \lambda_\phi = \lambda_\phi. \tag{2.3}$$

By a result of Atiyah and Guillemin–Sternberg (see [13,56]) P_ϕ (resp. P_ω) is the convex hull of the image by m_ϕ (resp. m_ω) of the fixed points for the $\mathbb{T}$-action. It then follows from (2.3) that $\mathrm{P}_\phi = \mathrm{P}_\omega + \lambda_\phi$. Using $\mathrm{P}_\omega = \mathrm{P}_\phi$, we get $\lambda_\phi = 0$ which proves 3. For the inverse implication, if $m_\phi(x) - m_\omega(x) = (d^c\phi)_x$ for any $x \in X$, then $m_\phi(x) = m_\omega(x)$ for any point $x \in X$ fixed by the $\mathbb{T}$-action and we have $\mathrm{P}_\phi = \mathrm{P}_\omega$ by [13,56]. $\square$

It follows from Lemma 1 that for each $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$ we can normalize m_ϕ such that the momentum polytope $\mathrm{P} = m_\phi(X) \subset \mathfrak{t}^*$ is ϕ-independent. This will be an overall assumption through this work.

Definition 2. For $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$ we define the v-*scalar curvature* of the Kähler metric $g_\phi = \omega_\phi(\cdot, J\cdot)$ for $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$ to be the function

$$\mathrm{Scal}_{\mathrm{v}}(\phi) := \mathrm{v}(m_\phi)\mathrm{Scal}(g_\phi) + 2\Delta_\phi(\mathrm{v}(m_\phi)) + \mathrm{tr}(\mathrm{G}_\phi \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\phi)), \tag{2.4}$$

where m_ϕ is the momentum map of ω_ϕ normalized as in Lemma 1, $\mathrm{Scal}(g_\phi)$ is the scalar curvature, Δ_ϕ is the Riemannian Laplacian on functions of the Kähler metric ω_ϕ and $\mathrm{Hess}(\mathrm{v})$ is the hessian of v, viewed as bilinear form on $\mathfrak{t}^*$ whereas G_ϕ is the bilinear form with smooth coefficients on $\mathfrak{t}$, given by the restriction of the Riemannian metric g_ϕ on fundamental vector fields.

In a basis $\boldsymbol{\xi} = (\xi_i)_{i=1,\cdots,\ell}$ of $\mathfrak{t}$ we have

$$\mathrm{tr}(\mathrm{G}_\phi \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\phi)) := \sum_{1 \leq i,j \leq \ell} \mathrm{v}_{,ij}(m_\phi) g_\phi(\xi_i, \xi_j),$$

where $\mathrm{v}_{,ij}$ stands for the partial derivatives of v with respect the dual basis of $\boldsymbol{\xi}$.

Lemma 2. *Let θ be a fixed $\mathbb{T}$-invariant closed $(1,1)$-form with momentum map m_θ and $\mathrm{v} \in C^\infty(X, \mathbb{R}_{>0})$, $\mathrm{w} \in C^\infty(X, \mathbb{R})$. Then with the normalization for m_ϕ given by Lemma 1, the following integrals are independent of the choice of $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$,*

$$\begin{aligned} A_{\mathrm{w}}(\phi) &:= \int_X \mathrm{w}(m_\phi)\omega_\phi^{[n]}, \\ B_{\mathrm{v}}^\theta(\phi) &:= \int_X \mathrm{v}(m_\phi)\theta \wedge \omega_\phi^{[n-1]} + \langle (d\mathrm{v})(m_\phi), m_\theta \rangle \omega_\phi^{[n]}, \\ C_{\mathrm{v}}(\phi) &:= \int_X \mathrm{Scal}_{\mathrm{v}}(\phi)\omega_\phi^{[n]}. \end{aligned}$$

Proof. The fact that $A_{\mathrm{w}}(\phi)$ is constant is well known, see e.g. [36, Theorem 3.14]. The constancy of $B^{\theta}_{\mathrm{v}}(\phi)$ can be easily established by a direct computation, but it also follows from the arguments in the proof of Lemma 4 below. Indeed, we note that $B^{\theta}_{\mathrm{v}}(\phi) = (\mathcal{B}^{\theta}_{\mathrm{v}})_{\phi}(1)$ where $\mathcal{B}^{\theta}_{\mathrm{v}}$ is the 1-form on $\mathcal{K}^{\mathbb{T}}_{\omega}$ given by (2.22). By taking $\dot{\phi} = 1$ in (2.23) we get $(\boldsymbol{\delta} B^{\theta}_{\mathrm{v}})_{\phi}(\dot{\psi}) = 0$ where $\dot{\psi}$ is a $\mathbb{T}$-invariant function on X defining a $\mathbb{T}$-invariant variation $\dot{\omega} = dd^c\dot{\psi}$ of ω_{ϕ}. From this we infer that $B^{\theta}_{\mathrm{v}}(\phi)$ is constant. For the last function $C_{\mathrm{v}}(\phi)$, we will calculate its variation $(\boldsymbol{\delta} C_{\mathrm{v}})_{\phi}(\dot{\phi})$ with respect to a $\mathbb{T}$-invariant variation $\dot{\omega} = dd^c\dot{\phi}$ of ω_{ϕ}. For this, we use that the variation of $\mathrm{Scal}_{\mathrm{v}}(\phi)$ is given by

$$(\boldsymbol{\delta}\mathrm{Scal}_{\mathrm{v}})_{\phi}(\dot{\phi}) = -2(D^{-}d)^{*}\left(\mathrm{v}(m_{\phi})(D^{-}d)\dot{\phi}\right) + (d\mathrm{Scal}_{\mathrm{v}}(\phi), d\dot{\phi})_{\phi}, \tag{2.5}$$

where D is the Levi-Civita connection of ω_{ϕ}, $(D^{-}d)\dot{\phi}$ denotes the $(2,0)+(0,2)$-type part of $(Dd\dot{\phi})$ and $(D^{-}d)^{*}$ is the formal adjoint operator of $(D^{-}d)$ (see [53, Section 1.23]). The above formula (2.5) is established in Lemma 9 below. By (2.5), we calculate

$$\begin{aligned}(\boldsymbol{\delta} C_{\mathrm{v}})_{\phi}(\dot{\phi}) = &\int_X -2(D^{-}d)^{*}\mathrm{v}(m_{\phi})(D^{-}d)(\dot{\phi})\omega_{\phi}^{[n]} + \int_X d\mathrm{Scal}_{\mathrm{v}}(\phi)\wedge d^c\dot{\phi}\wedge\omega_{\phi}^{[n-1]} \\ &+ \int_X \mathrm{Scal}_{\mathrm{v}}(\phi) dd^c\dot{\phi}\wedge\omega_{\phi}^{[n-1]}.\end{aligned}$$

Integration by parts yields $(\boldsymbol{\delta} C_{\mathrm{v}})_{\phi} = 0$. Thus C_{v} does not depend on the choice of $\phi \in \mathcal{K}^{\mathbb{T}}_{\omega}$. □

Definition 3. Let (X,α) be a compact Kähler manifold, $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ a real torus with momentum image $\mathrm{P} \subset \mathfrak{t}^{*}$ associated to α as in Lemma 1, and $\mathrm{v} \in C^{\infty}(\mathrm{P}, \mathbb{R}_{>0})$, $\mathrm{w} \in C^{\infty}(\mathrm{P}, \mathbb{R})$. The (v,w)-*slope* of (X,α) is the constant given by

$$c_{(\mathrm{v},\mathrm{w})}(\alpha) := \begin{cases} \frac{\int_X \mathrm{Scal}_{\mathrm{v}}(\omega)\omega^{[n]}}{\int_X \mathrm{w}(m_{\omega})\omega^{[n]}}, & \text{if } \int_X \mathrm{w}(m_{\omega})\omega^{[n]} \neq 0 \\ 1, & \text{if } \int_X \mathrm{w}(m_{\omega})\omega^{[n]} = 0, \end{cases} \tag{2.6}$$

which is independent from the choice of $\omega \in \alpha$ by virtue of Lemma 2.

Remark 1. If $\phi \in \mathcal{K}^{\mathbb{T}}_{\omega}$ defines a Kähler metric which satisfies $\mathrm{Scal}_{\mathrm{v}}(\phi) = c\mathrm{w}(m_{\phi})$ for some real constant c and $\int_X \mathrm{w}(m_{\omega})\omega^{[n]} \neq 0$, then we must have $c = c_{(\mathrm{v},\mathrm{w})}(\alpha)$ with $c_{\mathrm{v},\mathrm{w}}(\alpha)$ given by (2.6). ◇

Because of Remark 1 above, and to simplify the notation in the case when $\int_X \mathrm{w}(m_\omega)\omega^{[n]} = 0$, we adopt the following definition

Definition 4. Let (X, α) be a compact Kähler manifold, $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ a real torus with momentum image $\mathrm{P} \subset \mathfrak{t}^*$ associated to α as in Lemma 1, and $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$, $\mathrm{w} \in C^\infty(\mathrm{P}, \mathbb{R})$. A (v, w)-*cscK* metric $\omega \in \alpha$ is a $\mathbb{T}$-invariant Kähler metric satisfying (1.4), where $c_{\mathrm{v},\mathrm{w}}(\alpha)$ is given by (2.6).

2.2 Examples

We list below some geometrically significant examples of (v, w)-cscK metrics, obtained for special values of the weight functions v, w.

2.2.1 Constant scalar curvature and extremal Kähler metrics

When $\mathrm{v} \equiv 1$, $\mathrm{Scal}_{\mathrm{v}}(\phi) = \mathrm{Scal}(\phi)$ is the usual scalar curvature of the Kähler metric $\omega_\phi \in \mathcal{K}_\omega^{\mathbb{T}}$, so letting $\mathrm{w} \equiv 1$ the problem (1.4) reduces to the Calabi problem of finding a cscK metric in the Kähler class $\alpha = [\omega]$. In this case, we can take $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ to be a maximal torus by a result of Calabi [22]. More generally, for a fixed maximal torus $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ we can consider the more general problem of the existence of an *extremal Kähler metric* in $\mathcal{K}_\omega^{\mathbb{T}}$, i.e. a Kähler metric ω_ϕ such that $\mathrm{Scal}(\phi)$ is a Killing potential for ω_ϕ. As the Killing vector field ξ_{ext} generated by $\mathrm{Scal}(\phi)$ is $\mathbb{T}$-invariant, it belongs to the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ (by the maximality of $\mathbb{T}$). More generally, Futaki–Mabuchi [50] observed that for any $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$, the L^2 projection $\check{\mathrm{Scal}}(\phi)$ (with respect to the global inner product on smooth functions defined by ω_ϕ) of $\mathrm{Scal}(\phi)$ to the sub-space $\{m_\phi^\xi + c,\ c \in \mathbb{R}\}$ of Killing potentials for $\xi \in \mathfrak{t}$ defines a ϕ-independent element $\xi_{\mathrm{ext}} \in \mathfrak{t}$, i.e. $\check{\mathrm{Scal}}(\phi) = m_\phi^{\xi_{\mathrm{ext}}} + c_\phi$. The vector field ξ_{ext} is called the *extremal vector field* of $(X, \alpha, \mathbb{T})$. Furthermore, using the normalization for the moment map m_ϕ in Lemma 1, we see that

$$
\begin{aligned}
4\pi c_1(X)\cup\alpha^{[n-1]} &= \int_X \mathrm{Scal}(\phi)\omega_\phi^{[n]} = \int_X \check{\mathrm{Scal}}(\phi)\omega_\phi^{[n]}\\
&= \int_X m_\phi^{\xi_{\rm ext}}\omega_\phi^{[n]} + c_\phi \mathrm{Vol}(X,\alpha),
\end{aligned}
$$

showing that the real constant $c_{\rm ext} = c_\phi$ is independent of ω_ϕ too. Thus, there exists an affine-linear function $\mathrm{w}_{\rm ext}(p) = \langle \xi_{\rm ext}, p\rangle + c_{\rm ext}$ on $\mathfrak{t}^*$, such that $\omega_\phi \in \mathcal{K}_\omega^{\mathbb{T}}$ is extremal if and only if $\mathrm{Scal}_{\rm v}(\phi) = \mathrm{w}_{\rm ext}(m_\phi)$ i.e. if and only if ω_ϕ is $(1, \mathrm{w}_{\rm ext})$-cscK (as $c_{1,\mathrm{w}_{\rm ext}}(\alpha) = 1$ by definition of $\mathrm{w}_{\rm ext}$).

2.2.2 (v, w)-extremal Kähler metrics

As mentioned in the Introduction one can consider instead of (1.4) the more general problem (1.5) of finding a (v,w)-extremal Kähler metric ω_ϕ for $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$. It turns out that if $\mathrm{w}(p) > 0$ on P, similarly to the previous example, one can reduce the problem (1.5) to the problem (1.4) with the same v but a different w. This essentially follows from Theorem 5 below, which implies that for any $\mathbb{T}$-invariant, ω-compatible Kähler metric g, the orthogonal projection of $\mathrm{Scal}_{\rm v}(g)/\mathrm{w}(m_\omega)$ to the space of affine-linear functions in momenta with respect to the w-weighted global inner product (2.17) is independent of g. Using the $\mathbb{T}$-equivariant Moser lemma for a Kähler metric $\omega_\phi \in \mathcal{K}_\omega^{\mathbb{T}}$ and the normalization for m_ϕ given by Lemma 1, one can conclude as in the proof of [9, Cor. 2] that there exist a ϕ-independent affine-linear function $\mathrm{w}_{\rm ext}(p)$ such that $m_\phi^\xi + c = \mathrm{w}_{\rm ext}(m_\phi)$ for any metric in $\mathcal{K}_\omega^{\mathbb{T}}$ satisfying (1.5). In other words, if $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$ is (v,w)-extremal then ω is $(\mathrm{v}, \mathrm{w}\mathrm{w}_{\rm ext})$-cscK. Conversely if ω_ϕ is $(\mathrm{v},\mathrm{w}\mathrm{w}_{\rm ext})$-cscK, then $\mathrm{Scal}_{\rm v}(\omega_\phi) = c_{\mathrm{v},\mathrm{w}\mathrm{w}_{\rm ext}}(\alpha)\mathrm{w}(m_\phi)\mathrm{w}_{\rm ext}(m_\phi)$ where $c_{\mathrm{v},\mathrm{w}\mathrm{w}_{\rm ext}}(\alpha)$ is given by (2.6). We claim that $c_{\mathrm{w},\mathrm{w}\mathrm{w}_{\rm ext}}(\alpha) = 1$, which in turn implies that ω_ϕ is (v,w)-extremal. Indeed, if $\int_X \mathrm{w}(m_\phi)\mathrm{w}_{\rm ext}(m_\phi)\omega_\phi^{[n]} = 0$, then $c_{\mathrm{v},\mathrm{w}\mathrm{w}_{\rm ext}}(\alpha) = 1$ by Definition 3. Otherwise, if $\int_X \mathrm{w}(m_\phi)\mathrm{w}_{\rm ext}(m_\phi)\omega_\phi^{[n]} \neq 0$, we get

$$
\begin{aligned}
c_{\mathrm{v},\mathrm{w}\mathrm{w}_{\rm ext}}(\alpha)\int_X \mathrm{w}(m_\phi)\mathrm{w}_{\rm ext}(m_\phi)\omega_\phi^{[n]} &= \int_X (\mathrm{Scal}_{\rm v}(\phi)/\mathrm{w}(m_\phi))\mathrm{w}(m_\phi)\omega_\phi^{[n]}\\
&= \int_X \mathrm{w}_{\rm ext}(m_\phi)\mathrm{w}(m_\phi)\omega_\phi^{[n]},
\end{aligned}
$$

showing again that $c_{\mathrm{v},\mathrm{w}\mathrm{w}_{\rm ext}}(\alpha) = 1$.

2.2.3 The Kähler-Ricci solitons

This is the case when X is a smooth Fano manifold, $\alpha = 2\pi c_1(X)$ corresponds to the anti-canonical polarization, $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ is a maximal torus with momentum image P, and $\mathrm{v}(p) = \mathrm{w}(p) = e^{\langle \xi, p \rangle}$ for some $\xi \in \mathfrak{t}$. It was shown recently in [58] that a (v, w)-extremal metric with $\mathrm{w}_{\mathrm{ext}}(p) = 2(\langle \xi, p \rangle + c)$ (for some real constant c) corresponds to a Kähler metric $\omega \in \alpha$ which is a *gradient Kähler–Ricci soliton* with respect to ξ, i.e. satisfies

$$\mathrm{Ric}(\omega) - \omega = -\frac{1}{2}\mathcal{L}_{J\xi}\omega, \tag{2.7}$$

where $\mathrm{Ric}(\omega)$ is the Ricci form of ω. We include the verification of this claim for the convenience of the reader. We start by supposing that ω is a gradient Kähler–Ricci soliton, then we can rewrite (2.7) as

$$\mathrm{Ric}(\omega) - \omega = \frac{1}{2} dd^c m_\omega^\xi. \tag{2.8}$$

Taking the trace with respect to ω of (2.8), we get

$$\mathrm{Scal}(\omega) - 2n = -\Delta_\omega(m_\omega^\xi). \tag{2.9}$$

Taking the Lie derivative of (2.8), we obtain

$$-\frac{1}{2} dd^c \Delta_\omega(m_\omega^\xi) + dd^c m_\omega^\xi = -\frac{1}{2} dd^c |\xi|_\omega^2,$$

which yields the following identity

$$\Delta_\omega(m_\omega^\xi) - |\xi|_\omega^2 = 2m_\omega^\xi + c, \tag{2.10}$$

where c is a constant. Now, using (2.9) and (2.10), it follows that

$$\begin{aligned}\frac{\mathrm{Scal}_{\mathrm{v}}(\omega)}{\mathrm{w}(m_\omega)} &= \mathrm{Scal}(\omega) + 2\Delta_\omega(m_\omega^\xi) - |\xi|_\omega^2 \\ &= 2(m_\omega^\xi + c),\end{aligned}$$

i.e. ω is a (v, w)-extremal metric with $\mathrm{w}_{\mathrm{ext}}(p) = 2(\langle \xi, p \rangle + c)$. Conversely, suppose that ω is a (v, w)-extremal metric such that

$$\frac{\mathrm{Scal}_{\mathrm{v}}(\omega)}{\mathrm{w}(m_\omega)} = 2(\langle \xi, p \rangle + c), \tag{2.11}$$

and let p_ω be a $\mathbb{T}$-invariant function such that

$$\mathrm{Ric}(\omega) - \omega = \frac{1}{2} dd^c p_\omega. \tag{2.12}$$

As before, taking the trace and the Lie derivative of (2.12), we obtain the following identities

$$\mathrm{Scal}(\omega) - 2n = -\Delta_\omega(p_\omega), \quad \Delta_\omega(m_\omega^\xi) + \mathcal{L}_{J\xi} p_\omega = 2m_\omega^\xi + c', \tag{2.13}$$

where c' is a constant. Using (2.11) and (2.13) we get

$$2c = \frac{\mathrm{Scal}_{\mathrm{v}}(\omega)}{\mathrm{w}(m_\omega)} - 2m_\omega^\xi = (2n + c') + [\Delta_\omega + \mathcal{L}_{J\xi}](m_\omega^\xi - p_\omega).$$

Thus $[\Delta_\omega + \mathcal{L}_{J\xi}](m_\omega^\xi - p_\omega) = \mathrm{cst}$. It follows that $m_\omega^\xi - p_\omega$ is a constant function by the maximum principle. Consequently, ω is a gradient Kähler-Ricci soliton.

Thus, the theory of gradient Kähler-Ricci solitons (see e.g. [16,23,95,96]) fits in to our setting too. Further ramifications of this setting appear in [58].

2.2.4 Kähler metrics conformal to Einstein–Maxwell metrics (cKEM)

These are the metrics introduced by (1.2) in the Introduction. They have been studied in [9,10,51,52,60–62,64–66]. One can easily check that a Kähler metric satisfies (1.2) if and only if it is a (v, w)-cscK metrics with

$$\mathrm{v}(p) = (\langle \xi, p\rangle + a)^{-2m+1} \text{ and } \mathrm{w}(p) = (\langle \xi, p\rangle + a)^{-2m-1},$$

where $\langle \xi, p\rangle + a$ is positive affine-linear function on P. In this case, $\mathrm{Scal}_{\mathrm{v}}(\phi)/\mathrm{w}(m_\phi)$ equals to the usual scalar curvature of the Hermitian metric $\tilde{g}_\phi = \frac{1}{(m_\phi^\xi + a)^2} g_\phi$. Thus, a (v, w)-cscK metric ω_ϕ gives rise to a conformally Kähler, Hermitian metric $\tilde{g}_\phi$ which has Hermitian Ricci tensor and constant scalar curvature. The latter include the conformally Kähler, Einstein metrics classified in [28,35].

2.2.5 Extremal Sasaki metrics

Following [3], let (X, L) be a smooth compact polarized variety and $\alpha = 2\pi c_1(L)$ the corresponding Kähler class. Recall that for any Kähler metric $\omega \in \alpha$, there exits a unique

Hermitian metric h on L, whose curvature is ω. We denote by h^* the induced Hermitian metric on the dual line bundle L^*. It is well-known (see e.g. [21]) that the principal circle bundle $\pi : S \to X$ of vectors of unit norm of (L^*, h^*) has the structure of a Sasaki manifold, i.e. there exists a contact 1-form θ on S with $d\theta = \pi^*\omega$, defining a contact distribution $D \subset TS$ and a Reeb vector field χ given by the generator of the $\mathbb{S}^1$-action on the fibres of S, and a CR-structure J on D induced from the complex structure of L^*. The Sasaki structure (θ, χ, D, J) on S in turn defines a transversal Kähler structure (g_χ, ω_χ) on D by letting $\omega_\chi = (d\theta)_D$ and $g_\chi = -(d\theta)_D \circ J$, where the subscript D denotes restriction to $D \subset TS$; it is a well-known fact that (g_χ, ω_χ) coincides with the restriction to D of the pull-back of the Kähler structure (g, ω) on X or, equivalently, that (g_χ, ω_χ) induces the initial Kähler structures (g, ω) on the orbit space $X = S/\mathbb{S}^1_\chi$ for the $\mathbb{S}^1$-action $\mathbb{S}^1_\chi$ generated by χ.

Let $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ be a maximal torus, with a fixed momentum polytope $\mathrm{P} \subset \mathfrak{t}^*$ associated to the Kähler class α as in Lemma 1. We suppose that ω is a $\mathbb{T}$-invariant Kähler metric in α. For any positive affine-linear function $\langle \xi, p \rangle + a$ on P, we consider the corresponding Killing potential $f = m_\omega^\xi + a$ of ω and define the lift ξ_f of the Killing vector field $\xi \in \mathfrak{t}$ on X to S by

$$\xi_f = \xi^D + (\pi^* f)\chi,$$

where the super-scrip D stands for the horizontal lift. It is easily checked that ξ_f preserves the contact distribution D and the CR-structure J, and defines a new Sasaki structure $((\pi^* f)^{-1}\theta, \xi_f, D, J)$ on S. In general, the flow of ξ_f is not periodic, and the orbit space of ξ_f is not Hausdorff, but when it is, $X_f := S/\mathbb{S}^1_{\xi_f}$ is a compact complex orbifold endowed with a Kähler structure (g_f, ω_f). In [3], the triple (X_f, g_f, ω_f) is referred to as a *CR f-twist* of (X, ω, g) and it is shown there that (X_f, g_f, ω_f) is an extremal Kähler manifold or orbifold in the sense of Sect. 2.2.1 iff (X, ω, g) is (v, w)-extremal in the sense of Sect. 2.2.2 with

$$\mathrm{v}(p) = (\langle \xi, p \rangle + a)^{-m-1} \quad \text{and} \quad \mathrm{w}(p) = (\langle \xi, p \rangle + a)^{-m-3}. \tag{2.14}$$

In general, by using the $\mathbb{T}$-equivariant Moser lemma, a (v, w)-extremal metric $\omega_\phi \in \mathcal{K}_\omega^{\mathbb{T}}$

(with v, w given by (2.14)) gives rise to an extremal Sasaki structure (ξ_f, D, J_ϕ) on S within a class of (ξ_f, D)-compatible CR structures parametrized by $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$. By the discussion in Sect. 2.2.2, Theorem 4 provides an obstruction to the existence of such Sasaki structures. We note that a similar obstruction theory has been developed in [32] and [21] in terms of the complex affine variety $Y^{2n+1} := L^* \setminus \{0\}$ viewed as the cone over X.

2.2.6 The generalized Calabi construction and manifolds without multiplicities

In [8], the authors consider smooth compact manifolds X which are fibre-bundles over the product of cscK Hodge manifolds $(B, \omega_B) = (B_1, \omega_1) \times \cdots \times (B_N, \omega_N)$ with fibre a smooth ℓ-dimensional compact toric Kähler manifold $(V, \omega_V, \mathbb{T})$. More precisely, X is a V-fibre bundle associated to a certain principle $\mathbb{T}$-bundle over B. They introduce a class of $\mathbb{T}$-invariant Kähler metrics on X, compatible with the bundle structure, which are parametrized by ω_V-compatible toric Kähler metrics on V, and refer to them as Kähler metrics given by the *generalized Calabi construction.* The condition for the metric ω on X to be extremal is computed in [8] and can be re-written in our formulation as (see (5.3) below)

$$\mathrm{Scal}_{\mathrm{v}}(g_V) = \mathrm{w}(m), \tag{2.15}$$

where g_V is the corresponding toric Kähler metric on (V, ω_V), with

$$\begin{aligned} \mathrm{v}(p) &= \prod_{j=1}^{N} (\langle \xi_j, p \rangle + c_j)^{d_j}, \\ \mathrm{w}(p) &= (\langle \xi_0, p \rangle + c_0) \prod_{j=1}^{N} (\langle \xi_j, p \rangle + c_j)^{d_j} - \sum_{j=1}^{N} \mathrm{Scal}_j \Big(\frac{\prod_{k=1}^{N} (\langle \xi_k, p \rangle + c_k)^{d_j}}{(\langle \xi_j, p \rangle + c_j)} \Big). \end{aligned} \tag{2.16}$$

In the above expressions $m : V \to \mathfrak{t}^*$ stands for the momentum map of $(V, \omega_V, \mathbb{T})$, d_j and Scal_j denote the complex dimension and (constant) scalar curvature of (B_j, ω_j), respectively, whereas the affine-linear functions $(\langle \xi_k, p \rangle + c_k), k = 1, \cdots, N$ on $\mathfrak{t}^*$ are determined by the topology and the Kähler class $\alpha = [\omega]$ of X, and satisfy $(\langle \xi_j, p \rangle + c_j) > 0$ for $j = 1, \cdots, N$ on the Delzant polytope $\mathrm{P} = m(V)$. Thus, a Kähler metric ω on X given by the generalized Calabi ansatz is extremal if and only if the corresponding toric

Kähler metric g_V on V is (v,w)-extremal for the values of v,w given in (2.16). More generally, considering an arbitrary weight function w in (2.15) allows one to prescribe the scalar curvature of the Kähler metrics given by the geberalized Calabi construction on X. We note that a very similar equation for a toric Kähler metric on V appears in the construction of Kähler manifolds without multiplicities, see [45, 82]. We refer the Reader to [70, 71] for a comprehensive study of the equation (2.15) on a toric variety, for arbitrary weight functions $\mathrm{v}(p) > 0$ and $\mathrm{w}(p)$, which is referred to as the *generalized Abreu equation.*

2.3 A formal momentum map picture

In this section we extend the momentum map interpretation, originally introduced Donaldson [40] and Fujiki [48] in the cscK case and generalized by Apostolov–Maschler [9] to the case of conformally Einstein Maxwell, Kählerian metrics, to arbitrary positive weights v,w on P.

In the notation of Section 2.1, let $\mathcal{AC}_\omega^{\mathbb{T}}$ be the space of all ω-compatible, $\mathbb{T}$-invariant almost complex structures on (X,ω) and $\mathcal{C}_\omega^{\mathbb{T}} \subset \mathcal{AC}_\omega^{\mathbb{T}}$ the subspace of $\mathbb{T}$-invariant Kähler structures. We consider the natural action on $\mathcal{AC}_\omega^{\mathbb{T}}$ of the infinite dimensional group $\mathrm{Ham}^{\mathbb{T}}(X,\omega)$ of $\mathbb{T}$-equivariant Hamiltonian transformations of (X,ω), which preserves $\mathcal{C}_\omega^{\mathbb{T}}$. We identify $\mathrm{Lie}\left(\mathrm{Ham}^{\mathbb{T}}(X,\omega)\right) \cong C^\infty(X,\mathbb{R})^{\mathbb{T}}/\mathbb{R}$ where $C^\infty(X,\mathbb{R})^{\mathbb{T}}/\mathbb{R}$ is endowed with the Poisson bracket.

For any $\mathrm{v} \in C^\infty(\mathrm{P},\mathbb{R}_{>0})$, the space $\mathcal{AC}_\omega^{\mathbb{T}}$ carries a weighted formal Kähler structure $(\mathbf{J},\boldsymbol{\Omega}^{\mathrm{v}})$ given by ([9, 40, 48])

$$\begin{aligned}\boldsymbol{\Omega}^{\mathrm{v}}_J(\dot{J}_1,\dot{J}_2) :=& \frac{1}{2}\int_X \mathrm{Tr}(J\dot{J}_1\dot{J}_2)\mathrm{v}(m_\omega)\omega^{[n]},\\ \mathbf{J}_J(\dot{J}) :=& J\dot{J},\end{aligned}$$

in which the tangent space of $\mathcal{AC}_\omega^{\mathbb{T}}$ at J is identified with the space of smooth $\mathbb{T}$-invariant sections $\dot{J}$ of $\mathrm{End}(TX)$ satisfying

$$\dot{J}J + J\dot{J} = 0, \quad \omega(\dot{J}\cdot,\cdot) + \omega(\cdot,\dot{J}\cdot) = 0.$$

In what follows, we denote by $g_J := \omega(\cdot, J\cdot)$ the almost Kähler metric corresponding to $J \in \mathcal{AC}_\omega^{\mathbb{T}}$, and index all objects calculated with respect to J similarly. On $C^\infty(X,\mathbb{R})^{\mathbb{T}}$, for $\mathrm{w} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$, we consider the Ad-invariant scalar product given by,

$$\langle \phi, \psi \rangle_{\mathrm{w}} := \int_X \phi\psi \mathrm{w}(m_\omega)\omega^{[n]}, \tag{2.17}$$

Theorem 5. [9, 40, 48] *The action of* $\mathrm{Ham}^{\mathbb{T}}(X,\omega)$ *on* $\left(\mathcal{AC}_\omega^{\mathbb{T}}, \mathbf{J}, \mathbf{\Omega}^{\mathrm{v}}\right)$ *is a Hamiltonian action whose momentum map at* $J \in \mathcal{C}_\omega^{\mathbb{T}}$ *is the* $\langle .,. \rangle_{\mathrm{w}}$*-dual of* $\left(\frac{\mathrm{Scal}_{\mathrm{v}}(J)}{\mathrm{w}(m_\omega)} - c_{\mathrm{v},\mathrm{w}}([\omega])\right)$*, where* $\mathrm{Scal}_{\mathrm{v}}(J)$ *is the* v*-scalar curvature of* g_J *given by* (2.4) *and the real constant* $c_{\mathrm{v},\mathrm{w}}([\omega])$ *is given by* (2.6).

Proof. The proof follows from the computation of [9, 53] but we give the details for sake of clarity. Let $h \in C^\infty(X,\mathbb{R})^{\mathbb{T}}$. Integration by parts gives

$$\begin{aligned}
&\langle \mathrm{Scal}_{\mathrm{v}}(J)/\mathrm{w}(m_\omega), h\rangle_{\mathrm{w}} \\
&= \int_X \mathrm{Scal}(J) h \mathrm{v}(m_\omega)\omega^{[n]} + 2\int_X \Delta_J\left(\mathrm{v}(m_\omega)\right) h\omega^{[n]} + \int_X \mathrm{tr}\left[\mathrm{G}_J \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)\right] h\omega^{[n]} \\
&= \int_X \mathrm{Scal}(J)\mathrm{v}(m_\omega) h\omega^{[n]} + 2\int_X g_J\left(d(\mathrm{v}(m_\omega)), dh\right)\omega^{[n]} \\
&\quad + \int_X \mathrm{tr}\left[\mathrm{G}_J \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)\right] h\omega^{[n]}.
\end{aligned}$$

Let $J_t \in \mathcal{AC}^{\mathbb{T}}(X,\omega)$ be a path of almost complex structures with $J_0 = J$ and first variation $\dot{J} = \frac{d}{dt}J_t$. Then, $\dot{g}_J = g_J(\cdot, \dot{J}J\cdot)$. According to [53, Proposition 9.5.2], the first variation of Scal_{J_t} is given by $\dot{\mathrm{Scal}}_J = -\delta J(\delta\dot{J})$. It follows that

$$\begin{aligned}
&\frac{d}{dt}\langle \mathrm{Scal}_{\mathrm{v}}(J_t)/\mathrm{w}(m_\omega), h\rangle_{\mathrm{w}} \\
&= -\int_X (\delta J\delta\dot{J})\mathrm{v}(m_\omega) h\omega^{[n]} + 2\int_X g_J(d(\mathrm{v}(m_\omega)), \dot{J}Jdh)\omega^{[n]} \\
&\quad + \int_X \mathrm{tr}[\dot{\mathrm{G}}_J \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)] h\omega^{[n]} \\
&= \int_X g_J(\dot{J}, DJd(\mathrm{v}(m_\omega)h))\omega^{[n]} + 2\int_X g_J(d(\mathrm{v}(m_\omega)), \dot{J}Jdh)\omega^{[n]} \\
&\quad + \int_X \mathrm{tr}[\dot{\mathrm{G}}_J \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)] h\omega^{[n]},
\end{aligned} \tag{2.18}$$

where $\dot{\mathrm{G}}_J$ is the restriction of $\dot{g}_J$ to the fundamental vector fields of the $\mathbb{T}$-action, and D is the Levi-Civita connection of g. We have

$$\begin{aligned} g_J(\dot{J}, DJd(\mathrm{v}(m_\omega)h)) =& \mathrm{v}(m_\omega)g_J(\dot{J}, DJdh) + g_J(\dot{J}, d(\mathrm{v}(m_\omega)) \otimes Jdh) \\ &+ g_J(\dot{J}, dh \otimes Jd(\mathrm{v}(m_\omega))) + hg_J(\dot{J}, DJd(\mathrm{v}(m_\omega))) \\ =& \mathrm{v}(m_\omega)g_J(\dot{J}, DJdh) - 2g_J(d(\mathrm{v}(m_\omega)), \dot{J}Jdh) \\ &+ hg_J(\dot{J}, DJd(\mathrm{v}(m_\omega))). \end{aligned}$$

For a family of $\mathbb{S}^1$ generators $\boldsymbol{\xi} := (\xi_1, \cdots, \xi_\ell)$ of $\mathbb{T}$, we compute

$$\begin{aligned} g_J(\dot{J}, DJd(\mathrm{v}(m_\omega))) =& \sum_{i=1}^{\ell} \mathrm{v}_{,i}(m_\omega)g_J(\dot{J}, DJdm_\omega^{\xi_i}) + \sum_{1\le i,j\le \ell} \mathrm{v}_{,ij}(m_\omega)g_J(\dot{J}, dm_\omega^{\xi_i} \otimes Jdm_\omega^{\xi_j}) \\ =& \sum_{1\le i,j\le \ell} \mathrm{v}_{,ij}(m_\omega)g_J(\dot{J}, dm_\omega^{\xi_i} \otimes Jdm_\omega^{\xi_j}) \\ =& -\operatorname{tr}[\dot{\mathrm{G}}_J \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)]. \end{aligned}$$

where we have used $DJdm_\omega^{\xi_i} = 0$ (since the vector fields ξ_i are Killing with respect to g_J). It follows that

$$\begin{aligned} g_J(\dot{J}, DJd(\mathrm{v}(m_\omega)h)) =& \mathrm{v}(m_\omega)g_J(\dot{J}, DJdh) - 2g_J(d(\mathrm{v}(m_\omega)), \dot{J}Jdh) \\ &- \operatorname{tr}[\dot{\mathrm{G}}_J \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)]. \end{aligned}$$

Substituting back in (2.18), we obtain

$$\frac{d}{dt}\langle \mathrm{Scal}_{\mathrm{v}}(J_t)\mathrm{w}(m_\omega), h\rangle_{\mathrm{w}} = \int_X g_J(\dot{J}, DJdh)\mathrm{v}(m_\omega)\omega^{[n]}.$$

For any $h \in C^\infty(X, \mathbb{R})^{\mathbb{T}}$, the induced vector field on $\mathcal{AC}^{\mathbb{T}}(X,\omega)$ is given by $\hat{Z}_J = -\mathcal{L}_Z J = -2J(DJdh^\sharp)$, where $Z = Jdh^\sharp$ is the Hamiltonian vector field corresponding to h. Thus,

$$\boldsymbol{\Omega}^{\mathrm{v}}_J(\dot{J}, \hat{Z}_J) = -\int_X g_J(\dot{J}, DJdh)\mathrm{v}(m_\omega)\omega^{[n]} = -\frac{d}{dt}\langle \mathrm{Scal}_{\mathrm{v}}(J_t)/\mathrm{w}(m_\omega), h\rangle_{\mathrm{w}}.$$

The $\mathrm{Ham}^{\mathbb{T}}(X,\omega)$-equivariance of the map $J \mapsto \langle \mathrm{Scal}_{\mathrm{v}}(J)/\mathrm{w}(m_\omega), \cdot\rangle$ follows from the Ad-invariance of $\langle \cdot, \cdot \rangle_{\mathrm{w}}$ with respect to $\mathrm{Ham}^{\mathbb{T}}(X,\omega)$. $\square$

2.4 A variational setting

2.4.1 The (v,w)-Mabuchi energy

In this section we suppose that $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$ and $\mathrm{w} \in C^\infty(\mathrm{P}, \mathbb{R})$ is an arbitrary smooth function. We consider $\mathcal{K}_\omega^\mathbb{T}$ as a Fréchet space with tangent space $T_\phi \mathcal{K}_\omega^\mathbb{T} = C^\infty(X, \mathbb{R})^\mathbb{T}$ the space of $\mathbb{T}$-invariant smooth functions $\dot{\phi}$ on X.

Definition 5. The (v,w)-Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}} : \mathcal{K}_\omega^\mathbb{T} \to \mathbb{R}$ is defined by

$$\begin{cases} (d\mathcal{M}_{\mathrm{v},\mathrm{w}})_\phi(\dot{\phi}) = -\displaystyle\int_X \dot{\phi}\big(\mathrm{Scal}_\mathrm{v}(\phi) - c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_\phi)\big)\omega_\phi^{[n]}, \\ \mathcal{M}_{\mathrm{v},\mathrm{w}}(0) = 0, \end{cases} \tag{2.19}$$

for all $\dot{\phi} \in T_\phi \mathcal{K}_\omega^\mathbb{T}$, where $c_{(\mathrm{v},\mathrm{w})}(\alpha)$ is the constant given by (2.6).

Remark 2. The critical points of $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ are precisely the $\mathbb{T}$-invariant Kähler potentials $\phi \in \mathcal{K}_\omega^\mathbb{T}$ such that ω_ϕ is a solution to the equation (1.4). $\diamond$

We will show that the (v,w)-Mabuchi energy is well-defined by establishing in Theorem 6 below an analogue of the Chen-Tian formula (see [26,31,94]. We start with few lemmas.

Lemma 3. *The functional $\mathcal{E}_\mathrm{w} : \mathcal{K}_\omega^\mathbb{T} \to \mathbb{R}$ given by*

$$\begin{cases} (d\mathcal{E}_\mathrm{w})_\phi(\dot{\phi}) = \displaystyle\int_X \dot{\phi}\mathrm{w}(m_\phi)\omega_\phi^{[n]}, \\ \mathcal{E}_\mathrm{w}(0) = 0, \end{cases} \tag{2.20}$$

for any $\dot{\phi} \in T_\phi \mathcal{K}_\omega^\mathbb{T}$ is well-defined.

Proof. See e.g. [16, Lemma 2.14]. $\square$

Lemma 4. *Let θ be a fixed $\mathbb{T}$-invariant closed $(1,1)$-form and $m_\theta : X \to \mathfrak{t}^*$ a momentum map with respect to θ, see Definition 1. Then the functional $\mathcal{E}_\mathrm{v}^\theta : \mathcal{K}_\omega^\mathbb{T} \to \mathbb{R}$ given by*

$$\begin{cases} (d\mathcal{E}_\mathrm{v}^\theta)_\phi(\dot{\phi}) = \displaystyle\int_X \dot{\phi}\left[\mathrm{v}(m_\phi)\theta \wedge \omega_\phi^{[n-1]} + \langle (d\mathrm{v})(m_\phi), m_\theta\rangle \omega_\phi^{[n]}\right], \\ \mathcal{E}_\mathrm{v}^\theta(0) = 0, \end{cases} \tag{2.21}$$

for any $\dot{\phi} \in T_\phi \mathcal{K}_\omega^\mathbb{T}$ is well-defined.

Proof. As the Fréchet space $\mathcal{K}_\omega^{\mathbb{T}}$ is contractible, we have to show that the 1-form on $\mathcal{K}_\omega^{\mathbb{T}}$

$$\begin{aligned}(\mathcal{B}_{\mathrm{v}})_\phi(\dot\phi) &:= \int_X \dot\phi\left[\mathrm{v}(m_\phi)\theta\wedge\omega_\phi^{[n-1]} + \langle(d\mathrm{v})(m_\phi), m_\theta\rangle\omega_\phi^{[n]}\right]\\ &= \int_X \dot\phi\left[\mathrm{v}(m_\phi)\theta\wedge\omega_\phi^{[n-1]} + \sum_{j=1}^{\ell}\mathrm{v}_{,j}(m_\phi)m_\theta^{\xi_j}\omega_\phi^{[n]}\right]\end{aligned} \tag{2.22}$$

is closed. Let $\dot\phi, \dot\psi \in T_\phi\mathcal{K}_\omega^{\mathbb{T}}$. Using the identity

$$\left.\frac{d}{dt}\right|_{t=0}\mathrm{v}(m_{\phi+t\dot\psi}) = \sum_{j=1}^{\ell}\mathrm{v}_{,j}(m_\phi)(d^c\dot\psi)(\xi_j) = (d(\mathrm{v}(m_\phi)), d\dot\psi)_\phi,$$

we compute

$$\begin{aligned}(\boldsymbol{\delta}\mathcal{B}_{\mathrm{v}}(\dot\phi))_\phi(\dot\psi) &= \left.\frac{d}{dt}\right|_{t=0}(\mathcal{B}_{\mathrm{v}})_{\phi+t\dot\psi}(\dot\phi)\\
&= \int_X \dot\phi(d(\mathrm{v}(m_\phi)), d\dot\psi)_\phi\theta\wedge\omega_\phi^{[n-1]} + \int_X\dot\phi\mathrm{v}(m_\phi)\theta\wedge dd^c\dot\psi\wedge\omega_\phi^{[n-2]}\\
&\quad + \int_X\sum_{j=1}^{\ell}\dot\phi m_\theta^{\xi_j}(d(\mathrm{v}_{,j}(m_\phi)), d\dot\psi)_\phi\omega_\phi^{[n]} + \int_X\sum_{j=1}^{\ell}\dot\phi\mathrm{v}_{,j}(m_\phi)m_\theta^{\xi_j}dd^c\dot\psi\wedge\omega_\phi^{[n-1]}\\
&= \int_X \dot\phi(d(\mathrm{v}(m_\phi)), d\dot\psi)_\phi\theta\wedge\omega_\phi^{[n-1]} + \int_X\dot\phi\mathrm{v}(m_\phi)\theta\wedge dd^c\dot\psi\wedge\omega_\phi^{[n-2]}\\
&\quad + \int_X\sum_{j=1}^{\ell}\dot\phi m_\theta^{\xi_j}d(\mathrm{v}_{,j}(m_\phi))\wedge d^c\dot\psi\wedge\omega_\phi^{[n-1]} + \int_X\sum_{j=1}^{\ell}\dot\phi\mathrm{v}_{,j}(m_\phi)m_\theta^{\xi_j}dd^c\dot\psi\wedge\omega_\phi^{[n-1]}\\
&= \int_X \dot\phi(d(\mathrm{v}(m_\phi)), d\dot\psi)_\phi\theta\wedge\omega_\phi^{[n-1]} + \int_X\dot\phi\mathrm{v}(m_\phi)\theta\wedge dd^c\dot\psi\wedge\omega_\phi^{[n-2]}\\
&\quad - \int_X\sum_{j=1}^{\ell}\dot\phi\mathrm{v}_{,j}(m_\phi)(dm_\theta^{\xi_j}, d\dot\psi)_\phi\omega_\phi^{[n]} + \int_X(d\dot\phi, d\dot\psi)_\phi\langle(d\mathrm{v})(m_\phi), m_\theta\rangle\omega_\phi^{[n]},\end{aligned}$$

where $\boldsymbol{\xi} := (\xi_j)_{j=1,\cdots,\ell}$ is a basis of $\mathfrak{t}$. Integrating by parts, we obtain

$$\begin{aligned}&\int_X\dot\phi\mathrm{v}(m_\phi)\theta\wedge dd^c\dot\psi\wedge\omega_\phi^{[n-2]}\\
&= -\int_X\mathrm{v}(m_\phi)\theta\wedge d\dot\phi\wedge d^c\dot\psi\wedge\omega_\phi^{[n-2]} - \int_X\dot\phi\theta\wedge d(\mathrm{v}(m_\phi))\wedge d^c\dot\psi\wedge\omega_\phi^{[n-2]}\\
&= -\int_X(d\dot\phi, d\dot\psi)_\phi\mathrm{v}(m_\phi)\theta\wedge\omega_\phi^{[n-1]} + \int_X(\theta, d\dot\phi\wedge d^c\dot\psi)_\phi\mathrm{v}(m_\phi)\omega_\phi^{[n]}\\
&\quad - \int_X\dot\phi(d(\mathrm{v}(m_\phi)), d\dot\psi)_\phi\theta\wedge\omega_\phi^{[n-1]} - \int_X\sum_{j=1}^{\ell}\dot\phi\mathrm{v}_{,j}(m_\phi)(dm_\theta^{\xi_j}, d\dot\psi)_\phi\omega_\phi^{[n]},\end{aligned}$$

where we have used that

$$
\begin{aligned}
&\theta \wedge d(\mathrm{v}(m_\phi)) \wedge d^c\dot{\psi} \wedge \omega_\phi^{[n-2]} \\
=&(d(\mathrm{v}(m_\phi)), d\dot{\psi})_\phi \theta \wedge \omega_\phi^{[n-1]} - (\theta, d(\mathrm{v}(m_\phi)) \wedge d^c\dot{\psi})_\phi \omega_\phi^{[n]} \\
=&(d(\mathrm{v}(m_\phi)), d\dot{\psi})_\phi \theta \wedge \omega_\phi^{[n-1]} - \sum_{j=1}^{\ell} \mathrm{v}_{,j}(m_\phi)(\theta, dm_\phi^{\xi_j} \wedge d^c\dot{\psi})_\phi \omega_\phi^{[n]} \\
=&(d(\mathrm{v}(m_\phi)), d\dot{\psi})_\phi \theta \wedge \omega_\phi^{[n-1]} - \sum_{j=1}^{\ell} \mathrm{v}_{,j}(m_\phi)(dm_\theta^{\xi_j}, d\dot{\psi})_\phi \omega_\phi^{[n]}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(\boldsymbol{\delta}\mathcal{B}_\mathrm{v}(\dot{\phi}))_\phi(\dot{\psi}) = &-\int_X \mathrm{v}(m_\phi)(d\dot{\phi}, d\dot{\psi})_\phi \theta \wedge \omega_\phi^{[n-1]} \\
&- \int_X (d\dot{\phi}, d\dot{\psi})_\phi \langle (d\mathrm{v})(m_\phi), m_\theta \rangle \omega_\phi^{[n]} \\
&+ \int_X (\theta, d\dot{\phi} \wedge d^c\dot{\psi})_\phi \mathrm{v}(m_\phi) \omega_\phi^{[n]},
\end{aligned}
\tag{2.23}
$$

and hence

$$
(d\mathcal{B}_\mathrm{v})_\phi(\dot{\phi}, \dot{\psi}) = (\boldsymbol{\delta}\mathcal{B}_\mathrm{v}(\dot{\psi}))_\phi(\dot{\phi}) - (\boldsymbol{\delta}\mathcal{B}_\mathrm{v}(\dot{\phi}))_\phi(\dot{\psi}) - (\mathcal{B}_\mathrm{v})_\phi([\dot{\phi}, \dot{\psi}]) = 0,
$$

where $[\dot{\phi}, \dot{\psi}] = 0$, since $\dot{\phi}, \dot{\psi}$ are constant vector fields on $\mathcal{K}_\omega^{\mathbb{T}}$. Thus, $\mathcal{B}_\mathrm{v}$ is closed and therefore $\mathcal{E}_\mathrm{v}^\theta : \mathcal{K}_\omega^{\mathbb{T}} \to \mathbb{R}$ is well-defined. $\square$

Definition 6. We let

$$
\mathcal{H}_\mathrm{v}(\phi) := \int_X \log\left(\frac{\omega_\phi^n}{\omega^n}\right) \mathrm{v}(m_\phi)\omega_\phi^{[n]}
$$

be the v-*entropy* functional $\mathcal{H}_\mathrm{v} : \mathcal{K}_\omega^{\mathbb{T}} \to \mathbb{R}$.

Remark 3. If $\tilde{\mu}$ is an absolutely continuous measure with respect to $\mu_\omega := \omega^{[n]}$, then the entropy of $\tilde{\mu}$ relatively to μ is defined by,

$$
\mathrm{Ent}_{\mu_\omega}(\tilde{\mu}) := \int_X \log\left(\frac{d\tilde{\mu}}{d\mu_\omega}\right) d\tilde{\mu}.
$$

The entropy is convex on the space of finite measures $\tilde{\mu}$ endowed with its natural affine structure. In the case when $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$, the v-entropy functional in Definition 6 is given by

$$
\mathcal{H}_\mathrm{v}(\phi) = \mathrm{Ent}_\mu\left(\mathrm{v}(m_\phi)\omega_\phi^{[n]}\right) + c(\alpha, \mathrm{v})
$$

for all $\phi \in \mathcal{K}_{\omega}^{\mathbb{T}}$, where $c(\alpha, \mathrm{v}) = \int_X (\mathrm{v} \log \circ \mathrm{v})(m_\phi)\omega_\phi^{[n]}$ is a constant depending only on (α, v) (see Lemma 2). $\diamond$

Lemma 5. *1. For any $\mathbb{T}$-invariant Kähler form ω on X, we have*

$$\mathrm{Ric}(\omega)(\xi, \cdot) = -\frac{1}{2} d\langle \Delta_\omega(m_\omega), \xi\rangle.$$

2. For any $\phi \in \mathcal{K}_{\omega}^{\mathbb{T}}$ and $\xi \in \mathfrak{t}$, we have

$$\begin{aligned} \mathrm{Ric}(\omega_\phi) =& \mathrm{Ric}(\omega) - \frac{1}{2} dd^c \Psi_\phi, \\ m^{\xi}_{\mathrm{Ric}(\omega_\phi)} =& m^{\xi}_{\mathrm{Ric}(\omega)} - \frac{1}{2}(d^c\Psi_\phi)(\xi), \end{aligned}$$

where $m_{\mathrm{Ric}(\omega)} := \frac{1}{2}\Delta_\omega(m_\omega)$ is the $\mathrm{Ric}(\omega)$-momentum map of the action of $\mathbb{T}$ on X and $\Psi_\phi = \log\left(\frac{\omega_\phi^n}{\omega^n}\right)$.

Proof. The statement 1 is well known (see e.g. [53, Remark 8.8.2] and [90, Lemma 28]). We will give here a short argument for the statement 2. Let $\phi \in \mathcal{K}_{\omega}^{\mathbb{T}}$ and $\xi \in \mathfrak{t}$. Using that $\mathcal{L}_{J\xi}\omega_\phi = -dd^c m_\phi^\xi$ we obtain

$$\mathcal{L}_{J\xi}\omega_\phi^{[n]} = \Delta_\phi(m_\phi^\xi)\omega_\phi^{[n]}.$$

It follows that

$$-\frac{1}{2}(d^c\Psi_\phi)(\xi) = \frac{1}{2}\mathcal{L}_{J\xi}\Psi_\phi = \frac{1}{2}\frac{\mathcal{L}_{J\xi}\omega_\phi^{[n]}}{\omega_\phi^{[n]}} - \frac{1}{2}\frac{\mathcal{L}_{J\xi}\omega^{[n]}}{\omega^{[n]}} = m^{\xi}_{\mathrm{Ric}(\omega_\phi)} - m^{\xi}_{\mathrm{Ric}(\omega)}.$$

$\square$

We now extend a formula obtained in the case $\mathrm{v} = \mathrm{w} = 1$ by Chen-Tian (see [26,31,94]) to general values of v and $\mathrm{w} > 0$.

Theorem 6. *We have the following expression for the (v, w)-Mabuchi energy,*

$$\mathcal{M}_{\mathrm{v},\mathrm{w}} = \mathcal{H}_{\mathrm{v}} - 2\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)} + c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathcal{E}_{\mathrm{w}}. \tag{2.24}$$

Proof. We denote $\Psi_\phi := \log\left(\frac{\omega_\phi^n}{\omega^n}\right)$. We compute

$$\begin{aligned}(d\mathcal{H}_{\mathrm{v}})_\phi(\dot\phi) &= -\int_X \Delta_\phi(\dot\phi)\mathrm{v}(m_\phi)\omega_\phi^{[n]} + \int_X \Psi_\phi(d(\mathrm{v}(m_\phi)), d\dot\phi)_\phi\omega_\phi^{[n]} \\ &\quad - \int_X \Psi_\phi \mathrm{v}(m_\phi)\Delta_\phi(\dot\phi)\omega_\phi^{[n]} \\ &= -\int_X \dot\phi\Delta_\phi(\mathrm{v}(m_\phi))\omega_\phi^{[n]} - \int_X (d\Psi_\phi, d\dot\phi)_\phi \mathrm{v}(m_\phi)\omega_\phi^{[n]} \\ &= -\int_X \sum_{j=1}^{\ell} \dot\phi \mathrm{v}_{,j}(m_\phi)\Delta_\phi(m_\phi^{\xi_j})\omega^{[n]} + \int_X \sum_{i,j=1}^{\ell} \dot\phi \mathrm{v}_{,ij}(m_\phi)(\xi_i,\xi_j)_\phi\omega_\phi^{[n]} \\ &\quad + \int_X \sum_{j=1}^{\ell} \dot\phi(d\Psi_\phi, dm_\phi^{\xi_j})_\phi \mathrm{v}_{,j}(m_\phi)\omega_\phi^{[n]} - \int_X \dot\phi\Delta_\phi(\Psi_\phi)\mathrm{v}(m_\phi)\omega_\phi^{[n]},\end{aligned}$$

where $\boldsymbol{\xi} := (\xi_j)_{j=1,\cdots,\ell}$ is a basis for $\mathfrak{t}$. Using Lemma 5 and the fact that

$$\Delta_\phi(\Psi_\phi) = -\Lambda_{\omega_\phi} dd^c\Psi_\phi = 2\Lambda_{\omega_\phi}(\mathrm{Ric}(\omega_\phi) - \mathrm{Ric}(\omega)) = \mathrm{Scal}_\phi - 2\Lambda_{\omega_\phi}\mathrm{Ric}(\omega),$$

we get

$$\begin{aligned}(d\mathcal{H}_{\mathrm{v}})_\phi(\dot\phi) &= -\int_X \dot\phi \sum_{j=1}^{\ell} \mathrm{v}_{,j}(m_\phi)\Delta_\phi(m_\phi^{\xi_j})\omega^{[n]} + \int_X \sum_{i,j=1}^{\ell} \dot\phi \mathrm{v}_{,ij}(m_\phi)(\xi_i,\xi_j)_\phi\omega_\phi^{[n]} \\ &\quad + \int_X \sum_{j=1}^{\ell} \dot\phi\left(\Delta_\omega(m_\omega^{\xi_j}) - \Delta_\phi(m_\phi^{\xi_j})\right)\mathrm{v}_{,j}(m_\phi)\omega_\phi^{[n]} \\ &\quad - \int_X \dot\phi\left(\mathrm{Scal}_\phi - 2\Lambda_{\omega_\phi}\mathrm{Ric}(\omega)\right)\mathrm{v}(m_\phi)\omega_\phi^{[n]}.\end{aligned}$$

It follows that

$$d(\mathcal{H}_{\mathrm{v}} - 2\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)})_\phi(\dot\phi) = -\int_X \dot\phi\mathrm{Scal}_{\mathrm{v}}(\phi)\omega_\phi^{[n]}, \tag{2.25}$$

which yields (2.24) via (2.25) and (2.20). □

By the work of Mabuchi [76, 77], the space of $\mathbb{T}$-invariant Kähler potentials $\mathcal{K}_\omega^{\mathbb{T}}$ is an infinite dimensional Riemannian manifold with a natural Riemannian metric, called the *Mabuchi metric*, defined by

$$\langle\dot\phi_1, \dot\phi_2\rangle_\phi = \int_X \dot\phi_1\dot\phi_2\omega_\phi^{[n]},$$

for any $\dot{\phi}_1, \dot{\phi}_2 \in T_\phi \mathcal{K}^{\mathbb{T}}_\omega$. The equation of a geodesic $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{\mathbb{T}}_\omega$ connecting two points $\phi_0, \phi_1 \in \mathcal{K}^{\mathbb{T}}_\omega$ is given by (see e.g. [53, Section 4.6] for more references)

$$\ddot{\phi}_t = |d\dot{\phi}_t|^2_{\phi_t}.$$

The following result is a straightforward extension of an observation of Guan [54] (see e.g. [53, Proposition 4.6.3]).

Proposition 1. *Let X be a compact Kähler manifold with a fixed Kähler class α, $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ a real torus and suppose that $\omega \in \alpha$ is a (v, w)-cscK metric for smooth functions $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$, $\mathrm{w} \in C^\infty(\mathrm{P}, \mathbb{R})$ on the momentum image $\mathrm{P} \subset \mathfrak{t}^*$ associated to $(\mathbb{T}, \alpha)$. Then for any (v, w)-cscK metric $\omega_\phi \in \alpha$ connected to ω by a geodesic segment in $\mathcal{K}^{\mathbb{T}}_\omega$, there exists $\Phi \in \mathrm{Aut}_{\mathrm{red}}(X)$ commuting with the action of $\mathbb{T}$, such that $\omega_\phi = \Phi^*\omega$.*

Proof. By a straightforward calculation using the formula (2.40) in Lemma 9 below, we obtain the following expression for the second variation of the (v, w)-Mabuchi energy along a $\mathbb{T}$-invariant segment of Kähler potentials $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{\mathbb{T}}_\omega$:

$$\begin{aligned} \frac{d^2\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)}{dt^2} =& 2\int_X |D^- d\dot{\phi}_t|^2_{\phi_t} \mathrm{v}(m_{\phi_t})\omega^{[n]}_{\phi_t} \\ &- \int_X \left(\ddot{\phi}_t - |d\dot{\phi}_t|^2_{\phi_t}\right)(\mathrm{Scal}_{\mathrm{v}}(\phi_t) - \mathrm{w}(m_{\phi_t}))\,\omega^{[n]}_{\phi_t}. \end{aligned} \tag{2.26}$$

Suppose now that ω_ϕ, $\phi \in \mathcal{K}^{\mathbb{T}}_\omega$ is a (v, w)-cscK metric connected to ω by a smooth geodesic $(\phi_t)_{t\in[0,1]}$, such that $\phi_0 = 0$ and $\phi_1 = \phi$. Then $\frac{d\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)}{dt}\Big|_{t=0} = \frac{d\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)}{dt}\Big|_{t=1} = 0$, and using (2.26) we obtain

$$\frac{d^2\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)}{dt^2} = 2\int_X |D^- d\dot{\phi}_t|^2_{\phi_t} \mathrm{v}(m_{\phi_t})\omega^{[n]}_{\phi_t} \geq 0.$$

It follows that $\frac{d^2\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)}{dt^2} \equiv 0$ and $D^- d\dot{\phi}_t \equiv 0$. Thus, we have a family of real holomorphic vector vector fields $V_t := -\mathrm{grad}_{g_t}\dot{\phi}_t$, $t \in [0,1]$. By [53, Proposition 4.6.3], $V_t = V_0$ for all t, and $\omega_\phi = (\Phi^{V_0}_1)^*\omega$ where $\Phi^{V_0}_t \in \mathrm{Aut}_{\mathrm{red}}(X)$ is the flow of the real holomorphic vector field V_0. □

Remark 4. In general, the space $\mathcal{K}^{\mathbb{T}}_\omega$ is not geodesically convex by smooth geodesics (see [33, Theorem 1.2]). However, by a result of Chen [27], with complements of Blocki

[19], the space $\mathcal{K}_\omega^{\mathbb{T}}$ is geodesically convex by $\mathbb{T}$-invariant weak $C^{1,1}$-geodesics, i.e. in the space $\left(\mathcal{K}_\omega^{1,1}\right)^{\mathbb{T}}$ of $\mathbb{T}$-invariant real valued functions ϕ such that $\omega + dd^c\phi$ is a positive current with bounded coefficients. Using the formula $m_\phi = m_\omega + d^c\phi$ and Theorem 6, one can extend the (v,w)-Mabuchi energy to a functional $\mathcal{M}_{\mathrm{v},\mathrm{w}} : \left(\mathcal{K}_\omega^{1,1}\right)^{\mathbb{T}} \to \mathbb{R}$. $\diamond$

2.4.2 The relative (v,w)-Mabuchi energy

In this section we assume that both v and w are positive smooth functions on P.

Definition 7. The (v,w)-relative Mabuchi energy $\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}} : \mathcal{K}_\omega^{\mathbb{T}} \to \mathbb{R}$ is defined by

$$\begin{cases} (d\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}})_\phi(\dot\phi) = -\displaystyle\int_X \dot\phi\big(\mathrm{Scal}_{\mathrm{v}}(\phi)/\mathrm{w}(m_\phi) - \mathrm{w}_{\mathrm{ext}}(m_\phi)\big)\mathrm{w}(m_\phi)\omega_\phi^{[n]}, \\ \mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}}(0) = 0, \end{cases} \tag{2.27}$$

for any $\dot\phi \in T_\phi\mathcal{K}_\omega^{\mathbb{T}}$, where $\mathrm{w}_{\mathrm{ext}}$ is the affine linear function on P defined in Section 3.2.

Lemma 6. *We have* $\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}} = \mathcal{M}_{\mathrm{v},\mathrm{w}\mathrm{w}_{\mathrm{ext}}}$.

Proof. In Section 3.2, we will show that $c_{\mathrm{v},\mathrm{w}\mathrm{w}_{\mathrm{ext}}}(\alpha) = 1$. From the definitions of $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ and $\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}}$, it then follows that $\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}} = \mathcal{M}_{\mathrm{v},\mathrm{w}\mathrm{w}_{\mathrm{ext}}} + c$ and using $\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}}(0) = \mathcal{M}_{\mathrm{v},\mathrm{w}\mathrm{w}_{\mathrm{ext}}}(0) = 0$ we get $c = 0$. $\square$

2.4.3 Boundednes of the $(1,\mathrm{w})$-Mabuchi energy

Now we show how the results of Berman-Berndtsson in [15] can be extended to the $(1,\mathrm{w})$-cscK metrics.

Theorem 7. *Let X be a smooth compact Kähler manifold, $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ a real torus, and suppose that X admits a $(1,\mathrm{w})$-cscK metric ω in the the Kähler class α for some smooth function* w *on the momentum image* $\mathrm{P} \subset \mathfrak{t}^*$ *associated to $(\mathbb{T},\alpha)$. Then, ω is a global minima of $\mathcal{M}_{1,\mathrm{w}}$.*

Proof. We denote by $\mathcal{M}_{\mathrm{w}}$ the $(1,\mathrm{w})$-Mabuchi energy and by $\mathcal{M}$ the $(1, c_{1,\mathrm{w}}(\alpha))$-Mabuchi energy. From the definition of the Mabuchi energy we have the following relation

$$\mathcal{M}_{\mathrm{w}} = \mathcal{M} + \mathcal{E}_{\tilde{\mathrm{w}}},$$

where $\tilde{\mathrm{w}} := c_{1,\mathrm{w}}(\alpha)(1-\mathrm{w})$ and $\mathcal{E}_{\tilde{\mathrm{w}}}$ is the functional (2.20). Let $\phi_0, \phi_1 \in \mathcal{K}^{\mathbb{T}}_{\omega}$ be two smooth Kähler potentials and ϕ_t the weak geodesic connecting ϕ_0 and ϕ_1 (see [15, 31] and the references therein for the definition of a weak geodesic). By [17, Proposition 10.d] the function $t \mapsto \mathcal{E}_{\tilde{\mathrm{w}}}(\phi_t)$ is affine on $[0,1]$, whereas by [15, Theorem 3.4], the function $t \mapsto \mathcal{M}(\phi_t)$ is convex. It follows that $t \mapsto \mathcal{M}_{\mathrm{w}}(\phi_t)$ is convex. By [15, Lemma 3.5] and its proof, we get

$$\lim_{t\to 0^+} \frac{\mathcal{M}_{\mathrm{w}}(\phi_t) - \mathcal{M}_{\mathrm{w}}(\phi_0)}{t} \geq \int_X \left(\mathrm{Scal}(\phi_0) - c_{1,\mathrm{w}}(\alpha)\mathrm{w}(m_{\phi_0})\right)\dot{\phi}\,\omega^{[n]}_{\phi_0}.$$

where $\dot{\phi} := \frac{d\phi_t}{dt}_{|t=0^+}$. Using the sub-slope inequality for convex functions and the Cauchy–Shwartz inequality we get

$$\begin{aligned}
\mathcal{M}_{\mathrm{w}}(\phi_1) - \mathcal{M}_{\mathrm{w}}(\phi_0) &\geq \lim_{t\to 0^+} \frac{\mathcal{M}_{\mathrm{w}}(\phi_t) - \mathcal{M}_{\mathrm{w}}(\phi_0)}{t} \\
&\geq \int_X \left(\mathrm{Scal}(\phi_0) - c_{1,\mathrm{w}}(\alpha)\mathrm{w}(m_{\phi_0})\right)\dot{\phi}\,\omega^{[n]}_{\phi_0} \\
&\geq -\, d(\phi_0,\phi_1)\Big(\int_X \left(\mathrm{Scal}(\phi_0) - c_{(1,\mathrm{w})}(\alpha)\mathrm{w}(m_{\phi_0})\right)^2 \omega^{[n]}_{\phi_0}\Big)^{\frac{1}{2}},
\end{aligned}$$

where $d(\phi_0,\phi_1)^2 = \int_X \dot{\phi}^2 \omega^{[n]}_{\phi_0}$ is the Mabuchi distance between ϕ_0 and ϕ_1. In particular, if ω_{ϕ_0} is a $(1,\mathrm{w})$-cscK metric in the Kähler class α, then $\mathcal{M}_{\mathrm{w}}(\phi) \geq \mathcal{M}_{\mathrm{w}}(\phi_0)$ for any $\phi \in \mathcal{K}^{\mathbb{T}}_{\omega}$. □

2.5 The (v,w)-Futaki invariant for a Kähler class

Let (X,α) be a compact Kähler manifold and $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ a real torus with momentum polytope P with respect to α as in Lemma 1. For any $\phi \in \mathcal{K}^{\mathbb{T}}_{\omega}$ and $V \in \mathfrak{h}^{\mathbb{T}}_{\mathrm{red}}$ in the Lie algebra of the centralizer of $\mathbb{T}$ in $\mathrm{Aut}_{\mathrm{red}}(X)$, we denote by $h^V_\phi + \sqrt{-1}f^V_\phi \in C^\infty_{0,\phi}(X,\mathbb{C})$ the normalized holomorphy potential of ξ, i.e. h^V_ϕ and f^V_ϕ are smooth functions such that,

$$V = \mathrm{grad}_{g_\phi}(h_\phi^V) + J\mathrm{grad}_{g_\phi}(f_\phi^V),$$
$$\int_X f_\phi^V \omega_\phi^{[n]} = \int_X h_\phi^V \omega_\phi^{[n]} = 0.$$

Using that the tangent space in ϕ of $\mathcal{K}_\omega^{\mathbb{T}}$ is given by $T_\phi(\mathcal{K}_\omega^{\mathbb{T}}) \cong C_{0,\phi}^\infty(X,\mathbb{R})^{\mathbb{T}} \oplus \mathbb{R}$, the vector field JV defines a vector field $\widehat{JV}$ on $\mathcal{K}_\omega^{\mathbb{T}}$, given by:

$$\phi \mapsto \mathcal{L}_{JV}\omega_\phi = -dd^c f_\phi^V,$$

so that $\widehat{JV}_\phi = f_\phi^V$. We consider the 1-form σ on $\mathcal{K}_\omega^{\mathbb{T}}$, defined by

$$\sigma_\phi(\dot\phi) := (d\mathcal{M}_{\mathrm{v},\mathrm{w}})_\phi(\dot\phi)$$

where $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ is the (v,w)-Mabuchi energy associated to the smooth functions $\mathrm{v} \in C^\infty(\mathrm{P},\mathbb{R}_{>0})$ and $\mathrm{w} \in C^\infty(\mathrm{P},\mathbb{R})$ (see (2.19)). By the invariance of σ under the $\mathrm{Aut}_{\mathrm{red}}^{\mathbb{T}}(X)$-action and Cartan's formula, we get

$$\mathcal{L}_{\widehat{JV}}\sigma = d\big(\sigma(\widehat{JV})\big) = 0.$$

Then $\phi \mapsto \sigma_\phi(\widehat{JV})$ is constant on $\mathcal{K}_\omega^{\mathbb{T}}$, and we define

Definition 8. We let

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}^\alpha(V) := \sigma_\omega(\widehat{JV}) = \int_X \big(\mathrm{Scal}_{\mathrm{v}}(\omega) - c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_\omega)\big) f_\omega^V \omega^{[n]}, \qquad (2.28)$$

be the real constant associated to $V \in \mathfrak{h}_{\mathrm{red}}^{\mathbb{T}}$. We thus get a linear map $\mathcal{F}_{\mathrm{v},\mathrm{w}}^\alpha : \mathfrak{h}_{\mathrm{red}}^{\mathbb{T}} \to \mathbb{R}$ called the (v,w)-Futaki invariant associated to $(\alpha,\mathrm{P},\mathrm{v},\mathrm{w})$.

By its very definition, we have

Proposition 2. *If $(X,\alpha,\mathbb{T})$ admits a (v,w)-cscK metric then*

$$\int_X \mathrm{Scal}_{\mathrm{v}}(\omega)\omega^{[n]} = c_{(\mathrm{v},\mathrm{w})}(\alpha)\int_X \mathrm{w}(m_\omega)\omega^{[n]} \text{ and } \mathcal{F}_{\mathrm{v},\mathrm{w}}^\alpha \equiv 0. \qquad (2.29)$$

Remark 5. The first condition in (2.29) is satisfied when $\int_X \mathrm{w}(m_\omega)\omega^{[n]} \neq 0$ by the very definition of $c_{\mathrm{v},\mathrm{w}}(\alpha)$ (see Definition 3). Furthermore, in the case of a (v,w)-extremal Kähler metric considered in Section 3.1, both conditions in (2.29) hold true with respect to the weights v and $\mathrm{w}\mathrm{w}_{\mathrm{ext}}$. ◇

Following [50] and[53, Proposition 4.11.1] we have,

Definition 9. For any $V_1, V_2 \in \mathfrak{h}_{\mathrm{red}}^{\mathbb{T}}$, with normalized holomorphy potentials F_1^ω, F_2^ω, we define the w-Futaki-Mabuchi bilinear form by the following expression,

$$\mathcal{B}_{\mathrm{w}}^{\alpha}(V_1, V_2) := \int_X F_1^\omega F_2^\omega \mathrm{w}(m_\omega)\omega^{[n]}$$

which is independent from the $\mathbb{T}$-invariant $\omega \in \alpha$.

We have the following characterization of the extremal vector field ξ_{ext} of $(X, \alpha, \mathrm{P}_\alpha, \mathrm{v}, \mathrm{w})$ (see section 2.2.2),

Lemma 7. *The extremal vector field* $\xi_{\mathrm{ext}} \in \mathfrak{t}$ *of* $(\alpha, \mathrm{P}_\alpha, \mathrm{v}, \mathrm{w})$ *is the unique element of* $\mathfrak{t}$ *such that*

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}^{\alpha}(\xi) = \mathcal{B}_{\mathrm{w}}^{\alpha}(\xi_{\mathrm{ext}}, \xi), \tag{2.30}$$

for any $\xi \in \mathfrak{t}$.

2.6 The structure of the automorphism group of a manifold with weighted cscK metric

In the following section we give the proof of Theorem 1 from the Introduction. We need first to establish a couple of lemmas. In what follows, (X, α) is a compact Kähler manifold and $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ is a real torus with momentum image P_α, as in Lemma 1.

Lemma 8. *For any* $\mathbb{T}$*-invariant* 1*-form* θ *on* X*, and smooth function* $\mathrm{v} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R})$*, we have*

$$\begin{aligned} 2\delta\delta\left(\mathrm{v}(m_\omega)D^-\theta\right) =& 2\mathrm{v}(m_\omega)\delta\delta(D^-\theta) + 2\sum_{i=1}^{\ell} \mathrm{v}_{,i}(m_\omega)(\Delta\theta, (J\xi_i)^\flat) \\ & - \sum_{i=1}^{\ell} 2\mathrm{v}_{,i}(m_\omega)(\Delta(J\xi_i)^\flat, \theta) - \sum_{i=1}^{\ell} 2\mathrm{v}_{,i}(m_\omega)(\delta d\theta, (J\xi_i)^\flat) \\ & + \sum_{i,j=1}^{\ell} \mathrm{v}_{,ij}(m_\omega)(J\xi_i)(\theta(J\xi_j)) - \sum_{i,j=1}^{\ell} \mathrm{v}_{,ij}(m_\omega)(\theta, d(\xi_i, \xi_j)) \end{aligned} \tag{2.31}$$

where $(\cdot, \cdot)$ *stand for the inner product of tensors induced by the Kähler metric* ω*,*

$(\xi_1, \cdots, \xi_\ell)$ denote a basis for $\mathfrak{t}$ and $\mathrm{v}_{,i}$ (resp. $\mathrm{v}_{,ij}$) denotes (resp. mixed) partial derivatives of v.

Proof. In fact,

$$\begin{aligned}\delta\delta\left(\mathrm{v}(m_\omega)D^-\theta\right) &= \mathrm{v}(m_\omega)\delta\delta(D^-\theta) + \mathrm{v}_{,i}(m_\omega)(\delta D^-\theta)(J\xi_i) \\ &\quad + \mathrm{v}_{,i}(m_\omega)\delta\left((D^-\theta)(J\xi_i, \cdot)\right) - \mathrm{v}_{,ij}(m_\omega)(D^-\theta)(\xi_i, \xi_j).\end{aligned}$$

We consider the decomposition of the tensor $D^-\theta$ in symmetric and skew-symmetric parts Ψ and Φ, respectively,

$$D^-\theta = \Psi + \Phi.$$

For any vector field X on M we have

$$\begin{aligned}\delta\left(\Psi(X, .)\right) &= -(\Psi, DX^\flat) + (\delta\Psi)(X), \\ \delta\left(\Phi(X, .)\right) &= (\Phi, DX^\flat) - (\delta\Phi)(X).\end{aligned} \tag{2.32}$$

Using (2.32) for $X = J\xi_i$ we get

$$\begin{aligned}\delta\left(\Psi(J\xi_i, .)\right) &= (\delta\Psi)(J\xi_i), \\ \delta\left(\Phi(J\xi_i, .)\right) &= -(\delta\Phi)(J\xi_i).\end{aligned}$$

Thus,

$$\delta\delta\left(\mathrm{v}(m_\omega)D^-\theta\right) = \mathrm{v}(m_\omega)\delta\delta(D^-\theta) + 2\mathrm{v}'_i(m_\omega)(\delta\Psi)(J\xi_i) - \mathrm{v}''_{i,j}(m_\omega)(D^-\theta)(\xi_i, \xi_j). \tag{2.33}$$

Using [53, Lemma 1.23.4] and $2\Phi = d\theta - Jd\theta$ we have

$$\begin{aligned}(\delta\Psi)(J\xi_i) &= (\delta D^-\theta, (J\xi_i)^\flat) - (\delta\Phi)(J\xi_i) \\ &= \frac{1}{2}(\Delta\theta, (J\xi_i)^\flat) - \mathrm{Ric}(\omega)(J\xi_i, \theta^\sharp) - \frac{1}{2}(\delta d\theta, (J\xi_i)^\flat) + \frac{1}{2}(\delta Jd\theta, (J\xi_i)^\flat) \\ &= \frac{1}{2}(\Delta\theta, (J\xi_i)^\flat) - (\Delta(J\xi_i)^\flat, \theta) + \left(\delta D^+ d(J\xi_i)^\flat, \theta\right) - \frac{1}{2}(\delta d\theta, (J\xi_i)^\flat) \\ &= \frac{1}{2}(\Delta\theta, (J\xi_i)^\flat) - \frac{1}{2}(\Delta(J\xi_i)^\flat, \theta) - \frac{1}{2}(\delta d\theta, (J\xi_i)^\flat)\end{aligned} \tag{2.34}$$

where we have used the identity $(\delta J d\theta, (J\xi_i)^\flat) = -(\delta^c d\theta)(\xi_i) = \mathcal{L}_{\xi_i}\delta^c\theta = 0$ which holds since ξ_i is Killing. Furthermore,

$$\begin{aligned} 2(D^-\theta)(\xi_i,\xi_j) &= (D_{\xi_i}\theta)(\xi_j) - (D_{J\xi_i}\theta)(J\xi_j) \\ &= \xi_i(\theta(\xi_j)) - (J\xi_i)(\theta(J\xi_j)) - 2\theta\left(D_{\xi_j}\xi_i\right) \\ &= -J\xi_i(\theta(J\xi_j)) + (\theta, d(\xi_i,\xi_j)) \\ &= -(J\xi_i)(\theta(J\xi_j)) + (\theta, d(\xi_i,\xi_j)), \end{aligned} \tag{2.35}$$

since $\xi_i(\theta(\xi_j)) = 0$ by the $\mathbb{T}$-invariance of θ. The result follows by substituting (2.34) and (2.35) in (2.33). This completes the proof. $\square$

Corollary 3. *For any $\mathbb{T}$-invariant function $\phi \in C^\infty(X,\mathbb{R})^{\mathbb{T}}$, we have*

$$\mathcal{L}_{\xi_{\mathrm{ext}}}\phi = \frac{-2\delta\delta(\mathrm{v}(m_\omega)D^-(d^c\phi))}{\mathrm{w}(m_\omega)},$$

where $\mathcal{L}_{\xi_{\mathrm{ext}}}$ denotes the Lie derivative along the vector field $\xi_{\mathrm{ext}} := J\mathrm{grad}(\mathrm{Scal}_{\mathrm{v}}(\omega)/\mathrm{w}(m_\omega))$.

Proof. We have,

$$\mathrm{Scal}_{\mathrm{v}}(\omega) = \mathrm{v}(m_\omega)\mathrm{Scal}(\omega) + 2\sum_{i=1}^{\ell}\mathrm{v}_{,i}(m_\omega)\Delta_\omega(m_\omega^{\xi_i}) - \sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)(\xi_i,\xi_j).$$

By $\mathbb{T}$-invariance of ϕ, we obtain

$$\begin{aligned} &\mathrm{w}(m_\omega)\mathcal{L}_{\xi_{\mathrm{ext}}}\phi = -(d\mathrm{Scal}_{\mathrm{v}}(\omega), d^c\phi) \\ &= -\mathrm{v}(m_\omega)(d\mathrm{Scal}(\omega), d^c\phi) - 2\sum_{i=1}^{\ell}\mathrm{v}_{,i}(m_\omega)(d\Delta(m_\omega^{\xi_i}), d^c\phi) + \sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)(d^c\phi, d(\xi_i,\xi_j)). \end{aligned}$$

By taking $\theta = d^c\phi$ in (2.31) and using the fact that $\delta\delta(D^-d^c)\phi = (d\mathrm{Scal}(\omega), d^c\phi)$ (see [53, p.63, Eq.(1.23.15)]), we get,

$$\begin{aligned} &2\delta\delta\mathrm{v}(m_\omega)(D^-d^c)\phi \\ =&2\mathrm{v}(m_\omega)\delta\delta(D^-d^c)\phi + 2\sum_{i=1}^{\ell}\mathrm{v}_{,i}(m_\omega)(d\Delta(m_\omega^{\xi_i}), d^c\phi) - \sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)(d^c\phi, d(\xi_i,\xi_j)) \\ =&\mathrm{v}(m_\omega)(d\mathrm{Scal}(\omega), d^c\phi) + 2\sum_{i=1}^{\ell}\mathrm{v}_{,i}(m_\omega)(d\Delta(m_\omega^{\xi_i}), d^c\phi) - \sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)(d^c\phi, d(\xi_i,\xi_j)) \\ =&-\mathrm{w}(m_\omega)\mathcal{L}_{\xi_{\mathrm{ext}}}\phi. \end{aligned}$$

where we have used the $\mathbb{T}$-invariance of ϕ. $\square$

For a 1-form θ we denote by $D^{2,0}\theta$ (resp. $D^{0,2}\theta$) the $(2,0)$-part (resp. $(0,2)$-part) of the tensor $D\theta$. We define the (v,w)-Calabi's operators $\mathbb{L}^{\pm}_{(\mathrm{v},\mathrm{w})}$ on $C^{\infty}(M,\mathbb{C})^{\mathbb{T}}$ by

$$\begin{aligned}\mathbb{L}^{+}_{(\mathrm{v},\mathrm{w})}(F) &= \frac{2(D^{0,2}d)^{\star}\mathrm{v}(m_{\omega})D^{0,2}dF}{\mathrm{w}(m_{\omega})},\\ \mathbb{L}^{-}_{(\mathrm{v},\mathrm{w})}(F) &= \frac{2(D^{2,0}d)^{\star}\mathrm{v}(m_{\omega})D^{2,0}dF}{\mathrm{w}(m_{\omega})},\end{aligned}$$

and we define the (v,w)-Lichnerowicz operator by

$$\mathbb{L}_{(\mathrm{v},\mathrm{w})} := \mathbb{L}^{+}_{(\mathrm{v},\mathrm{w})} + \mathbb{L}^{-}_{(\mathrm{v},\mathrm{w})}. \tag{2.36}$$

Recall that the space of hamiltonian Killing vector fields is given by (see [53, Chapter 2])

$$\mathfrak{k}_{\mathrm{ham}} = \mathfrak{h}_{\mathrm{red}} \cap \mathfrak{k}.$$

The following proposition follows from the arguments in [53, Proposition 2.5.1], and will be left to the reader.

Proposition 3.

1. *Let* $V = \mathrm{grad}_g(h) + J\mathrm{grad}_g(f)$, *where* $f, h \in C^{\infty}(X,\mathbb{R})^{\mathbb{T}}$. *Then* $V \in \mathfrak{h}_{\mathrm{red}}$ *if and only if* $\mathbb{L}^{+}_{(\mathrm{v},\mathrm{w})}(h+\sqrt{-1}f) = 0$, *i.e. we have*

$$\mathfrak{h}^{\mathbb{T}}_{\mathrm{red}} \cong \ker(\mathbb{L}^{+}_{(\mathrm{v},\mathrm{w})}) \cap C^{\infty}_{\mathrm{w}}(M,\mathbb{C})^{\mathbb{T}}_{0}.$$

where $C^{\infty}_{\mathrm{w}}(M,\mathbb{C})^{\mathbb{T}}_{0}$ *is the space of smooth* $\mathbb{T}$*-invariant functions on* X *normalized by* $\int_X f\mathrm{w}(m_{\omega})\omega^n = 0$.

2. *The* (v,w)*-Lichnerowicz operator satisfies*

$$\mathbb{L}_{(\mathrm{v},\mathrm{w})} = \mathbb{L}^{\pm}_{(\mathrm{v},\mathrm{w})} \pm \frac{\sqrt{-1}}{2}\mathcal{L}_{\xi_{\mathrm{ext}}},$$

where $\xi_{\mathrm{ext}} := J\mathrm{grad}(\mathrm{Scal}_{\mathrm{v}}(\omega)/\mathrm{w}(m_{\omega}))$.

3. *Let* $\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}$ *be the Lie algebra of* $\mathbb{T}$*-equivariant Hamiltonian isometries of* X. *Then* $V \in \mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}$ *if and only if there exists* $h \in C^{\infty}(M,\mathbb{R})^{\mathbb{T}}$ *such that* $V = J\mathrm{grad}_g(h)$ *and* $\mathbb{L}_{(\mathrm{v},\mathrm{w})}(h) = 0$.

The next result was first established by Calabi in [22] in the case of extremal metrics, and was recently generalized independently by Futaki–Ono [51] and the author [61] to the case of manifolds admitting Kähler metrics conformally equivalent to Einstein Maxwell metrics.

Theorem 8. *If X admits a (v,w)-extremal Kähler metric with $\mathrm{v},\mathrm{w} \in C^\infty(\mathrm{P},\mathbb{R}_{>0})$. Then the complex Lie algebra of $\mathbb{T}$-equivariant automorphisms of X admits the following decomposition*

$$\mathfrak{h}^{\mathbb{T}} = (\mathfrak{a} \oplus \mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}} \oplus J\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}) \oplus \left(\bigoplus_{\lambda>0} \mathfrak{h}^{\mathbb{T}}_{(\lambda)} \right), \tag{2.37}$$

where $\mathfrak{a}$ is the abelian Lie algebra of parallel vector fields, $\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}$ is the real Lie algebra of $\mathbb{T}$-equivariant Hamiltonian isometries of X and $\mathfrak{h}^{\mathbb{T}}_{(\lambda)}$, $\lambda > 0$ denote the subspace of elements $V \in \mathfrak{h}^{\mathbb{T}}$ such that $\mathcal{L}_{\xi_{\mathrm{ext}}} V = \lambda J V$. Moreover, the Lie algebra of $\mathbb{T}$-equivariant isometries of X admits the following decomposition

$$\mathfrak{k}^{\mathbb{T}} = \mathfrak{a} \oplus \mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}. \tag{2.38}$$

Proof. The proof follows the arguments in [53, Theorem 3.4.1]. Let (g,ω) denote $V := V_H + \mathrm{grad}_g(h) + J\mathrm{grad}_g(f) \in \mathfrak{h}^{\mathbb{T}}$, where V_H is the dual of the harmonic part of $\theta := V^\flat$ denoted θ_H, and $f, g \in C^\infty(X,\mathbb{R})^{\mathbb{T}}$ with $\int_X f\mathrm{w}(m_\omega)\omega^n = \int_X h\mathrm{w}(m_\omega)\omega^n = 0$. By (2.31) in Lemma 8, the fact that θ_H is harmonic and the identity $2\delta\delta(D^-\theta_H) = (d\mathrm{Scal}(\omega), \theta_H)$ which follows from [53, Lemma 1.23.5], we obtain

$$\begin{aligned}
2\delta\delta(\mathrm{v}(m_\omega)D^-\theta_H) =& 2\mathrm{v}(m_\omega)\delta\delta(D^-\theta_H) - \sum_{i=1}^{\ell} 2\mathrm{v}_{,i}(m_\omega)(\Delta(J\xi_i)^\flat, \theta_H) \\
& - \sum_{i,j=1}^{\ell} \mathrm{v}_{,ij}(m_\omega)(\theta_H, d(\xi_i,\xi_j)) \\
=& \mathrm{v}(m_\omega)(d\mathrm{Scal}(\omega), \theta_H) + \sum_{i=1}^{\ell} 2\mathrm{v}_{,i}(m_\omega)(d\Delta(m^{\xi_i}_\omega), \theta_H) \\
& - \sum_{i,j=1}^{\ell} \mathrm{v}_{,ij}(m_\omega)(\theta_H, d(\xi_i,\xi_j)) \\
=& J\mathcal{L}_{\xi_{\mathrm{ext}}}\theta_H = 0 \text{ since } (g,\omega) \text{ is } (\mathrm{v},\mathrm{w})\text{-extremal.}
\end{aligned}$$

It follows that

$$0=\frac{2\delta\delta(\mathrm{v}(m_\omega)D^-\theta)}{\mathrm{w}(m_\omega)}=\frac{2\delta\delta(\mathrm{v}(m_\omega)D^-(dh+d^cf)}{\mathrm{w}(m_\omega)}=\mathrm{Re}\left(\mathbb{L}^+_{(\mathrm{v},\mathrm{w})}(h+\sqrt{-1}f)\right).$$

Starting from JV instead of V we similarly get

$$\mathrm{Im}\left(\mathbb{L}^+_{(\mathrm{v},\mathrm{w})}(h+\sqrt{-1}f)\right)=0.$$

It follows that $\mathbb{L}^+_{(\mathrm{v},\mathrm{w})}(h+\sqrt{-1}f)=0$, then by 1 in Proposition 3 we have that V_H and $\mathrm{grad}_g(h)+J\mathrm{grad}_g(f)$ are real holomorphic vector fields, which proves that

$$\mathfrak{h}^{\mathbb{T}}=\mathfrak{a}\oplus\mathfrak{h}^{\mathbb{T}}_{\mathrm{red}}. \tag{2.39}$$

Using the fact that $\mathfrak{k}_{\mathrm{ham}}:=\mathfrak{k}\cap\mathfrak{h}_{\mathrm{red}}$ and $\mathfrak{k}\cap\mathfrak{a}=\mathfrak{a}$, we obtain the decomposition (2.38).

Since ξ_{ext} is Killing and commutes with $\mathbb{T}$, the operators $\mathbb{L}^\pm_{(\mathrm{v},\mathrm{w})}$ commute. Then $\mathbb{L}^-_{(\mathrm{v},\mathrm{w})}$ acts on $\mathfrak{h}^{\mathbb{T}}_{red}$ and by Proposition 3 2, this action is given by $-\sqrt{-1}\mathcal{L}_{\xi_{\mathrm{ext}}}$. Since $\mathbb{L}^-_{(\mathrm{v},\mathrm{w})}$ is $\langle\cdot,\cdot\rangle_{\mathrm{w}}$-self-adjoint and semi-positive, $\mathfrak{h}^{\mathbb{T}}_{\mathrm{red}}$ splits as

$$\mathfrak{h}^{\mathbb{T}}_{\mathrm{red}}=\mathfrak{h}^{\mathbb{T}}_{\mathrm{red},(0)}\oplus\left(\bigoplus_{\lambda>0}\mathfrak{h}^{\mathbb{T}}_{(\lambda)}\right),$$

where $\mathfrak{h}^{\mathbb{T}}_{\mathrm{red},(0)}$ is the kernel of $\mathcal{L}_{\xi_{\mathrm{ext}}}$ in $\mathfrak{h}^{\mathbb{T}}_{\mathrm{red}}$ whereas, for each $\lambda>0$, $\mathfrak{h}^{\mathbb{T}}_{(\lambda)}$ is the subspace of elements $V\in\mathfrak{h}^{\mathbb{T}}$ such that $\mathcal{L}_{\xi_{\mathrm{ext}}}V=\lambda JV$. Using the splitting (2.39) we get (2.37) (Notice that $\mathfrak{h}^{\mathbb{T}}_{(\lambda)}=\mathfrak{h}^{\mathbb{T}}_{\mathrm{red},(\lambda)}$ since ξ_{ext} is Killing and commutes with $\mathbb{T}$).

We have $\mathfrak{a}\oplus\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}\oplus J\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}\subset\mathfrak{h}^{\mathbb{T}}_{(0)}$. By 2 in Proposition 3 the restriction of $\mathcal{L}_{\xi_{\mathrm{ext}}}$ to $\ker\left(\mathbb{L}^+_{(\mathrm{v},\mathrm{w})}\right)\cap C^\infty_{\mathrm{w}}(M,\mathbb{C})^{\mathbb{T}}$ coincides with the restriction of $\mathbb{L}_{(\mathrm{v},\mathrm{w})}$ to the same space. Then, using 3 in Proposition 3, we obtain the converse inclusion, which proves that

$$\mathfrak{h}^{\mathbb{T}}_{(0)}=\mathfrak{a}\oplus\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}\oplus J\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}.$$

This completes the proof. □

Now we are in position to give a proof for Theorem 1 from the introduction.

Proof of Theorem 1. This is done as in the case $\mathrm{v}=1$, $\mathrm{w}=1$ (see [53, Section 3.5]). Let $\mathfrak{s}$ be the Lie algebra of a connected, compact Lie subgroup, $S\subset\mathrm{Aut}^{\mathbb{T}}_0(X)$ containing

$\mathrm{Isom}_0^{\mathbb{T}}(X,g)$. Suppose, for a contradiction, that there exists $V \in \mathfrak{s}$ that doesn't belong to $\mathfrak{k}^{\mathbb{T}}$. By Theorem 8, we have the splitting

$$\mathfrak{h}^{\mathbb{T}} = \mathfrak{k}^{\mathbb{T}} \oplus J\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}} \oplus \left(\bigoplus_{\lambda>0} \mathfrak{h}^{\mathbb{T}}_{(\lambda)}\right),$$

then we can assume that $V \in J\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}} \oplus \left(\bigoplus_{\lambda>0} \mathfrak{h}^{\mathbb{T}}_{(\lambda)}\right)$. Let $V = V_0 + \sum_{\lambda>0} V_\lambda$ be the corresponding decomposition of V, then for any positive integer r we have

$$\left(\mathcal{L}_{\xi_{\mathrm{ext}}}\right)^{2r} V = -\sum_{\lambda>0} \lambda^{2r} V_\lambda \in \mathfrak{s}.$$

It follows that each component V_λ of V is in $\mathfrak{s}$. We can therefore assume that $V \in \mathfrak{s}_\lambda := \mathfrak{s} \cap \mathfrak{h}^{\mathbb{T}}_{(\lambda)}$ or $V \in J\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}} \subset \mathfrak{s}_0$. Suppose that $V \in \mathfrak{s}_\lambda$ for some $\lambda > 0$. Let B denote the Killing form of $\mathfrak{s}$. Since S is a compact Lie group, B is semi-negative and it's kernel coincides with the center of $\mathfrak{s}$. On the other hand V belongs to the kernel of B, indeed for any $W \in \mathfrak{s}_{\lambda_1}$ and $U \in \mathfrak{s}_{\lambda_2}$, by Jacobi identity we can easily show that $[V,[W,U]] \in \mathfrak{s}_{\lambda+\lambda_1+\lambda_2} \neq \mathfrak{s}_{\lambda_2}$ then $\mathfrak{s}_{\lambda+\lambda_1+\lambda_2} = \{0\}$ and by consequence $[V,[W,U]] = 0$. It follows that for any $W \in \mathfrak{s}$ we have $B(V,W) = 0$. Hence V belongs to the center of $\mathfrak{s}$, but we have $\xi_{\mathrm{ext}} \in \mathfrak{t}^{\mathbb{T}} \subset \mathfrak{s}$ and $[V, \xi_{\mathrm{ext}}] = -\lambda JV \neq 0$, a contradiction.

It follows that $V \in J\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}}$. Then $V = \mathrm{grad}_g(h)$ for some real function h. By the hypothesis, the flow Φ_t^V of V is contained in a compact connected subgroup of $\mathrm{Aut}_0^{\mathbb{T}}(X)$. It follows that V is quasi-periodic with a flow closure in $\mathrm{Aut}_0^{\mathbb{T}}(X)$ given by a torus T^k of dimension $k \geq 1$. Note that $k \neq 1$ since a gradient vector field does not admit any non-trivial closed integral curve, as $\frac{d}{dt} h\left(\Phi_t^X(x)\right) = |V|^2_{\Phi_t^V(x)} \geq 0$. It follows that $k > 1$. Let $x \in X$ such that $V_x \neq 0$. We have that $h(\Phi_t^X(x))$ is an increasing function of t, so that $h(\Phi_t^V(x)) - h(x) > c$, for $t > 1$, where $c > 0$. But by density of Φ_t^V in the torus T^k, Φ_t^V meets any small neighborhood U of x, which is a contradiction. We conclude that $\mathfrak{s} = \mathfrak{k}^{\mathbb{T}}$.

If the (v, w)-scalar curvature is constant then by Theorem 8, $\mathfrak{h}^{\mathbb{T}}$ splits as

$$\mathfrak{h}^{\mathbb{T}} = \mathfrak{a} \oplus \mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}} \oplus J\mathfrak{k}^{\mathbb{T}}_{\mathrm{ham}},$$

since $\xi_{\text{ext}} = 0$ and by consequence $\mathfrak{h}^{\mathbb{T}}_{(\lambda)} = \{0\}$. In particular $\mathfrak{h}^{\mathbb{T}}$ is a reductive complex Lie algebra. $\square$

We have the following immediate consequences of Theorem 1.

Corollary 4. *Any* (v, w)*-extremal metric is invariant under the action of some maximal torus* $\mathbb{T}_{\max}$ *of* $\mathrm{Aut}_{\mathrm{red}}(X)$.

Using an argument of Guan [54] (applied originally to the case $(\mathrm{v}, \mathrm{w}) = (1, 1)$), we obtain as in [9] the following uniqueness result in the toric case (i.e. when $\dim(\mathbb{T}) = \dim_{\mathbb{C}}(X) = n$).

Corollary 5. *Let* g *and* $\tilde{g}$ *be two* (v, w)*-extremal metrics on* X*. Then there is* $\Phi \in \mathrm{Aut}_0^{\mathbb{T}}(X)$ *such that* $\mathrm{Isom}_0^{\mathbb{T}}(X, g) = \mathrm{Isom}_0^{\mathbb{T}}(X, \Phi^\star \tilde{g})$*. Furthermore if* X *is a toric manifold and* g *and* $\tilde{g}$ *are two* (v, w)*-extremal metrics in the same Kähler class* α*, then they are isometric.*

2.7 Deformations of weighted cscK metrics

In this section we give the proof of Theorem 2 from the Introduction. Let X be a compact Kähler manifold, α a Kähler class, $\mathbb{T}_{\max} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ a maximal torus and $\mathrm{P}_\alpha \subset \mathfrak{t}^*$ a momentum polytope for α as in Lemma 1. Let $\beta \in H^{1,1}(X)$ and U an open subset of $\mathfrak{t}^*$ with $\mathrm{P}_\alpha \subset U$. Then there exist $a > 0$ such that for any $|r| < a$ we can choose $\mathrm{P}_{\alpha + r\beta} \subset U$ to be the momentum polytope of $\mathbb{T}_{\max}$ with respect to $\alpha + r\beta$. With these choices, we now suppose that v, w are positive smooth functions on U and $\tilde{\mathrm{v}}, \tilde{\mathrm{w}}$ are arbitrary smooth non vanishing functions on U. Let θ be a $\mathbb{T}-$invariant g-harmonic representative of β and ω a $\mathbb{T}$-invariant Kähler metric in α. We take $(\omega, \theta) = 0$ to avoid trivial deformations of the form $\theta = \lambda\omega$. We denote by

$$\omega_{r,\phi} := \omega + r\theta + dd^c\phi,$$

a $\mathbb{T}-$invariant deformations of ω for $r \in \mathbb{R}$ and $\phi \in C^\infty(X, \mathbb{R})^{\mathbb{T}}$. We consider the following map,

$$\mathcal{S} : \mathbb{R}^3 \times C^\infty(X, \mathbb{R})^{\mathbb{T}} \to C^\infty(X, \mathbb{R})^{\mathbb{T}}$$

defined by,

$$\mathcal{S}(t,s,r,\phi) := \frac{\mathrm{Scal}_{\mathrm{v}+t\tilde{\mathrm{v}}}(\omega_{r,\phi})}{(\mathrm{w}+s\tilde{\mathrm{w}})(m_{r,\phi})},$$

where $m_{(r,\phi)} := m_{\omega_{r,\phi}} : X \to \mathrm{P}_{\alpha+r\beta}$ denotes the $\omega_{t,\phi}$-momentum map with momentum image $\mathrm{P}_{\alpha+r\beta}$ and $\mathrm{v},\tilde{\mathrm{v}},\mathrm{w},\tilde{\mathrm{w}} \in C^\infty(U,\mathbb{R}_{>0})$. We take $k > n$ such that the Sobolev space $L^{2,k}(X,\mathbb{R})^{\mathbb{T}}$ form an algebra for the usual multiplication of functions, embadded in $C^4(X,\mathbb{R})^{\mathbb{T}}$. Then $\mathcal{S}$ defines a map

$$\mathcal{S} : \mathbb{R}^3 \times L^{2,k+4}(X,\mathbb{R})^{\mathbb{T}} \to L^{2,k}(X,\mathbb{R})^{\mathbb{T}}.$$

We will start by giving the linearization of the w-scalar curvature with respect to the Kähler potential ϕ.

Lemma 9. *For any $\mathbb{T}$-invariant Kähler metric $\omega \in \alpha$ and any variation $\dot{\phi} \in T_\phi \mathcal{K}_\omega^{\mathbb{T}}$ we have*

$$\delta\Big(\frac{\mathrm{Scal}_{\mathrm{v}}(\omega)}{\mathrm{w}(m_\omega)}\Big)(\dot{\phi}) = -2\mathbb{L}_{\mathrm{v},\mathrm{w}}(\dot{\phi}) + d^c\dot{\phi}(\xi_{\mathrm{ext}}), \tag{2.40}$$

where $\xi_{\mathrm{ext}} := J\mathrm{grad}_g\left(\frac{\mathrm{Scal}_{\mathrm{v}}(\omega)}{\mathrm{w}(m_\omega)}\right)$ and $\mathbb{L}_{\mathrm{v},\mathrm{w}}$ is the elliptic fourth order differential operator given by

$$\mathbb{L}_{\mathrm{v},\mathrm{w}}\dot{\phi} = \frac{\delta\delta\big(\mathrm{v}(m_\omega)(D^-d)\dot{\phi}\big)}{\mathrm{w}(m_\omega)},$$

here, D is the Levi-Civita connection of g and $D^-(d\dot{\phi})$ is the J-anti-invariant part of the tensor $D(d\dot{\phi})$.

Proof. Let $(\xi_1,\cdots,\xi_\ell)$ be a familly of $\mathbb{S}^1$-generators of $\mathbb{T}$. For a $\mathbb{T}$-invariant variation $\dot{\omega} = dd^c\dot{\phi}$, the corresponding variations of m_ω, Δ_ω, Scal_ω are given by (see e.g. [53]):

$$\begin{aligned} \dot{m}_\omega =& d\dot{\phi} \\ \dot{\Delta}_\omega =& (dd^c\dot{\phi}, dd^c\cdot) \\ \dot{\mathrm{Scal}}_\omega =& -2\mathbb{L}^g(\dot{\phi}) + (d\,\mathrm{Scal}(\omega), d\dot{\phi}), \end{aligned} \tag{2.41}$$

where $\mathbb{L}^g(\dot{\phi}) = \delta\delta(D^-d\dot{\phi})$ is the usual Lichnerowicz operator. Then the first variation of the v-scalar curvature in the direction $\dot{\phi}$ is given by:

$$\begin{aligned}\boldsymbol{\delta}\mathrm{Scal}_{\mathrm{v}}(\dot{\phi}) =& -2\mathrm{v}(m_\omega)\mathbb{L}^g(\dot{\phi}) + \mathrm{v}(m_\omega)(d\mathrm{Scal}(\omega), d\dot{\phi}) + \mathrm{Scal}(\omega)(d(\mathrm{v}(m_\omega)), d\dot{\phi}) \\ &+ 2\sum_{i=1}^{\ell}\mathrm{v}_{,i}(m_\omega)\Delta_\omega(dm_\omega^{\xi_i}, d\dot{\phi}) + 2\sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)(dm_\omega^{\xi_i}, d\dot{\phi})\Delta_\omega(m_\omega^{\xi_i}) \\ &+ 2\sum_{i=1}^{\ell}\mathrm{v}_{,i}(m_\omega)(dd^c m_\omega^{\xi_i}, dd^c\dot{\phi}) - \sum_{i,j,k=1}^{\ell}\mathrm{v}_{,ijk}(m_\omega)(dm_\omega^{\xi_k}, d\dot{\phi})(\xi_i, \xi_j) \\ &+ \sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)\xi_j((d^c\dot{\phi})(\xi_i)).\end{aligned}$$

By (2.32) and the $\mathbb{T}$-invariance of ϕ we have

$$(dd^c\dot{\phi}, dd^c m_\omega^{\xi_i}) = -\Delta(dm_\omega^{\xi_i}, d\dot{\phi}) + (d\Delta\dot{\phi}, dm_\omega^{\xi_i}).$$

Thus,

$$\begin{aligned}\boldsymbol{\delta}\mathrm{Scal}_{\mathrm{v}}(\dot{\phi}) =& -2\mathrm{v}(m_\omega)\mathbb{L}^g(\dot{\phi}) + \mathrm{v}(m_\omega)(d\mathrm{Scal}(\omega), d\dot{\phi}) + \mathrm{Scal}(\omega)(d(\mathrm{v}(m_\omega)), d\dot{\phi}) \\ &+ 2\sum_{i=1}^{\ell}\mathrm{v}_{,i}(m_\omega)(d\Delta\dot{\phi}, dm_\omega^{\xi_i}) + 2\sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)(dm_\omega^{\xi_i}, d\dot{\phi})\Delta_\omega(m_\omega^{\xi_i}) \\ &- \sum_{i,j,k=1}^{\ell}\mathrm{v}_{,ijk}(m_\omega)(dm_\omega^{\xi_k}, d\dot{\phi})(\xi_i, \xi_j) + \sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)\xi_j((d^c\dot{\phi})(\xi_i)).\end{aligned} \tag{2.42}$$

On the other hand we have

$$\begin{aligned}(d\mathrm{Scal}_{\mathrm{v}}(\omega), d\dot{\phi}) =& \mathrm{v}(m_\omega)(d\mathrm{Scal}(\omega), d\dot{\phi}) + (d(\mathrm{v}(m_\omega)), d\dot{\phi})\mathrm{Scal}(\omega) \\ &+ 2\sum_{i,j=1}^{\ell}\Delta(m_\omega^{\xi_i})\mathrm{v}_{,ij}(m_\omega)(dm_\omega^{\xi_j}, d\dot{\phi}) + 2\sum_{i=1}^{\ell}\mathrm{v}_{,i}(m_\omega)(d\Delta(m_\omega^{\xi_i}), d\dot{\phi}) \\ &- \sum_{i,j,k=1}^{\ell}\mathrm{v}_{,ijk}(m_\omega)(dm_\omega^{\xi_k}, d\dot{\phi})(\xi_i, \xi_j) - \sum_{i,j=1}^{\ell}\mathrm{v}_{,ij}(m_\omega)(d\dot{\phi}, d(\xi_i, \xi_j)).\end{aligned} \tag{2.43}$$

By taking the difference (2.42)-(2.43) we get exactly (2.31) for $\alpha = d\dot{\phi}$, which, in turn, is equal to $-2\delta\delta\big(\mathrm{v}(m_\omega)(D^-d)\dot\phi\big)$. The expression (2.40), follows from the following

$$\begin{aligned}\boldsymbol{\delta}\Big(\frac{\mathrm{Scal}_{\mathrm{v}}(\omega)}{\mathrm{w}(m_\omega)}\Big)(\dot\phi) =&\frac{1}{\mathrm{w}(m_\omega)}\boldsymbol{\delta}\Big(\mathrm{Scal}_{\mathrm{v}}(\omega)\Big)(\dot\phi)+\mathrm{Scal}_{\mathrm{v}}(\omega)\boldsymbol{\delta}\Big(\frac{1}{\mathrm{w}(m_\omega)}\Big)(\dot\phi)\\ =&\frac{1}{\mathrm{w}(m_\omega)}[-2\delta\delta\big(\mathrm{v}(m_\omega)(D^-d)\dot\phi\big)+(d\mathrm{Scal}_{\mathrm{v}}(\omega),d\dot\phi)]+(d(\frac{1}{\mathrm{w}(m_\omega)}),d\dot\phi)\\ =&-2\mathbb{L}_{\mathrm{v},\mathrm{w}}(\dot\phi)+d^c\dot\phi(\xi_{\mathrm{ext}}).\end{aligned}$$

□

Using the Lemma 9, we obtain

Lemma 10. *The map $\mathcal{S}$ is C^1 with Fréchet derivative in* 0 *given by*

$$\mathrm{T}_0\mathcal{S}=\begin{pmatrix}S_1 & S_2 & S_3 & S_4\end{pmatrix}$$

where $S_1(\dot t)$, $S_2(\dot s)$, $S_3(\dot r)$, $S_4(\dot\phi)$ are the derivatives with respect to t, s, r, ϕ respectively and

$$\begin{aligned}S_1(\dot t)=&\left(\frac{\mathrm{Scal}_{\tilde{\mathrm{v}}}(\omega)}{\mathrm{w}(m_\omega)}\right)\dot t,\\ S_2(\dot s)=&-\Big(\frac{\tilde{\mathrm{w}}(m_\omega)\mathrm{Scal}_{\mathrm{v}}(\omega)}{(\mathrm{w}(m_\omega))^2}\Big)\dot s,\\ S_3(\dot r)=&\Big[-\Big(\frac{(\mathrm{w}(m_\omega))^\theta}{\mathrm{w}(m_\omega)}\Big)\mathrm{Scal}_{\mathrm{v}}(\omega)+(\mathrm{v}(m_\omega))^\theta\mathrm{Scal}(m_\omega)-2\mathrm{v}(m_\omega)(\theta,\mathrm{Ric}(\omega))\\ &+2\sum_{i=1}^{\ell}(\mathrm{v}_{,i}(m_\omega))^\theta\Delta_\omega(m_\omega^{\xi_i})-\sum_{i,j=1}^{\ell}(\mathrm{v}_{,ij}(m_\omega))^\theta\theta(\xi_i,J\xi_i)\Big]\frac{\dot r}{\mathrm{w}(m_\omega)},\\ S_4(\dot\phi)=&-2\mathbb{L}_{\mathrm{v},\mathrm{w}}(\dot\phi)+(d^c\dot\phi)(\xi_{\mathrm{ext}}).\end{aligned}$$

where $(\xi_1,\cdots\xi_\ell)$ is a family of $\mathbb{S}^1$-generators of $\mathbb{T}$, and for $\mathrm{u}\in C^\infty(U,\mathbb{R})$ we denote

$$(\mathrm{u}(m_\omega))^\theta:=\frac{d}{dr}\Big|_{r=0}[\mathrm{u}(m_{\omega+r\theta})]=-\sum_{i=1}^{\ell}\mathrm{u}_{,i}(m_\omega)\mathbb{G}_\omega(\theta,dd^cm_\omega^{\xi_i}),\qquad(2.44)$$

with $\mathbb{G}_\omega$ is the Green operator relative to ω for the Laplacian Δ_ω.

Proof. The expressions of S_1 and S_2 are straightforward. The expression of S_4 follows from Lemma 9. For S_3, we use the following variation formulas contained in [53, Chapter

5] and [68],

$$\left.\frac{d}{dr}\right|_{r=0} \mathrm{Scal}(\omega + r\theta) = -2(\theta, \mathrm{Ric}(\omega)),$$
$$\left.\frac{d}{dr}\right|_{r=0} \Delta_{\omega+r\theta}(f) = -(dd^c f, \theta),$$
$$\left.\frac{d}{dr}\right|_{r=0} m^{\xi}_{\omega+r\theta} = -\mathbb{G}_{\omega}(\theta, dd^c m^{\xi}_{\omega}),$$
$$\left.\frac{d}{dr}\right|_{r=0} |\xi|^2_{\omega+r\theta} = \theta(\xi, J\xi).$$

□

We consider the inner product $\langle\cdot,\cdot\rangle_{\mathrm{w},\omega}$, given by

$$\langle f, h\rangle_{\mathrm{w},\omega} = \int_X f h \mathrm{w}(m_\omega)\omega^{[n]}.$$

Let $\mathfrak{t}_\omega$ be the space of ω-Killing potentials of the elements of $\mathfrak{t} := \mathrm{Lie}(\mathbb{T}_{\max})$ normalized by $\langle f, 1\rangle_{\mathrm{w},\omega} = 0$. We denote $L^{2,k}_{\perp}(X,\mathbb{R})^{\mathbb{T}}$ the orthogonal complement $\mathfrak{t}_\omega$ in $L^{2,k}(X,\mathbb{R})^{\mathbb{T}}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathrm{w},\omega}$. Let $\Pi_{s,r,\phi}$ (resp. $\Pi_{\mathrm{w},\omega} := \Pi_{(0,0,0)}$) denote the orthogonal projections on $\mathfrak{t}_{\omega_{r,\phi}}$ (resp. $\mathfrak{t}_\omega$) with respect to $\langle\cdot,\cdot\rangle_{\mathrm{w}+s\tilde{\mathrm{w}},\omega_{r,\phi}}$ (resp. $\langle\cdot,\cdot\rangle_{\mathrm{w},\omega}$). As in [68], taking $(s,r) \in (-\epsilon,\epsilon)^2$ close to zero, and $\phi \in \mathcal{U}$ in a neighborhood $\mathcal{U} \subset L^{2,k+4}_{\perp}(X,\mathbb{R})^{\mathbb{T}}$ of the origin, we have

$$\ker\left(\mathrm{Id} - \Pi_{\mathrm{w},\omega}\right) \circ \left(\mathrm{Id} - \Pi_{s,r,\phi}\right) = \ker\left(\mathrm{Id} - \Pi_{s,r,\phi}\right).$$

Now, we consider the Lebrun-Simanca map

$$\Psi : (-\epsilon,\epsilon)^3 \times \mathcal{U} \to (-\epsilon,\epsilon)^3 \times L^{2,k}_{\perp}(X,\mathbb{R})^{\mathbb{T}},$$

defined by

$$\Psi(t,s,r,\phi) := \left(t,s,r,\left(\mathrm{Id} - \Pi_{\mathrm{w},\omega}\right) \circ \left(\mathrm{Id} - \Pi_{s,r,\phi}\right) \mathcal{S}(t,s,r,\phi)\right). \tag{2.45}$$

We have $\Psi(0,0,0,0) = 0$ and $\Psi(t,s,r,\phi) = (t,s,r,0)$ if and only if $\omega_{r,\phi}$ is $(\mathrm{v}+t\tilde{\mathrm{v}}, \mathrm{w}+s\tilde{\mathrm{w}})$-extremal. We shall thus use the inverse function theorem for the map Ψ.

To calculate the derivative of Ψ in 0, we will need some technical Lemmas.

We denote by $\mathcal{F}^r_{t,s}$ the $(\mathrm{v}+t\tilde{\mathrm{v}}, \mathrm{w}+s\tilde{\mathrm{w}})$-Futaki invariant of $(\alpha + r\beta, \mathrm{P}_{\alpha+r\beta})$ (see Definition 8), and by $\mathcal{B}^r_s$ the $\mathrm{w}+s\tilde{\mathrm{w}}$-Futaki-Mabuchi bilinear form of $(\alpha + r\beta, \mathrm{P}_{\alpha+r\beta})$ (see Definition 9).

Lemma 11. *Let $\xi, \eta \in \mathfrak{t}$, with killing potentials $f^\xi_\omega, f^\eta_\omega$ normalized by having zero mean value. The partial derivatives of $\mathcal{F}^r_{t,s}(\xi)$, with respect to the variables (t,s,r) in $(0,0,0)$ are given by*

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_0 \mathcal{F}^r_{t,s}(\xi) =& \mathcal{F}^\alpha_{\tilde{\mathrm{v}},\mathrm{w}}(\xi),\\
\frac{\partial}{\partial s}\Big|_0 \mathcal{F}^r_{t,s}(\xi) =& 0,\\
\frac{\partial}{\partial r}\Big|_0 \mathcal{F}^r_{t,s}(\xi) =& \langle S_3(1), f^\xi_\omega\rangle_{\mathrm{w},\omega} + \langle \theta, (\mathrm{w}(m_\omega))^{-1}\mathbb{G}_\omega(\mathcal{S}(0)\mathrm{w}(m_\omega))dd^c f^\xi_\omega\rangle_{\mathrm{w},\omega}\\
&+\sum_{i=1}^{\ell}\langle \theta, (\mathrm{w}(m_\omega))^{-1}\mathbb{G}_\omega(\mathcal{S}(0)f^\xi_\omega \mathrm{w}_{,i}(m_\omega))dd^c m^{\xi_i}_\omega\rangle_{\mathrm{w},\omega}.
\end{aligned}$$

where S_3 is given in Lemma 10.

The partial derivatives of $\mathcal{B}^r_s(\xi,\eta)$, $\xi,\eta \in \mathfrak{t}$ with respect to the variables (s,r) in $(0,0,0)$ are given by

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_0 \mathcal{B}^r_s(\xi,\eta) =& \mathcal{B}^\alpha_{\tilde{\mathrm{w}}}(\xi,\eta),\\
\frac{\partial}{\partial r}\Big|_0 \mathcal{B}^r_s(\xi,\eta) =& \langle \theta, (\mathrm{w}(m_\omega))^{-1}[f^\xi_\omega dd^c\mathbb{G}_\omega(f^\eta_\omega \mathrm{w}(m_\omega))]\rangle_{\mathrm{w},\omega}\\
&+\langle \theta, (\mathrm{w}(m_\omega))^{-1}\mathbb{G}_\omega(f^\xi_\omega \mathrm{w}(m_\omega))dd^c f^\eta_\omega\rangle_{\mathrm{w},\omega}\\
&+\sum_{i=1}^{\ell}\langle \theta, (\mathrm{w}(m_\omega))^{-1}\mathbb{G}_\omega(f^\xi_\omega f^\eta_\omega \mathrm{w}_{,i}(m_\omega))dd^c m^{\xi_i}_\omega\rangle_{\mathrm{w},\omega}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\mathcal{F}^r_{t,s}(\xi) &= \int_X \mathcal{S}(t,s,r,0)f^\xi_{\omega+r\theta}(\mathrm{w}+s\tilde{\mathrm{w}})(m_{\omega+r\theta})(\omega+r\theta)^{[n]},\\
\mathcal{B}^r_s(\xi,\eta) &= \int_X f^\xi_{\omega+r\theta}f^\eta_{\omega+r\theta}(\mathrm{w}+s\tilde{\mathrm{w}})(m_{\omega+r\theta})(\omega+r\theta)^{[n]}.
\end{aligned}$$

The partial derivatives with respect to t,s of $\mathcal{F}^r_{t,s}$ and $\mathcal{B}^r_s$ are straightforward. The

r-derivative of $\mathcal{F}^r_{t,s}(\xi)$ is given by

$$\begin{aligned}
\frac{\partial}{\partial r}\bigg|_0 \mathcal{F}^r_{t,s}(\xi) &= \int_X S_3(1) f^\xi_\omega \mathrm{w}(m_\omega)\omega^{[n]} - \int_X \mathcal{S}(0)\mathbb{G}_\omega(\delta(\theta(J\xi)))\mathrm{w}(m_\omega)\omega^{[n]} \\
&\quad - \sum_{i=1}^{\ell} \int_X \mathcal{S}(0) f^\xi_\omega \mathrm{w}_{,i}(m_\omega)\mathbb{G}_\omega(\delta(\theta(J\xi_i)))\omega^{[n]} \\
&= \int_X S_3(1) f^\xi_\omega \mathrm{w}(m_\omega)\omega^{[n]} - \int_X (d\mathbb{G}_\omega(\mathcal{S}(0)\mathrm{w}(m_\omega)), \theta(J\xi))\omega^{[n]} \\
&\quad - \sum_{i=1}^{\ell} \int_X (\mathbb{G}_\omega[\mathcal{S}(0) f^\xi_\omega \mathrm{w}_{,i}(m_\omega)], \theta(J\xi_i))\omega^{[n]} \\
&= \int_X S_3(1) f^\xi_\omega \mathrm{w}(m_\omega)\omega^{[n]} - \int_X (d^c f^\xi_\omega \wedge d\mathbb{G}_\omega(\mathcal{S}(0)\mathrm{w}(m_\omega)), \theta)\omega^{[n]} \\
&\quad - \sum_{i=1}^{\ell} \int_X (d^c m^{\xi_i}_\omega \wedge d\mathbb{G}_\omega[\mathcal{S}(0) f^\xi_\omega \mathrm{w}_{,i}(m_\omega)], \theta)\omega^{[n]} \\
&= \int_X \Big(S_3(1) f^\xi_\omega \mathrm{w}(m_\omega) + \mathcal{S}(0)(\mathrm{w}(m_\omega))^\theta + (\theta, dd^c f^\xi_\omega)\mathbb{G}_\omega(\mathcal{S}(0)\mathrm{w}(m_\omega))\Big)\,\omega^{[n]} \\
&\quad + \sum_{i=1}^{\ell} \int_X (dd^c m^{\xi_i}_\omega, \theta)\mathbb{G}_\omega[\mathcal{S}(0) f^\xi_\omega \mathrm{w}_{,i}(m_\omega)]\omega^{[n]}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial}{\partial r}\bigg|_0 \mathcal{F}^r_{t,s}(\xi) = &\langle S_3(1), f^\xi_\omega\rangle_{\mathrm{w},\omega} + \langle \theta, (\mathrm{w}(m_\omega))^{-1}\mathbb{G}_\omega(\mathcal{S}(0)\mathrm{w}(m_\omega)) dd^c f^\xi_\omega\rangle_{\mathrm{w},\omega} \\
&+ \sum_{i=1}^{\ell} \langle \theta, (\mathrm{w}(m_\omega))^{-1}\mathbb{G}_\omega(\mathcal{S}(0) f^\xi_\omega \mathrm{w}_{,i}(m_\omega)) dd^c m^{\xi_i}_\omega\rangle_{\mathrm{w},\omega}.
\end{aligned}$$

Now, we consider the r-derivative of $\mathcal{B}^r_s(\xi,\eta)$. We compute

$$\begin{aligned}
\frac{\partial}{\partial r}\bigg|_0 \mathcal{B}^r_s(\xi,\eta) &= -\int_X f^\eta_\omega \mathbb{G}_\omega(\delta\theta(J\xi))\mathrm{w}(m_\omega)\omega^{[n]} - \int_X f^\xi_\omega \mathbb{G}_\omega(\delta\theta(J\eta))\mathrm{w}(m_\omega)\omega^{[n]} \\
&\quad - \sum_{i=1}^{\ell} \int_X f^\xi_\omega f^\eta_\omega \mathrm{w}_{,i}(m_\omega)\mathbb{G}_\omega(\delta\theta(J\xi_i))\omega^{[n]} \\
&= -\int_X (d^c f^\xi_\omega \wedge d\mathbb{G}_\omega(f^\eta_\omega \mathrm{w}(m_\omega)), \theta)\omega^{[n]} - \int_X (d^c f^\eta_\omega \wedge d\mathbb{G}_\omega(f^\xi_\omega \mathrm{w}(m_\omega)), \theta)\omega^{[n]} \\
&\quad - \sum_{i=1}^{\ell} \int_X (d^c m^{\xi_i}_\omega \wedge d\mathbb{G}_\omega(f^\xi_\omega f^\eta_\omega \mathrm{w}_{,i}(m_\omega)), \theta)\omega^{[n]} \\
&= \int_X (\theta, dd^c\mathbb{G}_\omega(f^\eta_\omega \mathrm{w}(m_\omega))) f^\xi_\omega \omega^{[n]} + \int_X (\theta, dd^c f^\eta_\omega)\mathbb{G}_\omega(\mathrm{w}(m_\omega) f^\xi_\omega)\omega^{[n]} \\
&\quad + \sum_{i=1}^{\ell} \int_X (dd^c m^{\xi_i}_\omega, \theta)\mathbb{G}_\omega(f^\xi_\omega f^\eta_\omega \mathrm{w}_{,i}(m_\omega))\omega^{[n]}.
\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial r}\Big|_0 \mathcal{B}^r_s(\xi,\eta) =& \langle \theta, (\mathrm{w}(m_\omega))^{-1}[f^\xi_\omega dd^c \mathbb{G}_\omega(f^\eta_\omega \mathrm{w}(m_\omega))]\rangle_{\mathrm{w},\omega} \\ &+ \langle \theta, (\mathrm{w}(m_\omega))^{-1}\mathbb{G}_\omega(f^\xi_\omega \mathrm{w}(m_\omega)) dd^c f^\eta_\omega\rangle_{\mathrm{w},\omega} \\ &+ \sum_{i=1}^{\ell} \langle \theta, (\mathrm{w}(m_\omega))^{-1}\mathbb{G}_\omega(f^\xi_\omega f^\eta_\omega \mathrm{w}_{,i}(m_\omega)) dd^c m^{\xi_i}_\omega\rangle_{\mathrm{w},\omega}.\end{aligned}$$

□

Now, we compute the derivatives of the $(\mathrm{v}+t\tilde{\mathrm{v}}, \mathrm{w}+s\tilde{\mathrm{w}})$-extremal vector field $\xi_{\mathrm{ext}}(t,s,r)$ of $(\alpha + r\beta, \mathrm{P}_{\alpha+r\beta})$.

Lemma 12. *Suppose that (g,ω) is a (v,w)-extremal metric. The partial derivatives of $\xi_{\mathrm{ext}}(t,s,r)$ are given by*

$$\begin{aligned}\frac{\partial}{\partial t}\Big|_0 \xi_{\mathrm{ext}}(t,s,r) =& J\mathrm{grad}_g\left(\Pi_{\mathrm{w},\omega}\left(\frac{\mathrm{Scal}_{\tilde{\mathrm{v}},\mathrm{w}}(\omega)}{\mathrm{w}(m_\omega)}\right)\right), \\ \frac{\partial}{\partial s}\Big|_0 \xi_{\mathrm{ext}}(t,s,r) =& \xi_{\mathrm{ext}}, \\ \frac{\partial}{\partial r}\Big|_0 \xi_{\mathrm{ext}}(t,s,r) =& J\mathrm{grad}_g(\Pi_{\mathrm{w},\omega}(S_3(1) - \mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0)))).\end{aligned}$$

Proof. We denote the r-partial derivative of $\xi_{\mathrm{ext}}(t,s,r)$ in 0 by $\dot{\xi}_{\mathrm{ext}}$ and z_ω its Killing potential with zero mean value. Using (2.30), we have

$$\mathcal{F}^r_{t,s}(\xi) = \mathcal{B}^r_s(\xi_{\mathrm{ext}}(t,s,r),\xi)$$

for any $\xi \in \mathfrak{t}$. Differentiating with respect to r, we obtain

$$\frac{\partial}{\partial r}\Big|_0 \mathcal{F}^r_{t,s}(\xi) = \left(\frac{\partial}{\partial r}\Big|_0 \mathcal{B}^r_s\right)(\xi_{\mathrm{ext}},\xi) + \mathcal{B}^\alpha_{\mathrm{w}}(\dot{\xi}_{\mathrm{ext}},\xi)$$

Since ω is extremal, using (11) we obtain

$$\begin{aligned}\langle S_3(1) - z_\omega, f^\xi_\omega\rangle_{\mathrm{w},\omega} =& \langle \theta, \mathrm{w}(m_\omega)^{-1}\mathcal{S}(0) dd^c\mathbb{G}_\omega(f^\xi_\omega \mathrm{w}(m_\omega))\rangle_{\mathrm{w},\omega} \\ =& \int_X \mathcal{S}(0)(\theta, dd^c\mathbb{G}_\omega(f^\xi_\omega \mathrm{w}(m_\omega))\omega^{[n]} \\ =& \int_X \mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0)) f^\xi_\omega \mathrm{w}(m_\omega)\omega^{[n]} \\ =& \langle \mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0)), f^\xi_\omega\rangle_{\mathrm{w},\omega}.\end{aligned}$$

Thus $z_\omega = S_3(1) - \mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0))$, and by consequence

$$\dot{\xi}_{\text{ext}} = J\text{grad}_g(S_3(1) - \mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0))).$$

The remaining derivatives follows using the same argument. $\square$

Lemma 13. *Suppose that ω is a (v,w)-extremal metric. The Lebrun-Simanca map (2.45) is C^1, with Fréchet derivative at the origin given by*

$$\mathrm{T}_0\Psi \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{Id} - \Pi_{\mathrm{w},\omega} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_1 & S_2 & S_3 + \mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0)) & -2\mathbb{L}_{\mathrm{v},\mathrm{w}} \end{pmatrix}$$

where S_1, S_2, S_3 are given in Lemma 10, and $\mathbb{G}_\omega$ is the Green operator relative to ω.

Proof. We calculate the partial derivatives in 0 of $Z(t,s,r,\phi) := (\mathrm{Id} - \Pi_{s,r,\phi})\,\mathcal{S}(t,s,r,\phi)$. For the derivative with respect to ϕ, using the fact that

$$\xi_{\text{ext}}(t,s,r) := J\text{grad}_{g_{r,\phi}}(\Pi_{s,r,\phi}\mathcal{S}(t,s,r,\phi)) = J\text{grad}_{g_{r,\phi}}((\mathcal{S} - Z)(t,s,r,\phi)),$$

is the $(\mathrm{v} + t\tilde{\mathrm{v}}, \mathrm{w} + s\tilde{\mathrm{w}})$-extremal vector field of $(X, \alpha + r\beta)$, we obtain

$$\left.\frac{\partial}{\partial\phi}\right|_0 (\mathcal{S} - Z)(\dot{\phi}) = \mathcal{L}_{\xi_{\text{ext}}}\dot{\phi} = (d\dot{\phi}, d\Pi_{\mathrm{w},\omega}\mathcal{S}(0)) = (d^c\dot{\phi})(\xi_{\text{ext}})(\dot{\phi}).$$

From this we deduce that

$$\left.\frac{\partial}{\partial\phi}\right|_0 [(\mathrm{Id} - \Pi_{\mathrm{w},\omega})\,Z](\dot{\phi}) = (\mathrm{Id} - \Pi_{\mathrm{w},\omega})\,(S_4(\dot{\phi}) - (d^c\dot{\phi})(\xi_{\text{ext}})) = -2\mathbb{L}_{\mathrm{v},\mathrm{w}},$$

where S_4 is the ϕ-derivative of $\mathcal{S}$ (see Lemma 10). Now we compute the r-derivative of Z in 0. Differentiating the relation (2.7) with respect to r, we obtain

$$\begin{aligned}\left.\frac{d}{dr}\right|_0 \xi_{\text{ext}} =& J(\left.\frac{d}{dr}\right|_0 \text{grad})((\mathcal{S} - Z)(0)) + J\text{grad}_g(\left.\frac{d}{dr}\right|_0 (\mathcal{S} - Z)) \\ =& (\theta(\xi_{\text{ext}}(0)))^\sharp + J\text{grad}_g(\left.\frac{d}{dr}\right|_0 (\mathcal{S} - Z)) \\ =& - J\text{grad}_g(\mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0)) + \left.\frac{d}{dr}\right|_0 (\mathcal{S} - Z)).\end{aligned}$$

Using Lemma 12, we obtain

$$\frac{d}{dr}\bigg|_0 (\mathcal{S} - Z) = \Pi_{\mathrm{w},\dot{\omega}}(S_3(1)) + (\mathrm{Id} - \Pi_{\mathrm{w},\omega})\mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0))).$$

It follows that

$$\frac{d}{dr}\bigg|_0 [(\mathrm{Id} - \Pi_{\mathrm{w},\omega})Z](\dot{r}) = (\mathrm{Id} - \Pi_{\mathrm{w},\omega})(S_3(\dot{r}) + \mathbb{G}_\omega(\theta, dd^c\mathcal{S}(0))\dot{r}).$$

The remaining derivatives with respect to t, s follows using a similar argument. $\square$

The operator $\mathbb{L}_{\mathrm{v},\mathrm{w}}$ is a fourth order $\langle\cdot,\cdot\rangle_{\mathrm{w},\omega}$-self adjoint $\mathbb{T}$-invariant elliptic linear operator. By standard elliptic theory we have the following decomposition $\langle\cdot,\cdot\rangle_{\mathrm{w},\omega}$-orthogonal decomposition

$$L^{2,k}(X,\mathbb{R})^{\mathbb{T}} = \mathrm{Ker}(\mathbb{L}_{\mathrm{v},\mathrm{w}}) \oplus \mathrm{Im}(\mathbb{L}_{\mathrm{v},\mathrm{w}}). \tag{2.46}$$

We have $\mathrm{Ker}(\mathbb{L}_{\mathrm{v},\mathrm{w}}) = \mathfrak{t}_\omega$ since $\mathbb{T}$ is a maximal torus, and $\mathrm{Im}(\mathbb{L}_{\mathrm{v},\mathrm{w}}) = L^{2,k}_{\perp}(X,\mathbb{R})^{\mathbb{T}}$. Thus,

$$\mathbb{L}_{\mathrm{v},\mathrm{w}} : L^{2,k+4}_{\perp}(X,\mathbb{R})^{\mathbb{T}} \to L^{2,k}_{\perp}(X,\mathbb{R})^{\mathbb{T}}$$

is an isomorphism. By consequence,

$$\mathrm{T}_0\Psi : \mathbb{R}^3 \times L^{2,k+4}_{\perp}(X,\mathbb{R})^{\mathbb{T}} \to \mathbb{R}^3 \times L^{2,k+4}_{\perp}(X,\mathbb{R})^{\mathbb{T}}$$

is an isomorphism.

Corollary 6. *There exists $\varepsilon > 0$ s.t. for $|r| < \varepsilon$, $|s| < \varepsilon$, $|t| < \varepsilon$ there exist $\phi \in C^4(X,\mathbb{R})$ and ℓ a smooth affine linear function on $U \subset \mathfrak{t}^*$ ($\mathrm{P}_{\alpha+r\beta} \subset U$) such that*

$$\frac{\mathrm{Scal}_{\mathrm{v}+t\tilde{\mathrm{v}}}(\omega_{r,\phi})}{(\mathrm{w}+s\tilde{\mathrm{w}})(m_{r,\phi})} = \ell(m_{r,\phi}).$$

Proof. By the inverse function theorem, it follows that Ψ is an isomorphism in a neighborhood $(-\epsilon,\epsilon)^3 \times \mathcal{U}$ of $0 \in \mathbb{R}^3 \times L^{2,k+4}_{\perp}(X,\mathbb{R})^{\mathbb{T}}$, and using the Sobolev embbeding theorem, we can assume that $L^{2,k+4}(X,\mathbb{R}) \subset C^4(X,\mathbb{R})$ for $k \gg 1$. Thus, for any $|r| < \varepsilon$, $|s| < \varepsilon$, $|t| < \varepsilon$ there exist $\phi \in C^4(X,\mathbb{R})$ such that $\omega_{r,\phi}$ is $(\mathrm{v}+t\tilde{\mathrm{v}}, \mathrm{w}+s\tilde{\mathrm{w}})$-extremal. $\square$

To complete the proof of Theorem 2, we need to conclude that the metric is actually C^∞. This follows form a bootstraping argument similar to the case of extremal metrics [68, Proposition 4].

Lemma 14. *A* (v, w)*-extremal metric of regularity* C^4 *is smooth.*

Proof. If the Kähler metric (g, ω) is of regularity C^4, then the scalar curvature Scal_g has C^2 regularity. Let $(\xi_1, \cdots, \xi_\ell)$ be a family of $\mathbb{S}^1$ generators of the torus $\mathbb{T}$ acting holomorphicaly on X. Then the vector fields $(\xi_1, \cdots, \xi_\ell)$ are real analytic, being the real parts of holomorphic sections of $T^{1,0}X$. Therefore, there duals $(dm_\omega^{\xi_1}, \cdots, dm_\ell^{\xi_\ell})$ with respect to ω have C^4 regularity. Thus, the momentum map $m_\omega : X \to \mathfrak{t}^*$

$$m_\omega(x) = (m_\omega^{\xi_1}(x), \cdots, m_\ell^{\xi_\ell}(x))$$

has C^5 regularity in holomorphic coordinates. Using (2.4), the vector field $\mathrm{grad}_g\big(\frac{\mathrm{Scal}_\mathrm{v}(\omega)}{\mathrm{w}(m_\omega)}\big)$ is of regularity C^1. By (v, w)-extremality of (g, ω), the vector field $\mathrm{grad}_g\big(\frac{\mathrm{Scal}_\mathrm{v}(\omega)}{\mathrm{w}(m_\omega)}\big)$ is real analytic. It follows that $\frac{\mathrm{Scal}_\mathrm{v}(\omega)}{\mathrm{w}(m_\omega)}$ is of regularity C^5 in holomorphic coordinates. In holomorphic coordinates we have

$$\Delta_\omega \log\big(\frac{\omega^n}{\omega_{\mathrm{flat}}^n}\big) = \frac{\mathrm{w}(m_\omega)}{\mathrm{v}(m_\omega)}\Big(\frac{\mathrm{Scal}_\mathrm{v}(\omega)}{\mathrm{w}(m_\omega)} - 2\sum_{i=1}^{\ell} \frac{\mathrm{v}_{,i}(m_\omega)}{\mathrm{w}(m_\omega)}\Delta_\omega(m_\omega^{\xi_i}) + \sum_{i,j=1}^{\ell} \frac{\mathrm{v}_{,ij}(m_\omega)}{\mathrm{w}(m_\omega)}(\xi_i, \xi_j)\Big), \tag{2.47}$$

where ω_{flat} is the local flat Kähler metric. Since the RHS of (2.47) has regularity C^3, and Δ_ω is elliptic, then ω has C^5 regularity. It follows, that (g, ω) is smooth. $\square$

CHAPTER III

QUANTIZATION OF KÄHLER METRICS WITH CONSTANT WEIGHTED SCALAR CURVATURE AND BOUNDEDNESS OF THE WEIGHTED MABUCHI ENERGY

In this chapter we give the proof of Theorem 3 from the introduction. Our method relies on the approach introduced by Donaldson [41, 44] and developed by Li [72] and Sano–Tipler [85], via finite dimensional approximations and generalized balanced metrics.

3.1 The (v,w)-equivariant Bergman kernels and (v,w)-balanced metrics

Let (X, L) be a smooth compact polarized projective manifold, where L is an ample holomorphic line bundle on X and $\mathbb{T} \subset \mathrm{Aut}(X, L)$ is an ℓ-dimensional real torus acting on the total space of L, which covers an ℓ-dimensional torus action (still denoted by $\mathbb{T}$) in $\mathrm{Aut}_{\mathrm{red}}(X) \cong \mathrm{Aut}(X,L)/\mathbb{C}^\star$. Let $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_\ell) \in \mathfrak{t}$ be a basis of $\mathbb{S}^1$-generators of $\mathbb{T}$ and $\boldsymbol{A}^{(k)}_{\boldsymbol{\xi}} := (A^{(k)}_{\xi_1}, \dots, A^{(k)}_{\xi_\ell})$ the induced infinitesimal actions of ξ_i on the finite dimensional space $\mathcal{H}_k := H^0(X, L^k)$ of global holomorphic sections of L^k for $k \gg 1$. For a $\mathbb{T}$-invariant Hermitian metric h on L with curvature two form $\omega \in 2\pi c_1(L)$ we have (see e.g. [53, Proposition 8.8.2])

$$A^{(k)}_{\xi_i} + \sqrt{-1}\nabla_{\xi_i} = k m^{\xi_i}_\omega \mathrm{Id}_{\mathcal{H}_k}, \tag{3.1}$$

where ∇ is the Chern connection of $h^k := h^{\otimes k}$ and $m^{\xi_i}_\omega$ is a ω-Hamiltonian function of ξ_i. Using the basis $\boldsymbol{\xi}$ we identify $\mathfrak{t} \cong \mathbb{R}^\ell$ and get a natural momentum map $m_\omega := (m^{\xi_1}_\omega, \cdots, m^{\xi_\ell}_\omega) : X \to \mathbb{R}^\ell$ for the action of $\mathbb{T}$ on X with momentum image $\mathrm{P} := m_\omega(X)$.

Notice that if $h_\phi := e^{-2\phi}h$ is another $\mathbb{T}$-invariant Hermitian metric on L with positive curvature $\omega_\phi > 0$, the corresponding momentum map satisfies $m_\phi^{\xi_i} = m_\omega^{\xi_i} + (d^c\phi)(\xi_i)$, thus showing, by virtue of Lemma 1 3, that the image $m_\phi(X) = \mathrm{P}$ is independent of the metric h_ϕ. We thus have a polytope $\mathrm{P}_L \subset \mathfrak{t}^*$ associated to the polarized manifold (X, L) and the lifted action $\mathbb{T} \subset \mathrm{Aut}(X, L)$.

The spectrum of $k^{-1}A_{\xi_j}^{(k)}$ is given by $\{\lambda_i^{(k)}(\xi_j),\ \lambda_i^{(k)} \in W_k\}$ where $W_k := \{\lambda_i^{(k)},\ i = 1, \cdots, N_k\} \subset \Lambda^*$ is the finite set of weights of the complexified action of $\mathbb{T}$ on $\mathcal{H}_k$ and Λ^* is the dual of the lattice $\Lambda \subset \mathfrak{t}$ of circle subgroups of $\mathbb{T}$ (see e.g. [9,16]).

Lemma 15. *The set of weights W_k is contained in the momentum polytope P_L of the action of $\mathbb{T}$ on (X, L).*

Proof. This lemma is well known (see e.g. [9, Section 5], but we give the proof for the sake of clarity. Let $\lambda_i^{(k)} \in W_k$, $\xi_j \in \boldsymbol{\xi}$ an $\mathbb{S}^1$-generator for the $\mathbb{T}$-action on X, and $s_{j,i}^{(k)} \in \mathcal{H}_k$ an eigensection associated to the eigenvalue $\lambda_i^{(k)}(\xi_j)$ of $k^{-1}A_{\xi_j}^{(k)}$. Using (3.1), we have

$$\begin{aligned}(\lambda_i^{(k)}(\xi_j) - m_\omega^{\xi_j})|s_{j,i}^{(k)}|_{h^k}^2 &= (k^{-1}A_{\xi_j}^{(k)} s_{j,i}, s_{j,i})_{h^k} - m_\omega^{\xi_j}|s_{j,i}^{(k)}|_{h^k}^2 \\ &= -\frac{\sqrt{-1}}{2}(d|s_{j,i}^{(k)}|_{h^k}^2)(\xi_j) - \frac{1}{2}(d|s_{j,i}^{(k)}|_{h^k}^2)(J\xi_j).\end{aligned}$$

At a point of global maximum x_0 of the smooth function $|s_{j,i}^{(k)}|_{h^k}^2$ on X, we obtain

$$\lambda_i^{(k)}(\xi_j) = m_\omega^{\xi_j}(x_0) \in \mathrm{P}_L.$$

It follows that $W_k \subset \mathrm{P}_L$. □

Using the weight decomposition of $\mathcal{H}_k$

$$\mathcal{H}_k = \bigoplus_{\lambda_i^{(k)} \in W_k} \mathcal{H}(\lambda_i^{(k)}), \tag{3.2}$$

and Lemma 15, for any smooth function $\mathrm{v} \in C^\infty(\mathrm{P}_L, \mathbb{R})$ we can define the operator $\mathrm{v}(k^{-1}\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)}) : \mathcal{H}_k \to \mathcal{H}_k$ by

$$\mathrm{v}(k^{-1}\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)})_{|\mathcal{H}(\lambda_i^{(k)})} := \mathrm{v}(k^{-1}\lambda_i^{(k)})\mathrm{Id}_{\mathcal{H}(\lambda_i^{(k)})}. \tag{3.3}$$

Let h be a $\mathbb{T}$-invariant Hermitian metric on L with curvature 2-form $\omega \in 2\pi c_1(L)$. We identify the space of $\mathbb{T}$-invariant Hermitian metrics $h_\phi := e^{-2\phi}h$ with positive curvature forms ω_ϕ with the space $\mathcal{K}_\omega^{\mathbb{T}}$ of $\mathbb{T}$-invariant Kähler potentials ϕ on X.

For $\mathrm{v} \in C^\infty(\mathrm{P}_L, \mathbb{R}_{>0})$ we consider the following weighted L^2-inner product on $C^\infty(X, L^k)$

$$\langle s, s' \rangle_{\mathrm{v},k\phi} := k^n \int_X (s, s')_{k\phi} \mathrm{v}(m_\phi)\omega_\phi^{[n]}.$$

where $(s, s')_{k\phi} := h_\phi^k(s, s')$. The operators $(A_{\xi_j}^{(k)})_{j=1,\cdots,\ell}$ are Hermitian with respect to $\langle \cdot, \cdot \rangle_{\mathrm{v},k\phi}$. Following [16, 89, 100], we have the following definition.

Definition 10. Let $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$, $\{s_i \mid i = 0, \cdots, N_k\}$ be a $\langle \cdot, \cdot \rangle_{\mathrm{v},k\phi}$-orthonormal basis of $\mathcal{H}$ and $\mathrm{w} \in C^\infty(\mathrm{P}_L, \mathbb{R})$. Then the (v, w)-equivariant Bergman kernel of the Hermitian metric h_ϕ^k on L^k, is the function defined on X by,

$$B_{\mathrm{w}}(\mathrm{v}, k\phi) := \mathrm{v}(m_\phi) \sum_{i=0}^{N_k} \Big(\mathrm{w}\big(k^{-1}\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)}\big)(s_i), s_i \Big)_{k\phi}, \tag{3.4}$$

where $\mathrm{w}\big(k^{-1}\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)}\big)$ is given by (3.3).

Equivalently, $B_{\mathrm{w}}(\mathrm{v}, k\phi)$ is the restriction to the diagonal $\{x = x'\} \subset X \times X$ of the Schwartz kernel of the operator $\mathrm{w}\big(k^{-1}\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)}\big)\Pi_{\mathrm{v}}^{k\phi}$, where $\Pi_{\mathrm{v}}^{k\phi} : L^2(X, L^k) \to \mathcal{H}_k$ denote the orthogonal projection with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathrm{v},k\phi}$.

Asymptotic expansions of (3.4) in $k \gg 1$ are known to exist in many special cases, see e.g. [16, 74, 89, 100]. For the general case, we will use results on the functional calculus of Toeplitz operators which follows essentially from [25], with a ramification from [38, 74, 75].

We start by recalling the definition and properties of Toeplitz operator (see [74, Chapter 7] and [25]).

Definition 11. Let $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$ and $\mathrm{v} \in C^\infty(\mathrm{P}_L, \mathbb{R}_{>0})$. A v-*Toeplitz operator* is a family $T^{(k)} := T(\mathrm{v}, k\phi)$ of operators $T^{(k)} : L^2(X, L^k) \to L^2(X, L^k)$ such that

$$T^{(k)} = \Pi_{\mathrm{v}}^{k\phi} f^{(k)} \Pi_{\mathrm{v}}^{k\phi} + R^{(k)},$$

where

- $f^{(k)} \in C^\infty(X,\mathbb{R})$ is a sequence of smooth functions which admits an asymptotic expansion $\sum_{j\geq 0} k^{-j} f_j$ in the C^∞-topology with $f_j \in C^\infty(X,\mathbb{R})$ i.e. for any $d, \ell \geq 0$ there exist a constant $C_{d,\ell} > 0$ such that for any $k > 0$

$$\parallel f^{(k)} - \sum_{j=0}^{d} k^{-j} f_j \parallel_{C^\ell} \leq C_{d,\ell}/k^{d+1}.$$

- $R^{(k)} := \mathcal{O}(k^{-\infty})$ is a *negligible* v-*Toeplitz operator*, that is there exist a sequence of $r^{(k)} \in C^\infty(X,\mathbb{R})$ such the $R^{(k)} = \Pi_{\mathrm{v}}^{k\phi} r^{(k)} \Pi_{\mathrm{v}}^{k\phi}$ and for any $j, \ell > 0$ there exist a constant $C_{j,\ell} > 0$ such that

$$\parallel r^{(k)} \parallel_{C^\ell} \leq C_{j,\ell}/k^j.$$

We denote the space of v-Toeplitz operators by $\mathcal{T}_{\mathrm{v}}$ and the space of negligible v-Toeplitz operator by $\mathcal{T}_{\mathrm{v}} \cap \mathcal{O}(k^{-\infty})$.

A v-Toeplitz operator $\{T^{(k)}\} \in \mathcal{T}_{\mathrm{v}}$ on $L^2(X, L^k)$ we defined above is a Toeplitz operator on $L^2(X, L^k \otimes E_0)$, as defined in [74, Definition 7.2.1], where the twisting bundle E_0 (in the notation of [74]) is the trivial line bundle $X \times \mathbb{C}$ on X, endowed with Hermitian metric $|\cdot|_{E_0} := \mathrm{v}(m_\phi)|\cdot|$, where $|\cdot|$ is the hermitian product of $\mathbb{C}$. Using [74, Section 4.1.1], the restriction to the diagonal $\{x = x'\} \subset X \times X$ of the Schawrtz kernel of the projection operator $\Pi_{\mathrm{v}}^{k\phi} : L^2(X, L^k) \to \mathcal{H}_k$, seen as projection operator from $L^2(X, L^k \otimes E_0)$ to $\mathcal{H}_k$, admits an asymptotic expansion in C^∞-topology given by

$$\Pi_{\mathrm{v}}^{k\phi}(x,x) = 1 + \frac{S_{\mathrm{v}}(\phi)}{k} + \mathcal{O}(k^{-2}), \tag{3.5}$$

where $S_{\mathrm{v}}(\phi)$ is defined by

$$S_{\mathrm{v}}(\phi) := \frac{1}{4}\big(\mathrm{Scal}_\phi + 2\Delta_\phi(\log(\mathrm{v}(m_\phi)))\big), \tag{3.6}$$

for any $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$. It follows from (3.5) that the restriction to the diagonal $\{x = x'\} \subset X \times X$ of the Schawrtz kernel of a v-Toeplitz operator $T^{(k)}$ admits an asymtotic expansion in the C^∞-topology

$$(2\pi)^n T^{(k)}(x,x) = \sum_i k^{-i} a_i(x) + \mathcal{O}(k^{-\infty}), \tag{3.7}$$

where $a_i \in C^\infty(X,\mathbb{R})$ are smooth functions.

Definition 12. The full symbol map $\sigma : \mathcal{T}_{\mathrm{v}} \to C^\infty(X,\mathbb{R})[[\hbar]]$ with values in the algebra of formal series with coefficients in $C^\infty(X,\mathbb{R})$, is defined by

$$\sigma(T^{(k)}) := \sum_{i\geq 0} a_i(x)\hbar^i, \tag{3.8}$$

for any $T^{(k)} \in \mathcal{T}_{\mathrm{v}}$, such that $T^{(k)}(x,x)$ is given by (3.7).

The following proposition is a simple application of [75, Theorem 0.2] to v-Toeplitz operators.

Proposition 4. *For any $f,g \in C^\infty(X,\mathbb{R})$, we have $\Pi_{\mathrm{v}}^{k\phi} f \Pi_{\mathrm{v}}^{k\phi} g \Pi_{\mathrm{v}}^{k\phi} \in \mathcal{T}_{\mathrm{v}}$, and the restriction to the diagonal $\{x = x'\} \subset X \times X$ of its Schawrtz kernel admits a C^∞-asymptotic expansion given by*

$$(\Pi_{\mathrm{v}}^{k\phi} f \Pi_{\mathrm{v}}^{k\phi} g \Pi_{\mathrm{v}}^{k\phi})(x,x) = fg + \big[\frac{1}{2}(df,dg)_\phi + S_{\mathrm{v}}(\phi) fg\big] k^{-1} + \mathcal{O}(k^{-2}),$$

where $S_{\mathrm{v}}(\phi)$ is given by (3.6).

For every $f,g \in C^\infty(X,\mathbb{R})$, we define the v-star product $f \star_{\mathrm{v}} g$ of f and g by

$$\begin{aligned} f \star_{\mathrm{v}} g :=& \sigma\big(\Pi_{\mathrm{v}}^{k\phi} f \Pi_{\mathrm{v}}^{k\phi} g \Pi_{\mathrm{v}}^{k\phi}\big) \\ =& fg + \hbar\left[\frac{1}{2}(df,dg)_\phi + S_{\mathrm{v}}(\phi) fg\right] + \mathcal{O}(\hbar^2). \end{aligned} \tag{3.9}$$

We define the v-star product to $C^\infty(X,\mathbb{R})[[\hbar]]$, using the Cauchy product

$$\big(\sum_{j\geq 0} f_j \hbar^j\big) \star_{\mathrm{v}} \big(\sum_{j\geq 0} g_j \hbar^j\big) := \sum_{s\geq 0} \big(\sum_{j=0}^{s} f_j \star_{\mathrm{v}} g_{s-j}\big)\hbar^s.$$

The unit $1_{\star_{\mathrm{v}}}$ of $(C^\infty(X,\mathbb{R})[[\hbar]], \star_{\mathrm{v}})$ is given by the symbol $\sigma(\Pi_{\mathrm{v}}^{k\phi})$.

Theorem 9. *The full symbol map $\sigma : (\mathcal{T}_{\mathrm{v}}, +, \circ) \to (C^\infty(X,\mathbb{R})[[\hbar]], +, \star_{\mathrm{v}})$ defines an isomorphism from the algebra $\mathcal{T}_{\mathrm{v}}$ of v-Toeplitz operators modulo the ideal of negligible operators $\mathcal{T}_{\mathrm{v}} \cap \mathcal{O}(k^{-\infty})$, into the algebra $C^\infty(X,\mathbb{R})[[\hbar]]$ endowed with the associative star product $\star_{\mathrm{v}}$.*

Proof. The fact that $\sigma : \mathcal{T}_{\mathrm{v}} \to C^\infty(X,\mathbb{R})[[\hbar]]$ is surjective with kernel $\mathcal{T}_{\mathrm{v}} \cap \mathcal{O}(k^{-\infty})$ follows from [25, Proposition 3]. $\square$

Proposition 5. *Let $(T_j^{(k)})_{j=1,\cdots,\ell}$ be a family of $\langle\cdot,\cdot\rangle_{\mathrm{v},k\phi}$-self adjoint commuting Toeplitz operators, such that the set of joint eigenvalues of $(T_j^{(k)})_{j=1,\ell}$ is contained in* P. *Suppose that the symbol of $T_j^{(k)}$, $j=1,\cdots,\ell$ is given by*

$$\sigma(T_j^{(k)}) := \sum_{i\geq 0} \hbar^i f_i^{(j)} \in C^\infty(X)[[\hbar]].$$

Then for any smooth function w *with compact support containing* P, *the operator* $\mathrm{w}(T_1^{(k)},\cdots,T_\ell^{(k)})$ *is a Toeplitz operator with symbol*

$$\sigma(\mathrm{w}(T_1^{(k)},\cdots,T_\ell^{(k)})) = s_0(\mathrm{v},\mathrm{w}) + s_1(\mathrm{v},\mathrm{w})\hbar + \mathcal{O}(\hbar^2),$$

where $s_0(\mathrm{v},\mathrm{w}), s_1(\mathrm{v},\mathrm{w})$ are given by

$$\begin{aligned} s_0(\mathrm{v},\mathrm{w}) =& \mathrm{w}(f_0^{(1)},\cdots,f_0^{(\ell)}),\\ s_1(\mathrm{v},\mathrm{w}) =& \mathrm{w}(f_0^{(1)},\cdots,f_0^{(\ell)})S_{\mathrm{v}}(\phi) + \sum_{j=1}^{\ell} \mathrm{w}_{,j}(f_0^{(1)},\cdots,f_0^{(\ell)})(f_1^{(j)} - f_0^{(j)}S_{\mathrm{v}}(\phi))\\ &+\frac{1}{4}\sum_{i,j=1}^{\ell} \mathrm{w}_{,ij}(f_0^{(1)},\cdots,f_0^{(\ell)})(df_0^{(i)},df_0^{(j)})_\phi, \end{aligned}$$

with $S_{\mathrm{v}}(\phi)$ given by (3.6).

Proof. In the case of one $\langle\cdot,\cdot\rangle_{\mathrm{v},k\phi}$-self adjoint Toeplitz operator $T^{(k)}$ and a smooth function of one variable w, the fact that $\mathrm{w}(T^{(k)})$ is again a Toeplitz operator is established in [25, Proposition 12]. The proof given in [25] relies on the Helffer-Sjostrand formula, see e.g. [38, Theorem 8.1]. Using its multivariable generalization [38, Equation 8.18] the proof in [25] readily generalizes to show that $\mathrm{w}(T_1^{(k)},\cdots,T_\ell^{(k)})$ is a Toeplitz operator for any smooth function on P, and family of $\langle\cdot,\cdot\rangle_{\mathrm{v},k\phi}$-self adjoint commuting Toeplitz operators $(T_j^{(k)})_{j=1,\cdots,\ell}$ such that the set of joint eigenvalues of $(T_j^{(k)})_{j=1,\ell}$ is contained in P.

We shall now compute the symbol of the Toeplitz operator $\mathrm{w}(T_1^{(k)},\cdots,T_\ell^{(k)})$. Following [25], the symbol of $\mathrm{w}(T_1^{(k)},\cdots,T_\ell^{(k)})$ is given by the Taylor series expansion of w at the point $\boldsymbol{a} := (f_0^{(1)}(x), \cdots, f_0^{(\ell)}(x))$, $\boldsymbol{a}_j := f_0^{(j)}(x)$ as follows:

$$\begin{aligned}\sigma(\mathrm{w}(T_1^{(k)}, \cdots, T_\ell^{(k)})) &= \mathrm{w}(\boldsymbol{a})1_{\star_{\mathrm{v}}}(x) + \sum_{j=1}^{\ell} \mathrm{w}_{,j}(\boldsymbol{a}) \left(\sum_{i\geq 0} \hbar^i f_i^{(j)}(y) - \boldsymbol{a}_j 1_{\star_{\mathrm{v}}}(y) \right)_{|y=x} \\ &+ \frac{1}{2!} \sum_{p,q=1}^{\ell} \mathrm{w}_{,pq}(\boldsymbol{a}) \left(\sum_{i\geq 0} \hbar^i f_i^{(p)}(y) - \boldsymbol{a}_p 1_{\star_{\mathrm{v}}}(y) \right) \star_{\mathrm{v}} \left(\sum_{i\geq 0} \hbar^i f_i^{(q)}(y) - \boldsymbol{a}_q 1_{\star_{\mathrm{v}}}(y) \right)_{|y=x} + \cdots\end{aligned} \tag{3.10}$$

On the other hand, we compute

$$\begin{aligned}&\Big(\sum_{i\geq 0} \hbar^i f_i^{(p)}(y) - \boldsymbol{a}_p 1_{\star_{\mathrm{v}}}(y) \Big) \star_{\mathrm{v}} \Big(\sum_{i\geq 0} \hbar^i f_i^{(q)}(y) - \boldsymbol{a}_q 1_{\star_{\mathrm{v}}}(y) \Big)_{|y=x} \\ &= \Big((f_0^{(p)}(y) - \boldsymbol{a}_p) + \hbar(f_1^{(p)}(y) - S_{\mathrm{v}}(y)) \Big) \star_{\mathrm{v}} \Big((f_0^{(q)}(y) - \boldsymbol{a}_q) + \hbar(f_1^{(q)}(y) - S_{\mathrm{v}}(y)) \Big)_{|y=x} + \mathcal{O}(\hbar^2) \\ &= (f_0^{(p)}(y) - \boldsymbol{a}_p) \star_{\mathrm{v}} (f_0^{(q)}(y) - \boldsymbol{a}_q)_{|y=x} + \hbar(f_0^{(p)}(x) - \boldsymbol{a}_p)(f_1^{(q)}(x) - S_{\mathrm{v}}(x)) \\ &\quad + \hbar(f_0^{(q)}(x) - \boldsymbol{a}_q)(f_1^{(p)}(x) - S_{\mathrm{v}}(x)) + \mathcal{O}(\hbar^2) \\ &= \frac{\hbar}{2}(df_0^{(p)}, df_0^{(q)})_\phi + \mathcal{O}(\hbar^2).\end{aligned}$$

Substituting back in (3.10), we obtain the symbol $\sigma(\mathrm{w}(T_1^{(k)}, \cdots, T_\ell^{(k)}))$ up to $\mathcal{O}(\hbar^2)$. □

Lemma 16. *For any $\xi \in \mathfrak{t}$, we have*

$$T_\xi^{(k)} := k^{-1} A_\xi^{(k)} \Pi_{\mathrm{v}}^{k\phi} \in \mathcal{T}_{\mathrm{v}}.$$

Proof. This follows from the fact that

$$T_\xi^{(k)} = \Pi_{\mathrm{v}}^{k\phi} \left(m_\phi^\xi + \frac{\delta^\phi(\mathrm{v}(m_\phi) dm_\phi^\xi)}{2k\mathrm{v}(m_\phi)} \right) \Pi_{\mathrm{v}}^{k\phi},$$

where δ^ϕ is the g_ϕ-codifferential. To get the above equality, using (3.1) it is enough to check that for any $s \in \mathcal{H}_k$ we have

$$\Big\langle \Pi_{\mathrm{v}}^{k\phi} (\frac{1}{k\sqrt{-1}} \nabla_\xi^\phi) s, s \Big\rangle_{\mathrm{v},k\phi} = \Big\langle \frac{\delta^\phi(\mathrm{v}(m_\phi) dm_\phi^\xi)}{2k\mathrm{v}(m_\phi)} s, s \Big\rangle_{\mathrm{v},k\phi}.$$

The above equation follows from a straightforward integration by parts. □

Theorem 10. *Let* $\mathrm{w} \in C^{\infty}(\mathrm{P}, \mathbb{R})$. *The* (v, w)*-equivariant Bergman kernel of the* $\mathbb{T}$*-invariant Hermitian metric* h_{ϕ}^{k} *on* L^{k} *admits an asymptotic expansion when* $k \gg 1$, *given by*

$$(2\pi)^{n} B_{\mathrm{w}}(\mathrm{v}, k\phi) = \begin{cases} \mathrm{w}(m_{\phi}) + \mathcal{O}(\frac{1}{k}), & \\ \mathrm{v}(m_{\phi}) + \frac{1}{4k}\mathrm{Scal}_{\mathrm{v}}(\phi) + \mathcal{O}(\frac{1}{k^{2}}), & \textit{if } \mathrm{w} = \mathrm{v}. \end{cases}$$

Moreover, the above expansions holds in C^{∞}, *i.e. for any integer* $\ell \geq 0$ *there exist a constant* $C_{\ell}(\mathrm{v}, \mathrm{w}) > 0$ *such that,*

$$\|(2\pi)^{n} B_{\mathrm{w}}(\mathrm{v}, k\phi) - \mathrm{w}(m_{\phi})\|_{C^{\ell}} \leq \frac{C_{\ell}(\mathrm{v}, \mathrm{w})}{k},$$
$$\left\|(2\pi)^{n} B_{\mathrm{v}}(\mathrm{v}, k\phi) - \mathrm{v}(m_{\phi}) - \frac{1}{4k}\mathrm{Scal}_{\mathrm{v}}(\phi)\right\|_{C^{\ell}} \leq \frac{C_{\ell}(\mathrm{v}, \mathrm{v})}{k^{2}}.$$

Proof. Since the symbol map σ is surjective with kernel given by the ideal of negligible Toeplitz operators $\mathcal{O}(k^{-\infty}) \cap \mathcal{T}_{\mathrm{v}}$ it suffices to calculate $\sigma\big(\mathrm{w}\big(k^{-1}\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)}\big)\Pi_{\mathrm{v}}^{k\phi}\big)$. We consider the special case of $\langle \cdot, \cdot \rangle_{\mathrm{v}, k\phi}$-self-adjoint v-Toeplitz operators $T_{j}^{(k)} := k^{-1}A_{\xi_{j}}^{(k)}\Pi_{\mathrm{v}}^{k\phi}$. We have

$$T_{j}^{(k)}(x, x) = \mathrm{v}(m_{\phi}) \sum_{i=0}^{N_{k}} \left(k^{-1}A_{j}^{(k)}s_{i}, s_{i}\right)_{k\phi}.$$

By a straightforward calculation using (3.1) the symbol of $T_{j}^{(k)}$ is given by

$$\sigma(T_{j}^{(k)}) = m_{\phi}^{\xi_{j}} + \Big[m_{\phi}^{\xi_{j}} S_{\mathrm{v}}(\phi) - \frac{1}{2}\sum_{i=1}^{\ell}(\log \circ \mathrm{v})_{,i}(m_{\phi})(\xi_{i}, \xi_{j})_{\phi}\Big]\hbar + \cdots$$

Using Proposition 5 we get

$$\sigma(\mathrm{w}(\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)})\Pi_{\mathrm{v}}^{k\phi}) = s_{0}(\mathrm{v}, \mathrm{w}) + s_{1}(\mathrm{v}, \mathrm{w})\hbar + \cdots$$

where

$$s_{0}(\mathrm{v}, \mathrm{w}) = \mathrm{w}(m_{\phi}),$$
$$s_{1}(\mathrm{v}, \mathrm{v}) = \mathrm{v}(m_{\phi})S_{\mathrm{v}}(\phi) - \frac{1}{2}\sum_{i,j=1}^{\ell} \frac{\mathrm{v}_{,i}(m_{\phi})\mathrm{v}_{,j}(m_{\phi})}{\mathrm{v}(m_{\phi})}(\xi_{i}, \xi_{j})_{\phi} + \frac{1}{4}\sum_{i,j=1}^{\ell} \mathrm{v}_{,ij}(m_{\phi})(\xi_{i}, \xi_{j})_{\phi}.$$

Replacing $S_{\mathrm{v}}(\phi)$ by its expression (3.6), we obtain $s_{1}(\mathrm{v}, \mathrm{v}) = \mathrm{Scal}_{\mathrm{v}}(\phi)$. ☐

Definition 13. We define the v-*weight* of the action of $\mathbb{T}$ on $\mathcal{H}_k$ by

$$W_{\rm v}(L^k) := \mathrm{tr}(\mathrm{v}(k^{-1}\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)})). \tag{3.11}$$

Lemma 17. *The* v-*weight of the action of* $\mathbb{T}$ *on* $\mathcal{H}_k$ *admits the following asymptotic expansion*

$$(2\pi)^n W_{\rm v}(L^k) = k^n \int_X \mathrm{v}(m_\omega)\omega^{[n]} + \frac{k^{n-1}}{4}\int_X \mathrm{Scal}_{\rm v}(\omega)\omega^{[n]} + \mathcal{O}(k^{n-2}). \tag{3.12}$$

for any smooth function v *with compact support containing* P.

Proof. This is a direct consequence of Theorem 10, by letting w = v in (3.4), and integrating in both sides over X. □

3.2 The quantization maps

Let W_k denote the set of weights for the complexified action of $\mathbb{T}$ on $\mathcal{H}_k$. We consider the following direct sum decomposition of the space $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ of $\mathbb{T}$-invariant positive definite Hermitian forms on $\mathcal{H}_k$,

$$\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k) := \bigoplus_{\lambda_i^{(k)}\in W_k} \mathcal{B}^{\mathbb{T}}(\mathcal{H}(\lambda_i^{(k)})),$$

where $\mathcal{B}^{\mathbb{T}}(\mathcal{H}(\lambda_i^{(k)}))$ is the space of $\mathbb{T}$-invariant positive definite Hermitian forms on $\mathcal{H}(\lambda_i^{(k)})$

Definition 14. Let $\mathrm{v} \in C^\infty(\mathrm{P}_L, \mathbb{R}_{>0})$, $\mathrm{w} \in C^\infty(\mathrm{P}_L, \mathbb{R})$. We introduce the following quantization maps:

1. The (v, w)-Hilbert map $\mathrm{Hilb}^k_{\mathrm{v,w}} : \mathcal{K}^{\mathbb{T}}_\omega \to \mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ which associates to every $\mathbb{T}$-invariant Kähler potential, the $\mathbb{T}$-invariant Hermitian inner product on $\mathcal{H}_k$, given by

$$(\mathrm{Hilb}^k_{\mathrm{v,w}}(\phi))(\cdot,\cdot) := \sum_{\lambda_i^{(k)}\in W_k} \frac{(\langle\cdot,\cdot\rangle_{\mathrm{v},k\phi})_{|\mathcal{H}_k(\lambda_i^{(k)})}}{\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v,w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})},$$

where $c_{(\mathrm{v,w})}(\alpha)$ is given by (2.6) (Notice that for k big enough the expression $\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v,w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)}) > 0$ since $\mathrm{v} > 0$ and w are bounded functions on P_L).

2. The (v,w)-Fubini–Study map $\mathrm{FS}^k_{\mathrm{v},\mathrm{w}} : \mathcal{B}^{\mathbb{T}}(\mathcal{H}_k) \to \mathcal{K}^{\mathbb{T}}_\omega$ given by

$$\mathrm{FS}^k_{\mathrm{v},\mathrm{w}}(H) := \frac{1}{2k}\log\left(\sum_{i=0}^{N_k}|s_i|^2_{h^k}\right) - \frac{\log(c_k(\mathrm{v},\mathrm{w}))}{2k},$$

where $\{s_i\}$ is an adapted H-orthonormal basis of $\mathcal{H}_k$ and $c_k(\mathrm{v},\mathrm{w})$ is a constant given by:

$$c_k(\mathrm{v},\mathrm{w}) := \frac{1}{k^n\int_X \mathrm{v}(m_\omega)\omega^{[n]}}\left[W_{\mathrm{v}}(L^k) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}W_{\mathrm{w}}(L^k)\right], \tag{3.13}$$

with $W_{\mathrm{v}}(L^k)$ the v-weight of the action of $\mathbb{T}$ on L^k given by (3.11).

Theorem 10 yields

Lemma 18. *For $\phi \in \mathcal{K}^{\mathbb{T}}_\omega$, the Bergman kernel $\rho_{\mathrm{v},\mathrm{w}}(k\phi)$ of $\mathrm{Hilb}_{\mathrm{v},\mathrm{w}}(k\phi)$ satisfies*

$$\rho_{\mathrm{v},\mathrm{w}}(k\phi) = B_{\mathrm{v}}(\mathrm{v},k\phi) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}B_{\mathrm{w}}(\mathrm{v},k\phi),$$

and has an asymptotic expansion,

$$(2\pi)^n\rho_{\mathrm{v},\mathrm{w}}(k\phi) = \mathrm{v}(m_\phi) + \frac{1}{4k}\left(\mathrm{Scal}_{\mathrm{v}}(\phi) - c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_\phi)\right) + \mathcal{O}\left(\frac{1}{k^2}\right).$$

The above asymptotic expansion holds in C^∞, i.e. for any integer $\ell \geq 0$ we have,

$$\left\|(2\pi)^n\rho_{\mathrm{v},\mathrm{w}}(k\phi) - \mathrm{v}(m_\phi) - \frac{1}{4k}\left(\mathrm{Scal}_{\mathrm{v}}(\phi) - c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_\phi)\right)\right\|_{C^\ell} \leq \frac{C_\ell(\mathrm{v},\mathrm{w})}{k^2}.$$

where $C_\ell(\mathrm{v},\mathrm{w}) > 0$.

Following [41,85,101], we give the following definition

Definition 15. We say that a metric $\phi \in \mathcal{K}^{\mathbb{T}}_\omega$ is (v,w)-balanced of order k if it satisfies:

$$\mathrm{FS}^k_{\mathrm{v},\mathrm{w}} \circ \mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\phi) = \phi.$$

or equivalently

$$\rho_{\mathrm{v},\mathrm{w}}(k\phi) = c_k(\mathrm{v},\mathrm{w})\mathrm{v}(m_\phi),$$

where $c_k(\mathrm{v},\mathrm{w})$ is given by (3.13).

Similarly to [41,42] we have

Proposition 6. *Let $(\phi_j)_{j\geq 0}$ be a sequence in $\mathcal{K}_\omega^{\mathbb{T}}$ such that every ϕ_j is a (v,w)-balanced metric of order j and ϕ_j converge in C^∞ to ϕ. Then ω_ϕ is a (v,w)-cscK metric.*

Proof. By Lemma 18 for $k \gg 1$,

$$\left\|(2\pi)^n\rho_{(\mathrm{v},\mathrm{w})}(k\phi_j) - \mathrm{v}(m_{\phi_j}) - \frac{1}{4k}(\mathrm{Scal}_{\mathrm{v}}(\phi_j) - c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_{\phi_j}))\right\|_{C^\ell} \leq \frac{C_\ell(\mathrm{v},\mathrm{w})}{k^2}.$$

Letting $j = k$, we get

$$\left\|(2\pi)^n c_k(\mathrm{v},\mathrm{w})\mathrm{v}(m_{\phi_k}) - \mathrm{v}(m_{\phi_k}) - \frac{1}{4k}(\mathrm{Scal}_{\mathrm{v}}(\phi_k) - c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_{\phi_k}))\right\|_{C^\ell} \leq \frac{C_\ell(\mathrm{v},\mathrm{w})}{k^2}. \tag{3.14}$$

From (3.13) and Lemma 18 we get

$$\begin{aligned}(2\pi)^m c_k(\mathrm{v},\mathrm{w}) =& \frac{(2\pi)^m \int_X \rho_{(\mathrm{v},\mathrm{w})}(h^k)(k\omega)^{[n]}}{\int_X \mathrm{v}(m_\omega)(k\omega)^{[n]}} \\ =& 1 + \mathcal{O}(k^{-2}).\end{aligned}$$

Taking a limit when k goes to infinity in (3.14), we obtain that $\mathrm{Scal}_{\mathrm{v}}(\phi) = c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_\omega)$. □

3.3 Boundedness of the (v,w)-Mabuchi energy as an obstruction to the existence of (v,w)-cscK metrics

In this section we prove Theorem 3, following the method of [42, 72, 85]. To this end, for each $k \gg 1$, we introduce appropriate functionals on the finite dimensional space of Fubini–Study metrics on $\mathbb{P}(\mathcal{H}_k^*)$, which when identified with a subspace of $\mathcal{K}_\omega^{\mathbb{T}}$ via the Kodaira embedding, will quantize the (v,w)-Mabuchi functional of $\alpha = 2\pi c_1(L)$. Furthermore, following the main ideas of [42, 72, 85], we will show that the (v,w)-balanced metrics are minima of these functionals, and that a Kähler metric with constant (v,w)-scalar curvature induces *almost* (v,w)-balanced Fubini–Study metrics on $\mathbb{P}(\mathcal{H}_k^*)$ for $k \gg 1$, i.e. minimizes the corresponding functionals up to an error that goes to zero.

3.3.1 Quantization of the (v,w)-Mabuchi energy

We start with introducing finite dimensional analogues of the (v,w)-Mabuchi energy (2.19), given by (3.21) on the spaces $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ and $\mathrm{FS}^k_{(\mathrm{v},\mathrm{w})}\left(\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)\right)$ (see Definition 14), respectively, thus setting the framework for the proof of Theorem 3 along the lines of [42, 72, 85].

We introduce the functional $\mathcal{E}^k_{\mathrm{v},\mathrm{w}} : \mathcal{B}^{\mathbb{T}}(\mathcal{H}_k) \to \mathbb{R}$ by

$$\mathcal{E}^k_{\mathrm{v},\mathrm{w}}(H) = \sum_{\lambda_i^{(k)} \in W_k} \left(\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right) \log\left(\det H_{\lambda_i^{(k)}}\right). \tag{3.15}$$

where $H := (H_{\lambda_i^{(k)}})_{\lambda_i^{(k)} \in W_k} \in \mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$.

Lemma 19. *1. We have the following expression for the variation of $2k^{n+1}c_k(\mathrm{v},\mathrm{w})\mathcal{E}_{\mathrm{v}}$ (see Lemma 3) and $\mathcal{E}^k_{\mathrm{v},\mathrm{w}}$:*

$$\begin{aligned} 2k^{n+1}c_k(\mathrm{v},\mathrm{w})\left(\mathbf{d}\mathcal{E}_{\mathrm{v}}\right)_\phi(\dot\phi) =& 2kc_k(\mathrm{v},\mathrm{w})\int_X \dot\phi\left(1+\frac{\Delta_\phi}{2k}\right)\mathrm{v}(m_\phi)(k\omega_\phi)^{[n]} \\ &- c_k(\mathrm{v},\mathrm{w})\int_X (d\dot\phi, d(\log\circ \mathrm{v}(m_\phi)))_\phi \mathrm{v}(m_\phi)(k\omega_\phi)^{[n]}, \end{aligned} \tag{3.16}$$

$$\left(\mathbf{d}\,\mathcal{E}^k_{\mathrm{v},\mathrm{w}}\right)_H(\dot H) = \sum_{\lambda_i^{(k)} \in W_k}\left(\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right)\mathrm{tr}(H^{-1}_{\lambda_i^{(k)}}\dot H_{\lambda_i^{(k)}}), \tag{3.17}$$

where $\phi \in \mathcal{K}^{\mathbb{T}}_\omega$ and $H = (H_{\lambda_i^{(k)}})_{\lambda_i^{(k)} \in W_k} \in \mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$.

2. The second variation of $\mathcal{E}_{\mathrm{v}}$ along a path $\phi_t \in \mathcal{K}^{\mathbb{T}}_\omega$ is given by

$$\frac{d^2}{dt^2}\mathcal{E}_{\mathrm{v}}(\phi_t) = \int_X \left(\ddot\phi_t - |d\dot\phi_t|^2_{\phi_t}\right)\mathrm{v}(m_{\phi_t})\omega_{\phi_t}^{[n]}. \tag{3.18}$$

3. For $\phi \in \mathcal{K}^{\mathbb{T}}_\omega$ and $k \gg 1$, the functional $\mathcal{E}_{\mathrm{v}}$ is concave along the path $(\phi_k(t))_{t\in[0,1]}$ of $\mathcal{K}^{\mathbb{T}}_\omega$ given by:

$$\phi_k(t) := \phi + \frac{t}{2k}\log\left(\frac{\rho_{(\mathrm{v},\mathrm{w})}(k\phi)}{\mathrm{v}(m_\phi)}\right). \tag{3.19}$$

4. The variation of (v,w)*-Hilbert map* $\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}$ *is given by,*

$$\left(\mathbf{d}\,\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}\right)_\phi(\dot\phi)(s,s') =$$
$$-\sum_{\lambda_i^{(k)}\in W_k} \frac{\int_X (s(\lambda_i^{(k)}), s'(\lambda_i^{(k)}))_{k\phi}[2k\dot\phi - (d(\log\circ\mathrm{v}(m_\phi)), d\dot\phi)_\phi + \Delta_\phi\dot\phi]\mathrm{v}(m_\phi)(k\omega_\phi)^{[n]}}{\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})}, \tag{3.20}$$

where $\phi\in\mathcal{K}^{\mathbb{T}}_\omega$ *and* $s,s'\in\mathcal{H}_k$ *admitting decompositions adapted to* (3.2) $s=\sum_{\lambda_i^{(k)}\in W_k} s(\lambda_i^{(k)})$, $s'=\sum_{\lambda_i^{(k)}\in W_k} s'(\lambda_i^{(k)})$.

Proof. 1. The expression (3.16) follows from

$$\begin{aligned}
&2kc_k(\mathrm{v},\mathrm{w})\int_X \dot\phi\left(1+\frac{\Delta_\phi}{2k}\right)\mathrm{v}(m_\phi)(k\omega_\phi)^{[n]}\\
=&2kc_k(\mathrm{v},\mathrm{w})\int_X \dot\phi\mathrm{v}(m_\phi)(k\omega_\phi)^{[n]} + c_k(\mathrm{v},\mathrm{w})\int_X \dot\phi\Delta_\phi(\mathrm{v}(m_\phi))(k\omega_\phi)^{[n]}\\
=&2k^{n+1}\left(\mathbf{d}\mathcal{E}_{\mathrm{v}}\right)_\phi(\dot\phi) + c_k(\mathrm{v},\mathrm{w})\int_X (d\dot\phi, d(\mathrm{v}(m_\phi)))_\phi(k\omega_\phi)^{[n]}\\
=&2k^{n+1}\left(\mathbf{d}\mathcal{E}_{\mathrm{v}}\right)_\phi(\dot\phi) + c_k(\mathrm{v},\mathrm{w})\int_X (d\dot\phi, d(\log\circ\mathrm{v}(m_\phi)))_\phi\mathrm{v}(m_\phi)(k\omega_\phi)^{[n]},
\end{aligned}$$

in the second line we integrated by parts in the second integral. The variation of $\mathcal{E}^k_{\mathrm{v},\mathrm{w}}$ follows from the calculation

$$\begin{aligned}
\left(\mathbf{d}\,\mathcal{E}^k_{\mathrm{v},\mathrm{w}}\right)_H(\dot H) ==& \left.\frac{d}{dt}\right|_{t=0}\mathcal{E}^k_{\mathrm{v},\mathrm{w}}(H+t\dot H)\\
=&\sum_{\lambda_i^{(k)}\in W_k}\left(\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right)\left.\frac{d}{dt}\right|_{t=0}\log\left(\det\left(H_{\lambda_i^{(k)}}+t\dot H_{\lambda_i^{(k)}}\right)\right)\\
=&\sum_{\lambda_i^{(k)}\in W_k}\left(\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right)\left.\frac{d}{dt}\right|_{t=0}\det\left(\mathrm{Id}_{\lambda_i^{(k)}}+tH^{-1}_{\lambda_i^{(k)}}\dot H_{\lambda_i^{(k)}}\right)\\
=&\sum_{\lambda_i^{(k)}\in W_k}\left(\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right)\mathrm{tr}\left(H^{-1}_{\lambda_i^{(k)}}\dot H_{\lambda_i^{(k)}}\right).
\end{aligned}$$

2. Let $\phi_t \in \mathcal{K}^{\mathbb{T}}_{\omega}$, we compute

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{E}_{\mathrm{v}}(\phi_t) =& \frac{d}{dt}\int_X \dot{\phi}_t \mathrm{v}(m_{\phi_t})\omega_{\phi_t}^{[n]} \\ =& \int_X \ddot{\phi}_t \omega_{\phi_t}^{[n]} + \int_X \dot{\phi}_t (d\dot{\phi}_t, \mathrm{v}(m_{\phi_t}))_{\phi_t}\omega_{\phi_t}^{[n]} - \int_X \dot{\phi}_t(\Delta_{\phi_t}(\dot{\phi}_t))\mathrm{v}(m_{\phi_t})\omega_{\phi_t}^{[n]} \\ =& \int_X \left(\ddot{\phi}_t - |d\dot{\phi}_t|^2_{\phi_t}\right)\mathrm{v}(m_{\phi_t})\omega_{\phi_t}^{[n]}.\end{aligned}$$

This completes the proof of (3.18).

3. The second variation of $\mathcal{E}_{\mathrm{v}}$ along the path $\phi_k(t)$ is given by

$$2k\frac{d^2}{dt^2}\mathcal{E}_{\mathrm{v}}(\phi_k(t)) = -\int_X \left| d\log\left(\frac{\rho_{(\mathrm{v},\mathrm{w})}(k\phi)}{\mathrm{v}(m_\phi)}\right)\right|^2_{\phi_k(t)} \mathrm{v}(m_{\phi_k(t)})\omega_{\phi_k(t)}^{[n]} < 0,$$

showing that $\mathcal{E}_{\mathrm{v}}$ is concave along the path $(\phi_k(t))_{t\in[0,1]}$.

4. For $\phi, \dot{\phi} \in \mathcal{K}^{\mathbb{T}}_{\omega}$ and $s, s' \in \mathcal{H}_k$ admitting decompositions adapted to (3.2) $s = \sum_{\lambda_i^{(k)}\in W_k} s(\lambda_i^{(k)})$, $s' = \sum_{\lambda_i^{(k)}\in W_k} s'(\lambda_i^{(k)})$, we have

$$\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\phi + t\dot{\phi})(s,s') = \sum_{\lambda_i^{(k)}\in W_k} \frac{\int_X (s(\lambda_i^{(k)}), s'(\lambda_i^{(k)}))_{k(\phi+t\dot{\phi})}\mathrm{v}(m_{\phi+t\dot{\phi}})(k\omega_{\phi+t\dot{\phi}})^{[n]}}{\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})}$$

using the fact that

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}(s(\lambda_i^{(k)}), s'(\lambda_i^{(k)}))_{k(\phi+t\dot{\phi})} =& \left.\frac{d}{dt}\right|_{t=0} e^{-2k(\phi+t\dot{\phi})}h^k(s(\lambda_i^{(k)}), s'(\lambda_i^{(k)})) \\ =& (-2k\dot{\phi})(s(\lambda_i^{(k)}), s'(\lambda_i^{(k)}))_{k\phi},\end{aligned}$$

the equation (3.20) follows from a straightforward calculation. □

We now consider the functionals $\mathcal{L}^k_{\mathrm{v},\mathrm{w}} : \mathcal{K}^{\mathbb{T}}_{\omega} \to \mathbb{R}$ and $Z^k_{\mathrm{v},\mathrm{w}} : \mathcal{B}^{\mathbb{T}}(\mathcal{H}_k) \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{L}^k_{\mathrm{v},\mathrm{w}} &:= \mathcal{E}^k_{\mathrm{v},\mathrm{w}} \circ \mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}} + 2k^{n+1}c_k(\mathrm{v},\mathrm{w})\mathcal{E}_{\mathrm{v}}, \\ Z^k_{\mathrm{v},\mathrm{w}} &:= 2k^{n+1}c_k(\mathrm{v},\mathrm{w})\mathcal{E}_{\mathrm{v}} \circ \mathrm{FS}^k_{\mathrm{v},\mathrm{w}} + \mathcal{E}^k_{\mathrm{v},\mathrm{w}},\end{aligned} \tag{3.21}$$

where $\mathcal{E}^k_{\mathrm{v},\mathrm{w}}$ is given by (3.15) and $\mathcal{E}_{\mathrm{v}}$ is given in Lemma 3. In what follows we will relate these functionals to the (v,w)-balanced metrics, similarly to [43, 72, 85], and we will show that they quantize the (v,w)-Mabuchi energy.

Proposition 7. *The* (v,w)*-balanced metrics of order* k *are critical points of the functional* $\mathcal{L}^k_{\mathrm{v},\mathrm{w}}$*. Furthermore, there exist real constants* b_k *such that,*

$$\lim_{k\to\infty}\left[\frac{2}{k^n}\mathcal{L}^k_{\mathrm{v},\mathrm{w}}+b_k\right]=\mathcal{M}_{\mathrm{v},\mathrm{w}},$$

where the convergence holds in the C^∞*-norm.*

Proof. Let $\{s_\gamma(\lambda_i^{(k)})\mid \lambda_i^{(k)}\in W_k,\ \gamma=1,\cdots,n(\lambda_i^{(k)})\}$ of $\mathcal{H}_k$ be a $\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}$-orthonormal basis adapted to the decomposition (3.2). Using (3.17) and (3.20) we have,

$$\begin{aligned}
&\mathbf{d}\left(\mathcal{E}^k_{\mathrm{v},\mathrm{w}}\circ \mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}\right)_\phi(\dot\phi)\\
=&-\sum_{\lambda_i^{(k)}\in W_k}\sum_{\gamma=1}^{n(\lambda_i^{(k)})}\int_X |s_\gamma(\lambda_i^{(k)})|^2_{k\phi}[2k\dot\phi-(d(\log\circ\mathrm{v}(m_\phi)),d\dot\phi)_\phi+\Delta_\phi\dot\phi]\mathrm{v}(m_\phi)(k\omega_\phi)^{[n]}\\
=&\int_X \rho_{\mathrm{v},\mathrm{w}}(k\phi)[2k\dot\phi-(d(\log\circ\mathrm{v}(m_\phi)),d\dot\phi)_\phi+\Delta_\phi\dot\phi](k\omega_\phi)^{[n]}\\
=&-2k\int_X\dot\phi\left(1+\frac{\Delta_\phi}{2k}\right)\rho_{\mathrm{v},\mathrm{w}}(k\phi)(k\omega_\phi)^{[n]}+\int_X\rho_{\mathrm{v},\mathrm{w}}(k\phi)(d(\log\circ\mathrm{v}(m_\phi)),d\dot\phi)_\phi(k\omega_\phi)^{[n]}.
\end{aligned}$$

By (3.16) we get

$$\begin{aligned}
\left(\mathbf{d}\mathcal{L}^k_{\mathrm{v},\mathrm{w}}\right)_\phi(\dot\phi)=&-2k\int_X\dot\phi\left(1+\frac{\Delta_\phi}{2k}\right)[\rho_{\mathrm{v},\mathrm{w}}(k\phi)-c_k(\mathrm{v},\mathrm{w})\mathrm{v}(m_\phi)]\,\mathrm{vol}_{k\omega_\phi}\\
&+\int_X[\rho_{\mathrm{v},\mathrm{w}}(k\phi)-c_k(\mathrm{v},\mathrm{w})\mathrm{v}(m_\phi)]\,(d(\log\circ\mathrm{v}(m_\phi)),d\dot\phi)_\phi(k\omega_\phi)^{[n]}.
\end{aligned}$$

From the above expression it is clear that a (v,w)-balanced metric of order k is critical point of $\mathcal{L}^k_{\mathrm{v},\mathrm{w}}$. By the asymptotic expansion in Lemma 18 we get

$$\int_X[\rho_{\mathrm{v},\mathrm{w}}(k\phi)-c_k(\mathrm{v},\mathrm{w})\mathrm{v}(m_\phi)]\,(d(\log\circ\mathrm{v}(m_\phi),d\dot\phi)_\phi(k\omega_\phi)^{[n]}=\mathcal{O}(k^{n-1}),$$

and

$$\begin{aligned}
&2k\int_X\dot\phi\left(1+\frac{\Delta_\phi}{2k}\right)[\rho_{(\mathrm{v},\mathrm{w})}(k\phi)-c_k(\mathrm{v},\mathrm{w})\mathrm{v}(m_\phi)]\,(k\omega_\phi)^{[n]}\\
=&2k^n\int_X(\mathrm{Scal}_\mathrm{v}(\phi)-c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\phi))\dot\phi{\omega_\phi}^{[n]}+\mathcal{O}(k^{n-1})\\
=&2k^m\left(\mathbf{d}\mathcal{M}_{\mathrm{v},\mathrm{w}}\right)_\phi(\dot\phi)+\mathcal{O}(k^{n-1}).
\end{aligned}$$

The proof is complete. ∎

Lemma 20. *For all $\phi \in \mathcal{K}_\omega^{\mathbb{T}}$ we have (in the C^∞ sense),*

$$\lim_{k\to\infty} k^{-n}\left[\mathcal{L}^k_{\mathrm{v,w}}(\phi) - Z^k_{\mathrm{v,w}} \circ \mathrm{Hilb}^k_{\mathrm{v,w}}(\phi)\right] = 0. \tag{3.22}$$

The functional $Z^k_{\mathrm{v,w}}$ is convex along the geodesics of $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$.

Proof. Using (3.21), we get

$$\begin{aligned} k^{-n}\left[\mathcal{L}^k_{\mathrm{v,w}}(\phi) - Z^k_{\mathrm{v,w}} \circ \mathrm{Hilb}^k_{\mathrm{v,w}}(\phi)\right] &= -2kc_k(\mathrm{v,w})\left[\mathcal{E}_{\mathrm{v}}(\mathrm{FS}^k_{\mathrm{v,w}} \circ \mathrm{Hilb}^k_{\mathrm{v,w}}(\phi)) - \mathcal{E}_{\mathrm{v,w}}(\phi)\right] \\ &= -2kc_k(\mathrm{v,w})\left[\mathcal{E}_{\mathrm{v}}(\phi_k(1)) - \mathcal{E}_{\mathrm{v}}(\phi_k(0))\right] \end{aligned}$$

where $\phi_k(t) \in \mathcal{K}_\omega^{\mathbb{T}}$ is the path given by (3.19). Using that $\mathcal{E}_{\mathrm{v}}(\phi_k(t))$ is concave (see Lemma 19), we deduce

$$\begin{aligned} k^{-n}\left[\mathcal{L}^k_{\mathrm{v,w}}(\phi) - Z^k_{\mathrm{v,w}} \circ \mathrm{Hilb}^k_{\mathrm{v,w}}(\phi)\right] &\geq -2kc_k(\mathrm{v,w})\,(\mathbf{d}\mathcal{E}_{\mathrm{v}})_{\phi_k(0)}\left(\frac{1}{2k}\log\left(\frac{\rho_{\mathrm{v,w}}(k\phi)}{\mathrm{v}(m_\phi)}\right)\right) \\ &\sim \frac{1}{4k}\int_X \frac{\mathrm{Scal}_{\mathrm{v}}(\phi) - c_{\mathrm{v,w}}(\alpha)\mathrm{w}(m_\phi)}{\mathrm{v}(m_\phi)}\omega_\phi^{[n]}, \end{aligned}$$

where we used the following smooth expansions to get the second line (see Lemma 18)

$$\begin{aligned} \omega_{\phi_k(t)} &= \omega_\phi + \mathcal{O}(k^{-2}) \\ \omega^{[n]}_{\phi_k(t)} &= \omega^{[n]}_\phi + \mathcal{O}(k^{-2}) \\ m_{\phi_k(t)} &= m_\phi + \mathcal{O}(k^{-2}) \\ d\left(\frac{\rho_{(\mathrm{v,w})}(k\phi)}{\mathrm{v}(m_\phi)}\right) &= \frac{-1}{4k} d\left(\frac{\mathrm{Scal}_{\mathrm{v}}(\phi) - c_{\mathrm{v,w}}(\alpha)\mathrm{w}(m_\phi)}{\mathrm{v}(m_\phi)}\right) + \mathcal{O}(k^{-2}). \end{aligned}$$

On the other hand

$$\begin{aligned} k^{-n}\left[\mathcal{L}^k_{\mathrm{v,w}}(\phi) - Z^k_{\mathrm{v,w}} \circ \mathrm{Hilb}^k_{\mathrm{v,w}}(\phi)\right] &\leq -2kc_k(\mathrm{v,w})\,(\mathbf{d}\mathcal{E}_{\mathrm{v}})_{\phi_k(1)}\left(\frac{1}{2k}\log\left(\frac{\rho_{\mathrm{v,w}}(k\phi)}{\mathrm{v}(m_\phi)}\right)\right) \\ &\sim \frac{1}{4k}\int_X \frac{\mathrm{Scal}_{\mathrm{v}}(\phi) - c_{\mathrm{v,w}}(\alpha)\mathrm{w}(m_\phi)}{\mathrm{v}(m_\phi)}\omega_\phi^{[n]}, \end{aligned}$$

which completes the proof of (3.22). ∎

Lemma 21. *The functional $Z^k_{\mathrm{v,w}}$ is convex along the geodesics of $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$.*

Proof. We follow closely the arguments of [44, Propostion 1] and [85, Propostion 3.2.3]. Let $H(t), t \in \mathbb{R}$ be a geodesic in $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ such that $h = \mathrm{FS}^k_{\mathrm{v,w}}(H(0))$. For a choice of an $H(0)$-orthonormal basis of $\mathcal{H}_k$

$$\{s_\gamma(\lambda_i^{(k)}) \mid \lambda_i^{(k)} \in W_k,\ \gamma = 1, \cdots, n(\lambda_i^{(k)})\},$$

adapted to the splitting 3.2, we have the following expression for $H(t)$,

$$H(t) = \mathrm{diag}\left(e^{tA(\lambda_i^{(k)})}\right)_{\lambda_i^{(k)} \in W_k}$$

with $A(\lambda_i^{(k)}) = \mathrm{diag}(a_\gamma(\lambda_i^{(k)}))_{\gamma=1,n(\lambda_i^{(k)})}$, $a_i(\lambda) \in \mathbb{R}$ and $\mathrm{tr}(A_\gamma(\lambda_i^{(k)})) = 0$. We consider the family of Kähler potentials given by $\phi(t) := \mathrm{FS}^k_{\mathrm{v,w}}(H(t))$. The collection

$$\{e^{\frac{-ta_\gamma(\lambda_i^{(k)})}{2}} s_\gamma(\lambda_i^{(k)}) \mid \lambda_i^{(k)} \in W_k,\ \gamma = 1, \cdots, n(\lambda_i^{(k)})\}$$

is an $H(t)$-orthonormal base of $\mathcal{H}_k$ adapted to (3.2). So we have

$$Z^k_{\mathrm{v,w}}(H(t)) = 2k^{n+1} c_k(\mathrm{v}, \mathrm{w}) \mathcal{E}_{\mathrm{w}}(\phi(t)).$$

By (3.18), we get

$$\left.\frac{d^2}{dt^2}\right|_{t=0} Z^k_{\mathrm{v,w}}(H(t)) = 2k^{n+1} c_k(\mathrm{v}, \mathrm{w}) \int_X \left(\ddot{\phi} - |d\dot{\phi}|^2_\omega\right) \mathrm{v}(m_\omega)\omega^{[n]}.$$

Using $h = \mathrm{FS}^k_{\mathrm{v,w}}(H(0))$, we obtain

$$\sum_{\lambda_i^{(k)} \in W_k} \sum_{\gamma=1}^{n(\lambda_i^{(k)})} |s_\gamma(\lambda_i^{(k)})|^2_{h^k} = 1.$$

It follows that

$$\begin{aligned}
\dot{\phi} &= -\frac{1}{2k} \sum_{\lambda_i^{(k)} \in W_k} \sum_{\gamma=1}^{n(\lambda_i^{(k)})} a_\gamma(\lambda_i^{(k)}) |s_\gamma(\lambda_i^{(k)})|^2_{h^k} \\
\ddot{\phi} &= \frac{1}{2k}\Big[\sum_{\lambda_i^{(k)} \in W_k} \sum_{\gamma=1}^{n(\lambda_i^{(k)})} a_\gamma(\lambda_i^{(k)})^2 |s_\gamma(\lambda_i^{(k)})|^2_{h^k} - \Big(\sum_{\lambda_i^{(k)} \in W_k} \sum_{\gamma=1}^{n(\lambda_i^{(k)})} a_\gamma(\lambda_i^{(k)}) |s_\gamma(\lambda_i^{(k)})|^2_{h^k} \Big)^2 \Big].
\end{aligned}$$

We compute,

$$
\begin{aligned}
&\frac{1}{k^n c_k(\mathrm{v},\mathrm{w})}\,\frac{d^2}{dt^2}\bigg|_{t=0} Z^k_{\mathrm{v},\mathrm{w}}(H(t))\\
&= \int_X -2k|d\dot\phi|^2_\omega \mathrm{v}(m_\omega)\omega^{[n]}\\
&\quad + \int_X \sum_{\lambda_i^{(k)}\in W_k}\sum_{\gamma=1}^{n(\lambda_i^{(k)})} a_\gamma(\lambda_i^{(k)})^2 |s_\gamma(\lambda_i^{(k)})|^2_{h^k}\mathrm{v}(m_\omega)\omega^{[n]}\\
&\quad - \int_X \Big(\sum_{\lambda_i^{(k)}\in W_k}\sum_{\gamma=1}^{n(\lambda_i^{(k)})} a_\gamma(\lambda_i^{(k)})|s_\gamma(\lambda_i^{(k)})|^2_{h^k}\Big)^2 \mathrm{v}(m_\omega)\omega^{[n]}\\
&= \sum_{\lambda_i^{(k)}\in W_k}\sum_{\gamma=0}^{n(\lambda_i^{(k)})}\int_X \Big|4\sqrt{k}(\nabla\dot\phi,\nabla s_\gamma(\lambda_i^{(k)})) - (a_\gamma(\lambda_i^{(k)}) - 4\sqrt{k}\dot\phi)s_\gamma(\lambda_i^{(k)})\Big|^2_{h^k}\mathrm{v}(m_\omega)\omega^{[n]} \geq 0.
\end{aligned}
$$

To get the last equality we used that for any smooth function ϕ on X, we have (see [44, Proposition 1])

$$
|\nabla\phi|^2_\omega = 2\sum_{\lambda_i^{(k)}\in W_k}\sum_{\gamma=1}^{n(\lambda_i^{(k)})}\Big|\Big(\nabla\phi,\nabla s_\gamma(\lambda_i^{(k)})\Big)\Big|^2_{h^k}.
$$

$\square$

Corollary 7. *A* (v,w)*-balanced metric of order* k *minimizes the functional* $Z^k_{(\mathrm{v},\mathrm{w})}$ *on* $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$.

Proof. We show that (v,w)-balanced metrics of order k are critical points of $Z^k_{(\mathrm{v},\mathrm{w})}$. Let $H(t)$ be a geodesic in $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ as in the proof of Lemma 21, such that $h = \mathrm{FS}^k_{\mathrm{v},\mathrm{w}}(H(0))$

is (v,w)-balanced of order k. We have

$$
\begin{aligned}
&\frac{-1}{2k^{n+1}c_k(\mathrm{v},\mathrm{w})}\left.\frac{d}{dt}\right|_{t=0} Z^k_{\mathrm{v},\mathrm{w}}(H(t))\\
&=\int_X \sum_{\lambda_i^{(k)}\in W_k}\sum_{\gamma=1}^{n(\lambda_i^{(k)})} a_\gamma(\lambda_i^{(k)})|s_\gamma(\lambda_i^{(k)})|^2_{h^k}\mathrm{v}(m_\omega)\omega^{[n]}\\
&=\sum_{\lambda_i^{(k)}\in W_k}\sum_{\gamma=1}^{n(\lambda_i^{(k)})}\left(\mathrm{v}(\lambda_i^{(k)})-\frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right)a_\gamma(\lambda_i^{(k)})\parallel s_\gamma(\lambda_i^{(k)})\parallel^2_{\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\mathrm{FS}^k_{\mathrm{v},\mathrm{w}}(H))}\\
&=\sum_{\lambda_i^{(k)}\in W_k}\left(\mathrm{v}(\lambda_i^{(k)})-\frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right)\mathrm{tr}(A(\lambda_i^{(k)}))=0,
\end{aligned}
$$

where we used that H is (v,w)-balanced $\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\mathrm{FS}^k_{\mathrm{v},\mathrm{w}}(H))=H$ and $(s_\gamma(\lambda_i^{(k)}))$ is an H-orthonormal basis of $\mathcal{H}_k$. Thus, H is a critical point of $Z^k_{\mathrm{v},\mathrm{w}}$ and by the convexity, we deduce that H is a minimum. □

Now we suppose that $\mathcal{K}^{\mathbb{T}}_\omega$ contains a (v,w)-cscK metric $\phi^\star$. We will show in the following proposition that the metrics $\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\phi^\star)$ are almost balanced in the sense that they minimizes $Z^k_{\mathrm{v},\mathrm{w}}$, up to an error that goes to zero.

Proposition 8. *For all $\phi\in\mathcal{K}^{\mathbb{T}}_\omega$ there exists a smooth function $\varepsilon_\phi(k)$, such that $\lim_{k\to\infty}\varepsilon_\phi(k)=0$ in $C^\ell(X,\mathbb{R})$ and,*

$$k^{-n}Z^k_{\mathrm{v},\mathrm{w}}\circ\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\phi)\geq k^{-n}Z^k_{\mathrm{v},\mathrm{w}}\circ\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\phi^\star)+\varepsilon_\phi(k).$$

Proof. We denote $H_k=\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\phi)$ and $H_k^\star=\mathrm{Hilb}^k_{\mathrm{v},\mathrm{w}}(\phi^\star)$. For a choice of an adapted $H_k^\star$-orthonormal basis $\{s_\gamma(\lambda_i^{(k)})|\lambda_i^{(k)}\in W_k,\ \gamma=1,\cdots,n(\lambda_i^{(k)})\}$ of $\mathcal{H}_k$ we can write $H_k=\mathrm{diag}(e^{A(\lambda_i^{(k)})})_{\lambda\in\Lambda_k(\hat\xi)}$ with $A(\lambda_i^{(k)})=\mathrm{diag}(a_\gamma(\lambda_i^{(k)}))_{\gamma=1,n(\lambda_i^{(k)})}$, $\mathrm{tr}(A(\lambda_i^{(k)}))=0$, and consider the geodesic that joins $H_k^\star$ to H_k,

$$H_k(t)=\mathrm{diag}\left(e^{tA(\lambda_i^{(k)})}\right)_{\lambda_i^{(k)}\in W_k}.$$

Let $P_k(t):=Z^k_{\mathrm{v},\mathrm{w}}(H_k(t))$. $P_k(t)$ is a convex function by Lemma 21. It follows that,

$$k^{-n}\left(Z^k_{\mathrm{v},\mathrm{w}}(H_k)-Z^k_{\mathrm{v},\mathrm{w}}(H_k^\star)\right)\geq k^{-n}P_k'(0).$$

Letting $\varepsilon_\phi(k) := k^{-n}P_k'(0)$, we have

$$P_k'(0) = 2k^{n+1}c_k(\mathrm{v},\mathrm{w})\int_X \dot{\phi}\mathrm{v}(m_{\phi^\star})\omega_{\phi^\star}^{[n]} = -c_k(\mathrm{v},\mathrm{w})k^n\int_X \frac{\rho_A(k\phi^\star)}{\rho_{\mathrm{v},\mathrm{w}}(k\phi^\star)}\mathrm{v}(m_{\phi^\star})\omega_{\phi^\star}^{[n]},$$

where

$$\rho_A(k\phi^\star) = \sum_{\lambda_i^{(k)}\in W_k}\sum_{\gamma=1}^{n(\lambda_i^{(k)})} a_\gamma(\lambda_i^{(k)})\mathrm{v}(m_{\phi^\star})|s_\gamma(\lambda_i^{(k)})|^2_{k\phi^\star}. \tag{3.23}$$

By Lemma 18, since $\phi^\star$ is a (v,w)-cscK metric we get

$$\rho_{\mathrm{v},\mathrm{w}}(k\phi^\star) = \mathrm{v}(m_{\phi^\star}) + \mathcal{O}(k^{-2}), \tag{3.24}$$

and therefore we obtain

$$P_k'(0) = -c_k(\mathrm{v},\mathrm{w})k^n\int_X \rho_A(k\phi^\star)\mathcal{O}(k^{-2})\omega_{\phi^\star}^{[n]}.$$

We have

$$\begin{aligned} e^{a_\gamma(\lambda_i^{(k)})} =& \parallel s_\gamma(\lambda_i^{(k)}) \parallel^2_{H_k} \\ =& k^n\sum_{\lambda_i^{(k)}\in W_k}\left(\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{(\mathrm{v},\mathrm{w})}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right)^{-1}\int_X |s_\gamma(\lambda_i^{(k)})|^2_{k\phi}\mathrm{v}(m_\phi)\omega_\phi^{[n]}. \end{aligned} \tag{3.25}$$

As $h_\phi^k = e^{-2k(\phi^\star-\phi)}h_{\phi^\star}^k$, there exists a constant $C_\phi > 0$ such that

$$e^{-2kC_\phi}h_{\phi^\star}^k \leq h_\phi^k \leq e^{2kC_\phi}h_{\phi^\star}^k. \tag{3.26}$$

By the fact that $\mathrm{v}(m_\phi)/\mathrm{v}(m_{\phi^\star})$ is bounded by positive constants (independent from ϕ), and $\omega_\phi^{[n]}/\omega_{\phi^\star}^{[n]}$ is bounded by positive constants depending only on ϕ, using (3.26) we obtain from (3.25) the following estimate

$$-2C_\phi k + B_\phi' \leq a_\gamma(\lambda_i^{(k)}) \leq 2C_\phi k + B_\phi, \tag{3.27}$$

where B_ϕ, B_ϕ' are real constants depending only on $\phi, \phi^\star$. We derive from (3.23) and (3.27) that,

$$(-2C_\phi k + B_\phi')\rho_{\mathrm{v},\mathrm{w}}(k\phi^\star) \leq \rho_A(k\phi^\star) \leq (2C_\phi k + B_\phi)\rho_{\mathrm{v},\mathrm{w}}(k\phi^\star).$$

Using (3.24) we infer

$$(-2C_\phi k + B_\phi')\mathrm{v}(m_{\phi^\star}) + \mathcal{O}(k^{-1}) \leq \rho_A(k\phi^\star) \leq (2C_\phi k + B_\phi)\mathrm{v}(m_{\phi^\star}) + \mathcal{O}(k^{-1}),$$

which shows that $\lim_{k\to\infty}\varepsilon_\phi(k) = 0$. $\square$

3.3.2 Proof of Theorem 3

Now we are in position to give the proof of Theorem 3 which is very similar to [85, Theorem 3.4.1].

Proof. Let $\phi^\star \in \mathcal{K}^{\mathbb{T}}_\omega$ the Kähler potential of a (v,w)-cscK metric. For any $\phi \in \mathcal{K}^{\mathbb{T}}_\omega$, by Corollary 7 we have

$$\begin{aligned}\mathcal{L}^k_{\mathrm{v},\mathrm{w}}(\phi) &= Z^k_{\mathrm{v},\mathrm{w}}(\mathrm{Hilb}_{\mathrm{v},\mathrm{w}}(k\phi)) + \left[\mathcal{L}^k_{\mathrm{v},\mathrm{w}}(\phi) - Z^k_{\mathrm{v},\mathrm{w}}(\mathrm{Hilb}_{\mathrm{v},\mathrm{w}}(k\phi))\right] \\ &\geq Z^k_{\mathrm{v},\mathrm{w}}(\mathrm{Hilb}_{\mathrm{v},\mathrm{w}}(k\phi^\star)) + k^m\varepsilon_\phi(k) + \left[\mathcal{L}^k_{\mathrm{v},\mathrm{w}}(\phi) - Z^k_{\mathrm{v},\mathrm{w}}(\mathrm{Hilb}_{\mathrm{v},\mathrm{w}}(k\phi))\right].\end{aligned}$$

Thus,

$$\begin{aligned}\frac{2}{k^n}\mathcal{L}^k_{\mathrm{v},\mathrm{w}}(\phi) + b_k &\geq \frac{2}{k^n}\mathcal{L}^k_{\mathrm{v},\mathrm{w}}(\phi^\star) + b_k + \frac{2}{k^n}\left[Z^k_{\mathrm{v},\mathrm{w}}(\mathrm{Hilb}_{\mathrm{v},\mathrm{w}}(k\phi^\star)) - \mathcal{L}^k_{\mathrm{v},\mathrm{w}}(\phi^\star)\right] \\ &\quad +\varepsilon_\phi(k) + \frac{2}{k^n}\left[\mathcal{L}^k_{\mathrm{v},\mathrm{w}}(\phi) - Z^k_{\mathrm{v},\mathrm{w}}(\mathrm{Hilb}_{\mathrm{v},\mathrm{w}}(k\phi))\right].\end{aligned}$$

Using Proposition 7 and Proposition 8 together with Lemma 20, by letting k go to infinity we get,

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi) \geq \mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi^\star).$$

□

3.3.3 A momentum map picture for the (v,w)-balanced metrics

There is a natural extension of the momentum map interpretation of balanced Fubini-Study metrics given by S. K. Donaldson in [41] to (v,w)-balanced metrics. Indeed, let us identify $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ with the space of bases of $\mathcal{H}_k$ compatible with the splitting (3.2), and denote by $\mathrm{Aut}^{\mathbb{T}}(X,L)$ the centralizer of $\mathbb{T}$ in the Lie group of automorphisms of the pair (X,L). Let θ_k denote the group representation of $\mathrm{Aut}^{\mathbb{T}}(X,L)$ in $\mathrm{GL}(\mathcal{H}_k)$, given by

$$\theta_k(\sigma)s := \sigma \circ s \circ p(\sigma)^{-1},$$

where $p : \mathrm{Aut}(X,L) \to \mathrm{Aut}_{\mathrm{red}}(X)$ is the natural projection. For each k we have the following group actions on $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$:

- $\mathbb{C}^*$ by scalar multiplications;
- $\mathcal{A}_k^{\mathbb{T}} := \theta_k\left(\mathrm{Aut}^{\mathbb{T}}(X,L)\right)$;
- $\mathcal{G}_k^{\mathbb{T}} := \left\{g \in \prod_{\lambda_i^{(k)}\in W_k} \mathbb{U}(\mathcal{H}_k(\lambda_i^{(k)})) \mid \prod_{\lambda_i^{(k)}} \det(g_{\lambda_i^{(k)}})^{\lambda_i^{(k)}(\mathrm{v},\mathrm{w})} = 1\right\}$, where we denote $\lambda_i^{(k)}(\mathrm{v},\mathrm{w}) := \mathrm{v}(\lambda_i^{(k)}) - \frac{c_{\mathrm{v},\mathrm{w}}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})$.

We consider the quotient space,

$$\mathcal{Z}^{\mathbb{T}}(\mathcal{H}_k) = \mathcal{B}^{\mathbb{T}}(\mathcal{H}_k) \Big/ (\mathbb{C}^* \times \mathcal{A}_k^{\mathbb{T}}),$$

on which we have a natural action of $\mathcal{G}_k^{\mathbb{T}}$. The quotient $\mathcal{Z}^{\mathbb{T}}(\mathcal{H}_k)$ carries a natural Kähler structure, defined as follows:

- The multiplication by $\sqrt{-1}$ defines an integrable complex structure on $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ invariant under the action of $\mathbb{C}^* \times \mathcal{A}_k^{\mathbb{T}}$, so it descends to a complex structure $J_{\mathcal{Z}}^{(k)}$ on the quotient $\mathcal{Z}^{\mathbb{T}}(\mathcal{H}_k)$.
- There is a natural Kähler form on $\mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ given by

  $$\varpi_{\mathcal{B}}^{(k)} := dd^c Z_{\mathrm{v},\mathrm{w}}^k,$$

  where $d^c := J_{\mathcal{B}}^{(k)} d$. The form $\varpi_{\mathcal{B}}^{(k)}$ is invariant under the group actions of $\mathbb{C}^* \times \mathcal{A}_k^{\mathbb{T}}$ and $\mathcal{G}_k^{\mathbb{T}}$, so it defines a $\mathcal{G}_k^{\mathbb{T}}$-invariant Kähler form on $\mathcal{Z}^{\mathbb{T}}(\mathcal{H}_k)$.

We endow $\mathrm{Lie}(\mathcal{G}_k^{\mathbb{T}})$ with the pairing

$$\langle a, b\rangle_{\mathrm{v},\mathrm{w},k} = \sum_{\lambda_i^{(k)}\in W_k} \left(\mathrm{v}(\lambda_i^{(k)}) - \frac{c_{\mathrm{v},\mathrm{w}}(\alpha)}{4k}\mathrm{w}(\lambda_i^{(k)})\right) \cdot \mathrm{tr}\left(a_{\lambda_i^{(k)}} b^*_{\lambda_i^{(k)}}\right),$$

and identify $\mathrm{Lie}(\mathcal{G}_k^{\mathbb{T}})$ with the dual vector space by using $\langle\cdot,\cdot\rangle_{\mathrm{v},\mathrm{w},k}$. For any $a \in \mathrm{Lie}(\mathcal{G}_k^{\mathbb{T}})$ we denote,

$$(a)_0 := a - \frac{\langle a, \mathrm{Id}\rangle_{\mathrm{v},\mathrm{w},k}}{\langle \mathrm{Id}, \mathrm{Id}\rangle_{\mathrm{v},\mathrm{w},k}}\mathrm{Id}.$$

The action of $\mathcal{G}_k^{\mathbb{T}}$ on $\mathcal{Z}^{\mathbb{T}}(\mathcal{H}_k)$ is Hamiltonian with $\varpi_{\mathcal{Z}}^{(k)}$-moment map $\underline{\boldsymbol{\mu}}_{\mathrm{v},\mathrm{w}}^{(k)} : \mathcal{Z}^{\mathbb{T}}(\mathcal{H}_k) \to \mathrm{Lie}(\mathcal{G}_k^{\mathbb{T}})$ given by

$$\underline{\boldsymbol{\mu}}_{\mathrm{v},\mathrm{w}}^{(k)}(\mathbf{s}) := \sqrt{-1}\left(\bigoplus_{\lambda_i^{(k)}\in W_k} \left(\mathrm{Hilb}_{\mathrm{v},\mathrm{w}}^k\left(\mathrm{FS}_{\mathrm{v},\mathrm{w}}^k(\mathbf{s})\right)(s_\gamma(\lambda_i^{(k)}), s_\eta(\lambda_i^{(k)}))\right)_{\gamma,\eta=1,n(\lambda_i^{(k)})}\right)_0,$$

where $n(\lambda_i^{(k)}) := \dim(\mathcal{H}(\lambda_i^{(k)}))$ and for any $\mathbf{s} \in \mathcal{B}^{\mathbb{T}}(\mathcal{H}_k)$ we identify $\mathbf{s}$ with the unique positive definite Hermitian form so that $\mathbf{s}$ is orthonormal. Thus the zeroes of the moment map $\underline{\boldsymbol{\mu}}^{(k)}_{\mathrm{v},\mathrm{w}}$ are the (v,w)-balanced elements of $\mathcal{Z}^{\mathbb{T}}(\mathcal{H}_k)$.

CHAPTER IV

WEIGHTED K-STABILITY AS AN OBSTRUCTION TO THE EXISTENCE KÄHLER METRICS WITH CONSTANT WEIGHTED SCALAR CURVATURE.

We are going to establish in this chapter Theorem 4 from the introduction.

4.1 The (v,w)-Futaki invariant of a smooth test configuration

Let X be a compact Kähler manifold endowed with an ℓ-dimensional real torus $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ and a Kähler class $\alpha \in H^{1,1}(X,\mathbb{R})$. Following [37,46,47] we give the following

Definition 16. A smooth $\mathbb{T}$-compatible Kähler test configuration for (X,α) is a compact smooth $(n+1)$-dimensional Kähler manifold $(\mathcal{X},\mathcal{A})$, endowed with a holomorphic action of a real torus $\hat{\mathbb{T}} \subset \mathrm{Aut}_{\mathrm{red}}(\mathcal{X})$ with Lie algebra $\hat{\mathfrak{t}}$ and

- a surjective holomorphic map $\pi : \mathcal{X} \to \mathbb{P}^1$ such that the torus action $\hat{\mathbb{T}}$ on $\mathcal{X}$ preserves each fiber $X_\tau := \pi^{-1}(\tau)$ and $(X_1, \mathcal{A}_{|X_1}, \hat{\mathbb{T}}) \cong (X, \alpha, \mathbb{T})$,
- a $\mathbb{C}^\star$-action ρ on $\mathcal{X}$ commuting with $\hat{\mathbb{T}}$ and covering the usual $\mathbb{C}^\star$-action on $\mathbb{P}^1$,
- a biholomorphism
$$\lambda : \mathcal{X} \setminus X_0 \simeq X \times \left(\mathbb{P}^1 \setminus \{0\}\right), \tag{4.1}$$
which is equivariant with respect to the actions of $\hat{\mathbb{G}} := \hat{\mathbb{T}} \times \mathbb{S}^1_\rho$ on $\mathcal{X} \setminus X_0$ and the action of $\mathbb{G} := \mathbb{T} \times \mathbb{S}^1$ on $X \times \left(\mathbb{P}^1 \setminus \{0\}\right)$.

In what follows we shall tacitly identify $\hat{\mathbb{T}}$ with $\mathbb{T}$ and $\hat{\mathbb{G}}$ with $\mathbb{G}$.

Definition 17. A *smooth* $\mathbb{T}$*-compatible Kähler test configuration* $(\mathcal{X}, \mathcal{A}, \rho, \mathbb{T})$ for $(X, \alpha, \mathbb{T})$ is called

- *trivial* if it is given by $(\mathcal{X}_0 = X \times \mathbb{P}^1, \mathcal{A}_0 = \pi_X^*\alpha + \pi_{\mathbb{P}^1}^*[\omega_{\mathrm{FS}}], \mathbb{T})$ and $\mathbb{C}^\star$-action $\rho_0(\tau)(x, z) = (x, \tau z)$ for any $\tau \in \mathbb{C}^\star$ and $(x, z) \in X \times \mathbb{P}^1$.
- *product* if it is given by $(\mathcal{X}_{\mathrm{prod}}, \mathcal{A}_{\mathrm{prod}}, \rho_{\mathrm{prod}}, \mathbb{T})$ where $\mathcal{X}_{\mathrm{prod}}$ is the compactification (in the sense of [80, 97], see also [20, Example 2.8] and [79, p. 12-13]) of $X \times \mathbb{C}$ with $\mathbb{C}^\star$-action $\rho_{\mathrm{prod}}(\tau)(x, z) = (\rho_X(\tau)x, \tau z)$ where ρ_X is a $\mathbb{C}^\star$-action on X and $\mathcal{A}_{\mathrm{prod}}$ is a Kähler class on $\mathcal{X}_{\mathrm{prod}}$ which restricts to α on $X_1 \cong X$.

Let $(\mathcal{X}, \mathcal{A}, \mathbb{T})$ be a smooth $\mathbb{T}$-compatible Kähler test configuration for $(X, \alpha, \mathbb{T})$ and $\Omega \in \mathcal{A}$ a $\mathbb{G}$-invariant Kähler form. The action of $\mathbb{T}$ on $\mathcal{X}$ is Hamiltonian with Ω-momentum map $m_\Omega : \mathcal{X} \to \mathfrak{t}^*$, normalized by $m_\Omega(X_1) = \mathrm{P}$, where P is a fixed momentum polytope for the induced $\mathbb{T}$-action on $X_1 \cong X$.

For any $\tau \in \mathbb{C}^\star$, we denote by

$$\Omega_\tau := \Omega_{|X_\tau}, \ \ \Omega_1 =: \omega \text{ and } \omega_\tau := \rho(\tau)^*\Omega_\tau, \tag{4.2}$$

where $\rho(\tau) : X_1 \xrightarrow{\sim} X_\tau$ is the restriction of $\rho(\tau) \in \mathrm{Aut}_{\mathrm{red}}(\mathcal{X})$ to X_1. The action of $\mathbb{T}$ on X_τ is Hamiltonian with Ω_τ-momentum map $(m_\Omega)_{|X_\tau}$. Pulling the structure on X_τ back to X_1 via $\rho(\tau)$, we get a ω_τ-momentum map for the $\mathbb{T}$-action on X_1, given by

$$m_\tau = m_{\Omega_\tau} \circ \rho(\tau). \tag{4.3}$$

Lemma 22. *For any* $\tau \in \mathbb{C}^\star$*, we have*

$$\int_{X_\tau} m_{\Omega_\tau} \Omega_\tau^{[n]} = \int_{X_1} m_\tau \omega_\tau^{[n]} = \int_{X_1} m_1 \omega^{[n]}.$$

It follows that $\mathrm{P}_\tau = \mathrm{P}$ *for any* $\tau \in \mathbb{C}^\star$*, where* $\mathrm{P}_\tau := m_\Omega(X_\tau) = m_\tau(X_1)$ *is the momentum polytope of the induced action of* $\mathbb{T}$ *on* X_τ *and* $\Omega_\tau := \Omega_{|X_\tau}$.

Proof. Since Ω is $\mathbb{S}^1_\rho$-invariant, the following integral depends only on $t = -\log|\tau|$,

$$\int_{X_\tau} m_{\Omega_\tau} \Omega_\tau^{[n]} = \int_{X_1} m_\tau \omega_\tau^{[n]} = \int_{X_1} m_\tau (\omega + dd^c\phi_\tau)^{[n]}.$$

Let V_ρ be the generator of the $\mathbb{S}^1_\rho$-action. By (4.3) we have

$$\frac{d}{dt}m_\tau = \frac{d}{dt}(m_\Omega \circ \varphi^t_{\mathcal{J}V_\rho}) = (\varphi^t_{\mathcal{J}V_\rho})^*(dm_\Omega, \mathcal{J}V_\rho)_\Omega = -(\varphi^t_{\mathcal{J}V_\rho})^*(dm_\Omega, dh^\rho)_\Omega,$$

where $\mathcal{J}$ denotes the complex structure on $\mathcal{X}$, $\varphi^t_{\mathcal{J}V_\rho} = \rho(e^{-t})$ is the flow of $\mathcal{J}V_\rho$ and h^ρ is a Ω-Hamiltonian function for V_ρ. On the other hand, we have

$$\frac{d}{dt}\rho(\tau)^*\Omega = (\varphi^t_{\mathcal{J}V_\rho})^*\mathcal{L}_{\mathcal{J}V_\rho}\Omega = -(\varphi^t_{\mathcal{J}V_\rho})^*dd^c h^\rho.$$

It follows that

$$\begin{aligned}
\frac{d}{dt}\int_{X_1} m_\tau \omega_\tau^{[n]} =& \frac{d}{dt}\int_{X_1} m_\tau((\rho(\tau)^*\Omega)_{|X_1})^{[n]} \\
=& -\int_{X_\tau}((dm_\Omega, dh^\rho)_\Omega)_{|X_\tau}\Omega_\tau^{[n]} - \int_{X_\tau} m_{\Omega_\tau} dd^c h^\rho_{|X_\tau} \wedge \Omega_\tau^{[n-1]} \\
=& -\int_{X_\tau}((dm_\Omega, dh^\rho)_\Omega)_{|X_\tau}\Omega_\tau^{[n]} + \int_{X_\tau} m_{\Omega_\tau}\Delta_{\Omega_\tau}(h^\rho_{|X_\tau})\Omega_\tau^{[n]} \\
=& -\int_{X_\tau}((dm_\Omega, dh^\rho)_\Omega)_{|X_\tau}\Omega_\tau^{[n]} + \int_{X_\tau}(dm_{\Omega_\tau}, dh^\rho_{|X_\tau})_{\Omega_\tau}\Omega_\tau^{[n]} = 0,
\end{aligned}$$

where we have used that $((dm_\Omega, dh^\rho)_\Omega)_{|X_\tau} = \left(dm_{\Omega_\tau}, dh^\rho_{|X_\tau}\right)_{\Omega_\tau}$ since the symplectic gradient of $m_\Omega : \mathcal{X} \to \mathfrak{t}^*$ is given by the $\mathfrak{t}$-valued fundamental vector field for the $\mathbb{T}$-action, and thus is tangent to the fibers. It follows that

$$\int_{X_1} m_\tau \omega_\tau^{[n]} = \int_{X_1} m_1 \omega^{[n]}.$$

□

Since $m_\Omega : \mathcal{X} \to \mathfrak{t}^*$ is continuous it follows from Lemma 22 that $m_\Omega(\mathcal{X}) = \mathrm{P}$.

Definition 18. Let $(\mathcal{X}, \mathcal{A}, \mathbb{T})$ be a smooth $\mathbb{T}$-compatible Kähler test configuration for the compact Kähler manifold (X, α) and $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$, $\mathrm{w} \in C^\infty(\mathrm{P}, \mathbb{R})$. The (v, w)-*Futaki invariant* of $(\mathcal{X}, \mathcal{A}, \mathbb{T})$ is defined to be the real number

$$\begin{aligned}
\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}, \mathcal{A}) =& -\int_{\mathcal{X}}(\mathrm{Scal}_\mathrm{v}(\Omega) - c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_\Omega))\Omega^{[n+1]} \\
&+ 2\int_{\mathcal{X}} \mathrm{v}(m_\Omega)\pi^*\omega_{\mathrm{FS}} \wedge \Omega^{[n]}
\end{aligned} \tag{4.4}$$

where $\Omega \in \mathcal{A}$ is a $\mathbb{T}$-invariant representative of $\mathcal{A}$, ω_{FS} is the Fubini-Study metric on $\mathbb{P}^1$ with $\mathrm{Ric}(\omega_{\mathrm{FS}}) = \omega_{\mathrm{FS}}$, and $c_{(\mathrm{v},\mathrm{w})}(\alpha)$ is the (v, w)-slope of (X, α) given by (2.6).

Remark 6. 1. By Lemma 2, (4.4) is independent from the choice of a $\mathbb{T}$-invariant Kähler form $\Omega \in \mathcal{A}$. For $\mathrm{v} = \mathrm{w} \equiv 1$ we also recover the Futaki invariant of a smooth test configuration introduced in [37, 46, 47].

2. It is easy to show that

$$\begin{aligned}
2\int_{\mathcal{X}} \mathrm{v}(m_\Omega)\pi^*\omega_{\mathrm{FS}} \wedge \Omega^{[n]} =& 2\int_{\mathcal{X}\setminus X_0} \mathrm{v}(m_\Omega)\pi^*\omega_{\mathrm{FS}} \wedge \Omega^{[n]} \\
=& 2\int_{\mathbb{P}^1\setminus\{0\}} \left(\int_{X_\tau} \mathrm{v}(m_{\Omega_\tau})\Omega_\tau^{[n]}\right)\omega_{\mathrm{FS}} \\
=& 2\mathrm{vol}(\mathbb{P}^1)\Big(\int_{X_1} \mathrm{v}(m_\omega)\omega^{[n]}\Big) \\
=& (8\pi)\int_X \mathrm{v}(m_\omega)\omega^{[n]},
\end{aligned}$$

where for passing from the second line to the third line we used that $\rho(\tau)^*\Omega_\tau$ and ω are in the same Kähler class $\mathcal{A}_{|X_1}$ on X_1, see Lemma 2. Thus, we obtain the following equivalent expression for the (v, w)-Futaki invariant

$$\begin{aligned}
\mathcal{F}_{\mathrm{v,w}}(\mathcal{X}, \mathcal{A}) =& -\int_{\mathcal{X}} \big(\mathrm{Scal}_{\mathrm{v}}(\Omega) - c_{(\mathrm{v,w})}(\alpha)\mathrm{w}(m_\Omega)\big)\Omega^{[n+1]} \\
&+ (8\pi)\int_X \mathrm{v}(m_\omega)\omega^{[n]}.
\end{aligned} \tag{4.5}$$

3. It is easy to compute the (v, w)-Futaki invariant of the trivial test configuration $(\mathcal{X}_0, \mathcal{A}_0)$ (see Definition 17), using that for a product Kähler form $\Omega_0 := \pi_X^*\omega + \pi_{\mathbb{P}^1}^*\omega_{\mathrm{FS}}$ we have $\mathrm{Scal}_{\mathrm{v}}(\Omega_0) = \mathrm{Scal}_{\mathrm{v}}(\omega) + 2\mathrm{v}(m_\omega)$, then (4.4) reduces to

$$\mathcal{F}_{\mathrm{v,w}}(\mathcal{X}_0, \mathcal{A}_0) = -4\pi\int_X \big(\mathrm{Scal}_{\mathrm{v}}(\omega) - c_{(\mathrm{v,w})}(\alpha)\mathrm{w}(m_\omega)\big)\omega^{[n]}.$$

◇

Definition 19. [36, 47] We say that $(X, \alpha, \mathbb{T})$ is

1. (v, w)-*K-semistable* on smooth Kähler test configurations if $\mathcal{F}_{\mathrm{v,w}}(\mathcal{X}, \mathcal{A}) \geq 0$ for any $\mathbb{T}$-compatible test configuration $(\mathcal{X}, \mathcal{A}, \mathbb{T})$ of $(X, \alpha, \mathbb{T})$ and $\mathcal{F}_{\mathrm{v,w}}(\mathcal{X}_0, \mathcal{A}_0) = 0$ for the trivial test configuration $(\mathcal{X}_0, \mathcal{A}_0)$.

2. (v, w)-*K-stable* on smooth Kähler test configuations if it is (v, w)-*K-semistable* and

$\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X},\mathcal{A}) = 0$ if and only if $(\mathcal{X},\mathcal{A}) = (\mathcal{X}_{\mathrm{prod}},\mathcal{A}_{\mathrm{prod}})$ is a product in the sense of Definition 17.

Following [37, 46], there is a family of $\mathbb{T}$-invariant Kähler potentials $\phi_\tau \in \mathcal{K}^{\mathbb{T}}_{\omega}(X_1)$, $\tau \in \mathbb{C}^* \subset \mathbb{P}^1$ given by the following lemma.

Lemma 23. *Let $\Omega \in \mathcal{A}$ be a $\mathbb{G}$-invariant Kähler form on $\mathcal{X}$.*

1. *On $\mathcal{X}^\star := \mathcal{X} \setminus X_0$ we have*
$$\Omega = \hat{\omega} + dd^c\Phi, \tag{4.6}$$
where $\hat{\omega} := (\pi_X \circ \lambda)^\star\omega$ with λ the map given by (4.1) and π_X is the projection on the first factor of $X \times (\mathbb{P}^1 \setminus \{0\})$, and Φ is a smooth $\mathbb{G}$-invariant function on $\mathcal{X}^\star$, such that for all $\tau \in \mathbb{C}^*$,
$$\phi_\tau := \rho(\tau)^*(\Phi_{|X_\tau}) \in \mathcal{K}^{\mathbb{T}}_{\omega}(X_1), \tag{4.7}$$
satisfies
$$\omega_\tau - \omega = dd^c\phi_\tau,$$
where we recall that ω_τ is defined in (4.2).

2. *$m^{\xi}_{\hat{\omega}} := m^{\xi}_{\Omega} - (d^c\Phi)(\xi)$, $\xi \in \mathfrak{t}$ is a moment map of $\hat{\omega}$ restrected to a fiber X_τ for the $\mathbb{T}$-action on $\mathcal{X}^\star$, satisfying $m_{\hat{\omega}}(\mathcal{X}^\star) = \mathrm{P}$.*

Proof. (i) Using [46, Proposition 3.10] we can find a smooth function Φ on $\mathcal{X}^\star$ such that $\Omega = \hat{\omega} + dd^c\Phi$ on $\mathcal{X}^\star$. Taking the restriction of the latter equality to X_τ ($\tau \neq 0$) we have $\Omega_\tau = \rho(\tau^{-1})^*\omega + dd^c(\Phi_{|X_\tau})$, pulling back by $\rho(\tau)$ yields $\omega_\tau - \omega = dd^c\phi_\tau$.

(ii) By the relation (4.6) and the fact that the action of $\mathbb{T}$ preserves the fibers we obtain that $m^{\xi}_{\hat{\omega}} := m^{\xi}_{\Omega} - (d^c\Phi)(\xi)$ is a momentum map of $(X_\tau, \hat{\omega}_{|X_\tau})$. It thus follows from Lemmas 1 and 22 that $m_{\hat{\omega}}(X_\tau) = \mathrm{P}$. $\square$

The main result of this section is the following theorem which extends the results from [37, 46] to arbitrary values of v, w:

Theorem 11. *Let $(\mathcal{X}, \mathcal{A}, \mathbb{T})$ be a smooth $\mathbb{T}$-compatible Kähler test configuration, for a compact Kähler manifold $(X, \alpha, \mathbb{T})$ and $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$, $\mathrm{w} \in C^\infty(\mathrm{P}, \mathbb{R})$ are weight functions. If the central fiber X_0 is reduced, then*

$$\lim_{t \to +\infty} \frac{\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)}{t} = \mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}, \mathcal{A}).$$

where $\phi_t := \phi_\tau$ with $\tau = e^{-t+is}$ is given by (4.7). In particular, if $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ is bounded from bellow, then $\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}, \mathcal{A}) \geq 0$.

Before we give the proof we need a couple of technical lemmas.

Lemma 24. *Under the hypotheses of Theorem 11, we have*

$$\lim_{t \to +\infty} \frac{\mathcal{E}_{\mathrm{w}}(\phi_t)}{t} = \int_{\mathcal{X}} \mathrm{w}(m_\Omega)\Omega^{[n+1]}. \tag{4.8}$$

Proof. We will start by showing as in [37, 46, 92] that,

$$\pi_\star\big(\mathrm{w}(m_\Omega)\Omega^{[n+1]}\big) = dd^c\mathcal{E}_{\mathrm{w}}(\phi_\tau) \tag{4.9}$$

on $\mathbb{C}^\star \subset \mathbb{P}^1$, in the sens of currents. From the very definition of the functional $\mathcal{E}_{\mathrm{w}}$ (see (2.20)) we have

$$\begin{aligned}
\mathcal{E}_{\mathrm{w}}(\phi_\tau) &= \int_0^1 \left(\int_X \phi_\tau \mathrm{w}(m_{\epsilon\phi_\tau})\omega_{\epsilon\phi_\tau}^{[n]}\right) d\epsilon \\
&= \int_0^1 \left(\int_X \phi_\tau \mathrm{w}(\epsilon m_\tau + (1-\epsilon)m_\omega)(\epsilon\omega_\tau + (1-\epsilon)\omega)^{[n]}\right) d\epsilon \\
&= \int_0^1 \left(\int_{X_\tau} (\Phi \mathrm{w}(m_{\Omega_\epsilon})\Omega_\epsilon^{[n]})_{|X_\tau}\right) d\epsilon
\end{aligned}$$

where $\Omega_\epsilon := \epsilon\Omega + (1-\epsilon)\hat{\omega}$, $m_{\Omega_\epsilon} := \epsilon m_\Omega + (1-\epsilon)m_{\hat{\omega}}$, and $\hat{\omega}$, Φ are given in Lemma 23. It thus follows that $\mathcal{E}_{\mathrm{w}}(\phi_\tau)$ extends to a smooth function on $\mathbb{P}^1 \setminus \{0\}$. Let $f(\tau)$ be a smooth function with compact support in $\mathbb{C}^\star \subset \mathbb{P}^1$. Letting $\hat{f} := \pi^\star f$ we have

$$
\begin{aligned}
\langle dd^c\mathcal{E}_{\mathrm{w}}(\phi_\tau), f\rangle &= \int_0^1 \left(\int_{\mathbb{C}^\star} dd^c f(\tau) \int_{X_\tau} (\Phi \mathrm{w}(m_{\Omega_\epsilon})\Omega_\epsilon^{[n]})_{|X_\tau}\right) d\epsilon \\
&= \int_0^1 \left(\int_{\mathcal{X}^\star} \Phi \mathrm{w}(m_{\Omega_\epsilon}) dd^c \hat{f} \wedge \Omega_\epsilon^{[n]}\right) d\epsilon \\
&= -\int_0^1 \left(\int_{\mathcal{X}^\star} \Phi d\hat{f} \wedge d^c \mathrm{w}(m_{\Omega_\epsilon}) \wedge \Omega_\epsilon^{[n]}\right) d\epsilon \\
&\quad - \int_0^1 \left(\int_{\mathcal{X}^\star} \mathrm{w}(m_{\Omega_\epsilon}) d\hat{f} \wedge d^c\Phi \wedge \Omega_\epsilon^{[n]}\right) d\epsilon \\
&= -\int_0^1 \left(\int_{\mathcal{X}^\star} \Phi d\hat{f} \wedge d^c \mathrm{w}(m_{\Omega_\epsilon}) \wedge \Omega_\epsilon^{[n]}\right) d\epsilon \\
&\quad + \int_0^1 \left(\int_{\mathcal{X}^\star} \hat{f} \mathrm{w}(m_{\Omega_\epsilon}) dd^c\Phi \wedge \Omega_\epsilon^{[n]}\right) d\epsilon \\
&\quad + \int_0^1 \left(\int_{\mathcal{X}^\star} \hat{f} d\mathrm{w}(m_{\Omega_\epsilon}) \wedge d^c\Phi \wedge \Omega_\epsilon^{[n]}\right) d\epsilon.
\end{aligned} \tag{4.10}
$$

The first integral in the last equality vanishes. Indeed, for a basis $(\xi_i)_{i=1,\cdots,\ell}$ of $\mathfrak{t}$ we have

$$
d\hat{f} \wedge d^c \mathrm{w}(m_{\hat{\Omega}_\epsilon}) \wedge \Omega_\epsilon^{[n]} = \sum_{i=1}^{\ell} \mathrm{w}_{,i}(m_{\Omega_\epsilon})(df)(\pi_*\xi_i)\Omega_\epsilon^{[n+1]} = 0,
$$

since the action of $\mathbb{T}$ preserves the fibers of $\mathcal{X} \to \mathbb{P}^1$. For the remaining integrals in the last equality in (4.10), integration by parts in the variable ϵ gives

$$
\begin{aligned}
&\int_0^1 \left(\int_{\mathcal{X}^\star} \hat{f} \mathrm{w}(m_{\Omega_\epsilon}) dd^c\Phi \wedge \Omega_\epsilon^{[n]}\right) d\epsilon \\
&= \int_0^1 \left(\int_{\mathcal{X}^\star} \hat{f} \mathrm{w}(m_{\Omega_\epsilon}) \frac{d}{d\epsilon}\Omega_\epsilon^{[n+1]}\right) d\epsilon \quad (\text{since } \Omega_\epsilon := \hat{\omega} + \epsilon dd^c\Phi) \\
&= \int_{\mathcal{X}^\star} \hat{f} \mathrm{w}(m_\Omega)\Omega^{[n+1]} - \int_0^1 \left(\int_{\mathcal{X}^\star} \hat{f} \left(\frac{d}{d\epsilon}\mathrm{w}(m_{\Omega_\epsilon})\right) \Omega_\epsilon^{[n+1]}\right) d\epsilon \\
&= \int_{\mathcal{X}^\star} \hat{f} \mathrm{w}(m_\Omega)\Omega^{[n+1]} - \int_0^1 \left(\int_{\mathcal{X}^\star} \hat{f} d\mathrm{w}(m_{\Omega_\epsilon}) \wedge d^c\Phi \wedge \Omega_\epsilon^{[n]}\right) d\epsilon,
\end{aligned} \tag{4.11}
$$

where for passing from the third line to the last line we used the following

$$
\begin{aligned}
\left(\frac{d}{d\epsilon}\mathrm{w}(m_{\Omega_\epsilon})\right) \Omega_\epsilon^{[n+1]} &= \sum_{i=1}^{\ell} \mathrm{w}_{,i}(m_{\Omega_\epsilon}) d^c\Phi(\xi_i)\Omega_\epsilon^{[n+1]} \\
&= \sum_{i=1}^{\ell} \mathrm{w}_{,i}(m_{\Omega_\epsilon}) dm_{\Omega_\epsilon}^{\xi_i} \wedge d^c\Phi \wedge \Omega_\epsilon^{[n]} \\
&= \mathrm{w}(m_{\Omega_\epsilon}) \wedge d^c\Phi \wedge \Omega_\epsilon^{[n]}.
\end{aligned}
$$

By substituting (4.11) in (4.10) we get (4.9).

Now we establish (4.8) using (4.9), following the proof [46, Theorem 4.9]. Let $\mathbb{D}_\epsilon \subset \mathbb{C}$ be the disc of center 0 and radius $\epsilon > 0$. Using the change of coordinates (t, s) given by $\tau = e^{-t+is} \in \mathbb{C}$ and the $\mathbb{S}^1$-invariance of $\mathcal{E}_{\mathrm{w}}(\phi_\tau)$ we calculate

$$\begin{aligned}
\int_{\mathcal{X}} \mathrm{w}(m_\Omega)\Omega^{[n+1]} =& \lim_{\epsilon\to 0} \int_{\mathcal{X}\setminus\pi^{-1}(\mathbb{D}_\epsilon)} \mathrm{w}(m_\Omega)\Omega^{[n+1]} \\
=& \lim_{\epsilon\to 0} \int_{\mathbb{P}^1\setminus\mathbb{D}_\epsilon} \pi_\star\big(\mathrm{w}(m_\Omega)\Omega^{[n+1]}\big) \\
=& \lim_{\epsilon\to 0} \int_{\mathbb{P}^1\setminus\mathbb{D}_\epsilon} dd^c\mathcal{E}_{\mathrm{w}}(\phi_\tau) \quad \text{by (4.9)} \\
=& \lim_{\epsilon\to 0} \left(\frac{d}{dt}\Big|_{t=-\log\epsilon} \mathcal{E}_{\mathrm{w}}(\phi_t)\right) \quad \text{by the Green-Riesz formula} \\
=& \lim_{t\to+\infty} \frac{d}{dt}\mathcal{E}_{\mathrm{w}}(\phi_t) = \lim_{t\to+\infty} \frac{\mathcal{E}_{\mathrm{w}}(\phi_t)}{t}.
\end{aligned}$$

□

Let $\Omega \in \mathcal{A}$ be $\mathbb{G}$-invariant Kähler form. We consider the Kähler metric on $\mathcal{X}^\star$ given by $\hat{\omega} + \pi^*\omega_{\mathrm{FS}} = \lambda^*(\pi_X^*\omega + \pi_{\mathbb{P}^1}^*\omega_{\mathrm{FS}})$ (by the equivariance of λ), where $\hat{\omega} := (\pi_X \circ \lambda)^*\omega$ with λ the map defined by (4.1) and $\pi_X, \pi_{\mathbb{P}^1}$ denote the projections on the factors of $X \times (\mathbb{P}^1 \setminus \{0\})$. Then we have on $\mathcal{X}^\star$

$$\mathrm{Ric}(\Omega) - \pi^*\omega_{\mathrm{FS}} - \widehat{\mathrm{Ric}(\omega)} = \frac{1}{2}dd^c\Psi, \tag{4.12}$$

where $\Psi = \log\left(\frac{\Omega^{n+1}}{\hat{\omega}^n\wedge\pi^*\omega_{\mathrm{FS}}}\right)$ and $\widehat{\mathrm{Ric}(\omega)} := (\pi_X \circ \lambda)^*\mathrm{Ric}(\omega)$. Using (4.12) and Lemma 5 2, we obtain on $\mathcal{X}^\star$

$$m^\xi_{\widehat{\mathrm{Ric}(\omega)}} = m^\xi_{\mathrm{Ric}(\Omega)} + \frac{1}{2}(d^c\Psi)(\xi), \tag{4.13}$$

for any $\xi \in \mathfrak{t}$, where $m_{\widehat{\mathrm{Ric}(\Omega)}} := (\pi_X \circ \lambda)^* m_{\mathrm{Ric}(\omega)}$.

Lemma 25. *Under the hypotheses of Theorem 11, we have*

$$dd^c\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_\tau) = \pi_*\left(\mathrm{v}(m_\Omega)\widehat{\mathrm{Ric}(\omega)} \wedge \Omega^{[n]} + \langle (d\mathrm{v})(m_\Omega), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle\Omega^{[n+1]}\right). \tag{4.14}$$

Proof. From the very definition of $\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}$ (see (2.21)) we have

$$\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_\tau) = \int_0^1 \left(\int_{X_\tau} \left[\Phi\big(\mathrm{v}(m_{\Omega_\epsilon})\widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n-1]} + \langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle \Omega_\epsilon^{[n]}\big)\right]_{|X_\tau}\right) d\epsilon,$$

where $\Omega_\epsilon := \epsilon\Omega + (1-\epsilon)\hat{\omega}$, $m_{\Omega_\epsilon} := \epsilon m_\Omega + (1-\epsilon)m_{\hat{\omega}}$, and $\hat{\omega}$, Φ are given in Lemma 23. As in the proof of Lemma 24, we see that $\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_\tau)$ extends to a smooth function on $\mathbb{P}^1 \setminus \{0\}$. Furthermore, for any smooth function $f(\tau)$ with compact support in $\mathbb{C}^\star \subset \mathbb{P}^1$, we have

$$
\begin{aligned}
&\langle dd^c\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_\tau), f\rangle = \int_{\mathbb{C}^\star} \mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_\tau) dd^c f = \\
=&\int_0^1 \Big(\int_{\mathbb{C}^\star} dd^c f \int_{X_\tau} \big[\Phi\big(\mathrm{v}(m_{\Omega_\epsilon})\widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n-1]} + \langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle \Omega_\epsilon^{[n]}\big)\big]_{|X_\tau}\Big) d\epsilon \\
=&\int_0^1 \Big(\int_{\mathcal{X}^\star} \hat{f} dd^c\big[\Phi\big(\mathrm{v}(m_{\Omega_\epsilon})\widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n-1]} + \langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle \Omega_\epsilon^{[n]}\big)\big]\Big) d\epsilon \\
=&-\int_0^1 \Big(\int_{\mathcal{X}^\star} \Phi\big[d\hat{f} \wedge d^c(\mathrm{v}(m_{\Omega_\epsilon})) \wedge \widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n-1]} + d\hat{f} \wedge d^c(\langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle) \wedge \Omega_\epsilon^{[n]}\big]\Big) d\epsilon \\
&+\int_0^1 \Big(\int_{\mathcal{X}^\star} \hat{f}\big[d(\langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle) \wedge d^c\Phi \wedge \Omega_\epsilon^{[n]} + d(\mathrm{v}(m_{\Omega_\epsilon})) \wedge d^c\Phi \wedge \widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n-1]}\big]\Big) d\epsilon \\
&+\int_0^1 \Big(\int_{\mathcal{X}^\star} \hat{f}\big[\mathrm{v}(m_{\Omega_\epsilon})\widehat{\mathrm{Ric}(\omega)} \wedge (dd^c\Phi) \wedge \Omega_\epsilon^{[n-1]} + \langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle (dd^c\Phi) \wedge \Omega_\epsilon^{[n]})\big]\Big) d\epsilon \\
=&I_1 + I_2 + I_3,
\end{aligned}
$$

where I_1, I_2, I_3 respectively denote the integrals on the first, second and third lines of the last equality. Now we compute each integral individually. We have

$$
\begin{aligned}
&d\hat{f} \wedge d^c(\langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle) \wedge \Omega_\epsilon^{[n]} + d\hat{f} \wedge d^c(\mathrm{v}(m_{\Omega_\epsilon})) \wedge \widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n-1]} \\
=&\sum_{i,j} \mathrm{v}_{,ij}(m_{\Omega_\epsilon})(d\hat{f})(\xi_j) m^{\xi_i}_{\widehat{\mathrm{Ric}(\omega)}} \Omega_\epsilon^{[n+1]} + \sum_i \mathrm{v}_{,i}(m_{\Omega_\epsilon}) d\hat{f} \wedge d^c m^{\xi_i}_{\widehat{\mathrm{Ric}(\omega)}} \wedge \Omega_\epsilon^{[n]} \\
&+\sum_i \mathrm{v}_{,i}(m_{\Omega_\epsilon})(d\hat{f})(\xi_i)(\Lambda_{\Omega_\epsilon}\widehat{\mathrm{Ric}(\omega)})\Omega_\epsilon^{[n+1]} - (d\hat{f} \wedge d^c(\mathrm{v}(m_{\Omega_\epsilon})), \widehat{\mathrm{Ric}(\omega)})_{\Omega_\epsilon}\Omega_\epsilon^{[n+1]} \\
=&\sum_i \mathrm{v}_{,i}(m_{\Omega_\epsilon})(d\hat{f} \wedge d^c m^{\xi_i}_{\Omega_\epsilon}, \widehat{\mathrm{Ric}(\omega)})\Omega_\epsilon^{[n+1]} - (d\hat{f} \wedge d^c(\mathrm{v}(m_{\Omega_\epsilon})), \widehat{\mathrm{Ric}(\omega)})\Omega_\epsilon^{[n+1]} = 0,
\end{aligned}
$$

where $\boldsymbol{\xi} = (\xi_i)_{i=1,\cdots,\ell}$ is a basis of $\mathfrak{t}$. It follows that $I_1 = 0$. For the integral I_2, a similar calculation gives

$$
I_2 = \int_0^1 \Big(\int_{\mathcal{X}^\star} \hat{f}\big[\sum_i \mathrm{v}_{,i}(m_{\Omega_\epsilon})(d^c\Phi)(\xi_i)\widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n]} + \sum_{i,j} \mathrm{v}_{,ij}(m_{\Omega_\epsilon}) m^{\xi_i}_{\widehat{\mathrm{Ric}(\omega)}}(d^c\Phi)(\xi_j)\Omega_\epsilon^{[n]}\big]\Big) d\epsilon.
$$

Now we consider the integral I_3. Using the fact that $\Omega_\epsilon = \hat{\omega} + \epsilon dd^c\Phi$, an integration by parts with respect to ϵ gives

$$
\begin{aligned}
I_3 =& \int_0^1 \Big(\int_{\mathcal{X}^\star} \hat{f}\big[\mathrm{v}(m_{\Omega_\epsilon})\widehat{\mathrm{Ric}(\omega)} \wedge (\frac{d}{d\epsilon}\Omega_\epsilon^{[n]}) + \langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle (\frac{d}{d\epsilon}\Omega_\epsilon^{[n+1]})\big]\Big) d\epsilon \\
=& \int_{\mathcal{X}^\star} \hat{f}\big[\mathrm{v}(m_\Omega)\widehat{\mathrm{Ric}(\omega)} \wedge \Omega^{[n]} + \langle (d\mathrm{v})(m_\Omega), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle \Omega^{[n+1]}\big] \\
&- \int_0^1 \Big(\int_{\mathcal{X}^\star} \hat{f}\big[(\frac{d}{d\epsilon}\mathrm{v}(m_{\Omega_\epsilon}))\widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n]} + (\frac{d}{d\epsilon}\langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle)\Omega_\epsilon^{[n+1]}\big]\Big) d\epsilon,
\end{aligned}
$$

By Lemma 23 2 the integral on the last line is given by of the last equality is given by

$$
\begin{aligned}
&\int_0^1 \Big(\int_{\mathcal{X}^\star} \hat{f}\big[(\frac{d}{d\epsilon}\mathrm{v}(m_{\Omega_\epsilon}))\widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n]} + (\frac{d}{d\epsilon}\langle (d\mathrm{v})(m_{\Omega_\epsilon}), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle)\Omega_\epsilon^{[n+1]}\big]\Big) d\epsilon \\
=&\int_0^1 \Big(\int_{\mathcal{X}^\star} \hat{f}\big[\sum_i \mathrm{v}_{,i}(m_{\Omega_\epsilon})(d^c\Phi)(\xi_i)\widehat{\mathrm{Ric}(\omega)} \wedge \Omega_\epsilon^{[n]} + \sum_{i,j} \mathrm{v}_{,ij}(m_{\Omega_\epsilon}) m^{\xi_i}_{\widehat{\mathrm{Ric}(\omega)}}(d^c\Phi)(\xi_j)\Omega_\epsilon^{[n]}\big]\Big) d\epsilon \\
=&I_2.
\end{aligned}
$$

It follows that

$$
I_1 + I_2 + I_3 = \int_{\mathcal{X}^\star} \hat{f}\big[\mathrm{v}(m_\Omega)\widehat{\mathrm{Ric}(\omega)} \wedge \Omega^{[n]} + \langle (d\mathrm{v})(m_\Omega), m_{\widehat{\mathrm{Ric}(\omega)}}\rangle \Omega^{[n+1]}\big].
$$

This completes the proof. □

Lemma 26. *Under the hypotheses of Theorem 11,*

$$
\begin{aligned}
&\lim_{t\to+\infty} \frac{1}{t}\Big(\int_{X_1} \psi_t \mathrm{v}(m_{\phi_t})\omega_{\phi_t}^{[n]} - 2\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_t)\Big) \\
=& -2\int_{\mathcal{X}} \mathrm{v}(m_\Omega)(\mathrm{Ric}(\Omega) - \pi^\star\omega_{\mathrm{FS}}) \wedge \Omega^{[n]} + \langle (d\mathrm{v})(m_\Omega), m_{\mathrm{Ric}(\Omega)}\rangle \Omega^{[n+1]}
\end{aligned}
\tag{4.15}
$$

where ϕ_t is given by (4.7) and $\psi_t = \psi_\tau$ with $\tau = e^{-t+is}$ is given by

$$
\psi_\tau := \rho(\tau)^*(\Psi_{|X_\tau}) \in C^\infty(X_1, \mathbb{R})^{\mathbb{T}}. \tag{4.16}
$$

Proof. We define on $\mathbb{C}^\star$ the function $\mathcal{H}(\tau) := \int_X \psi_\tau \mathrm{v}(m_\tau)\omega_\tau^{[n]}$. Let $f(\tau)$ be a test function with support in $\mathbb{C}^\star \subset \mathbb{P}^1$ and $\hat{f} := \pi^\star f$. We have

$$
\begin{aligned}
\langle dd^c\mathcal{H}, f\rangle =& \int_{\mathbb{C}^\star} dd^c f \int_{X_\tau} (\Psi \mathrm{v}(m_\Omega)\Omega^{[n]})_{|X_\tau} \\
=& \int_{\mathcal{X}^\star} \Psi \mathrm{v}(m_\Omega) dd^c\hat{f} \wedge \Omega^{[n]} \\
=& \int_{\mathcal{X}^\star} \Psi d(\mathrm{v}(m_\Omega)) \wedge d^c\hat{f} \wedge \Omega^{[n]} - \int_{\mathcal{X}^\star} \mathrm{v}(m_\Omega) d\Psi \wedge d^c\hat{f} \wedge \Omega^{[n]}
\end{aligned}
$$

Notice that $d(\mathrm{v}(m_\Omega)) \wedge d^c\hat{f} \wedge \Omega^n = 0$ since the 1-form $d^c\hat{f}$ is zero on the fundamental vector fields of the $\mathbb{T}$-action. Integration by parts gives

$$\langle dd^c\mathcal{H}, f\rangle = \int_{\mathcal{X}^\star} \hat{f} d\Psi \wedge d^c\mathrm{v}(m_\Omega) \wedge \Omega^{[n]} + \int_{\mathcal{X}^\star} \hat{f}\mathrm{v}(m_\Omega) dd^c\Psi \wedge \Omega^{[n]}.$$

Using the equations (4.12) and (4.13) we obtain

$$\begin{aligned}\langle dd^c\mathcal{H}, f\rangle =& -2\int_{\mathcal{X}^\star} \hat{f}\langle (d\mathrm{v})(m_\Omega), m_{\mathrm{Ric}(\Omega)} - m_{\widehat{\mathrm{Ric}(\omega)}}\rangle \Omega^{[n+1]} \\ & - \int_{\mathcal{X}^\star} \hat{f}\mathrm{v}(m_\Omega)(\mathrm{Ric}(\Omega) - 2\pi^\star\omega_{\mathrm{FS}} - \widehat{\mathrm{Ric}(\omega)}) \wedge \Omega^{[n]}.\end{aligned} \tag{4.17}$$

Combining (4.14) and (4.17) gives

$$\begin{aligned} & dd^c\big(\mathcal{H}(\tau) - 2\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_\tau)\big) \\ &\quad = -2\pi_\star\left(\mathrm{v}(m_\Omega)(\mathrm{Ric}(\Omega) - \pi^\star\omega_{\mathrm{FS}}) \wedge \Omega^{[n]} + \langle (d\mathrm{v})(m_\Omega), m_{\mathrm{Ric}(\Omega)}\rangle \Omega^{[n+1]}\right).\end{aligned}$$

We conclude in the same way as in the proof of Lemma 24. □

Let us now consider the following function on $\mathbb{C}^\star$:

$$\mathcal{M}_{\mathrm{v,w}}^{\Psi}(\phi_\tau) := \int_X \psi_\tau \mathrm{v}(m_{\phi_\tau})\omega_{\phi_\tau}^{[n]} - 2\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_\tau) + c_{(\mathrm{v,w})}(\alpha)\mathcal{E}_{\mathrm{v}}(\phi_\tau), \tag{4.18}$$

where ϕ_τ and ψ_τ are given by (4.7) and (4.16) respectively. From the definition of $\mathcal{M}_{\mathrm{v,w}}^{\Psi}(\phi_\tau)$ and Lemmas 24 and 26 we see that

$$\lim_{t\to+\infty} \frac{\mathcal{M}_{\mathrm{v,w}}^{\Psi}(\phi_t)}{t} = \mathcal{F}_{\mathrm{v,w}}(\mathcal{X}, \mathcal{A}). \tag{4.19}$$

Lemma 27. *If the central fiber X_0 is reduced, then the integral*

$$\Upsilon(\tau) := \int_{X_\tau} \log\left(\frac{\Omega^n \wedge \pi^\star\omega_{\mathrm{FS}}}{\Omega^{n+1}}\right) \mathrm{v}(m_\Omega)\Omega_\tau^{[n]},$$

is bounded on $\mathbb{C}^\star$.

Proof. The integral $\Upsilon(\tau)$ is bounded from above since $Z(\hat{x}) = \frac{\Omega^n\wedge\pi^\star\omega_{\mathrm{FS}}}{\Omega^{n+1}}$ is a non-negative smooth function on $\mathcal{X}$ and the integral $\int_{X_\tau} \mathrm{v}(m_\Omega)\Omega_\tau^{[n]}$ is independent from τ (see Lemma 22). Notice that $\Upsilon(\tau)$ is bounded if and only if $\int_{X_\tau} |\log(Z)|\mathrm{v}(m_\Omega)\Omega_\tau^{[n]}$ is bounded. Indeed, if $\Upsilon(\tau) = O(1)$ then

$$\int_{X_\tau} |\log(Z)|\mathrm{v}(m_\Omega)\Omega_\tau^{[n]} = \int_{X_\tau} (\log(Z) + |\log(Z)|)\mathrm{v}(m_\Omega)\Omega_\tau^{[n]} - \Upsilon(\tau) = \mathcal{O}(1).$$

It follows that $\int_{X_\tau} |\log(Z)|\mathrm{v}(m_\Omega)\Omega_\tau^{[n]} = \mathcal{O}(1)$. The converse follows from

$$|\Upsilon(\tau)| \leq \int_{X_\tau} |\log(Z)|\mathrm{v}(m_\Omega)\Omega_\tau^{[n]}.$$

Using that $\mathrm{v}(m_\Omega)$ is a smooth function on $\mathcal{X}$ we see that $\int_{X_\tau} |\log(Z)|\mathrm{v}(m_\Omega)\Omega_\tau^{[n]} = \mathcal{O}(1)$ if and only if $\int_{X_\tau} |\log(Z)|\Omega_\tau^{[n]} = \mathcal{O}(1)$, which is also equivalent to $\int_{X_\tau} \log(Z)\Omega_\tau^{[n]} = \mathcal{O}(1)$. By [37, Remark 4.12], if the central fiber X_0 is reduced then $\int_{X_\tau} \log(Z)\Omega_\tau^{[n]} = \mathcal{O}(1)$ which implies that $\Upsilon(\tau) = \mathcal{O}(1)$. □

Now we are in position to give a proof for Theorem 11.

Proof of Theorem 11. From the modified Chen-Tian formula in Theorem 6, (4.18) and by Lemma 27 we get

$$\begin{aligned}
\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau) - \mathcal{M}^{\Psi}_{\mathrm{v},\mathrm{w}}(\phi_\tau) &= \int_X \left(\log\left(\frac{\omega_\tau^n}{\omega^n}\right) - \psi_\tau\right)\mathrm{v}(m_\tau)\omega_\tau^{[n]} \\
&= \int_{X_\tau} \left(\log\left(\frac{\Omega^n \wedge \pi^*\omega_{\mathrm{FS}}}{\hat{\omega}^n \wedge \pi^*\omega_{\mathrm{FS}}}\right) - \Psi\right)\rho(\tau^{-1})^*(\mathrm{v}(m_\tau)\omega_\tau^{[n]}) \\
&= \int_{X_\tau} \log\left(\frac{\Omega^n \wedge \pi^*\omega_{\mathrm{FS}}}{\Omega^{n+1}}\right)\mathrm{v}(m_\Omega)\Omega_\tau^{[n]} = \mathcal{O}(1).
\end{aligned}$$

Dividing by t (where we recall $\tau = e^{-t+is}$) and passing to the limit when t goes to infinity concludes the proof. □

Proof of Theorems 4. This is a direct corollary of Theorem 3 from the introduction, together with Theorem 11. □

Proposition 9. *If $(\mathcal{X}, \mathcal{A}, \mathbb{T})$ is a Kähler test configuration of $(X, \alpha, \mathbb{T})$ such that $\pi : \mathcal{X} \to \mathbb{P}^1$ is a smooth submersion then*

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}, \mathcal{A}) = \mathcal{F}^{\alpha}_{\mathrm{v},\mathrm{w}}(V_\rho) - \frac{\mathrm{Vol}(\mathcal{X}, \mathcal{A})}{\mathrm{Vol}(X, \alpha)}\int_X \left(\mathrm{Scal}_{\mathrm{v}}(\omega) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\omega)\right)\omega^{[n]},$$

where V_ρ is the generator of the $\mathbb{S}^1_\rho$-action on X_0, and $\mathcal{F}^{\alpha}_{\mathrm{v},\mathrm{w}}(V_\rho)$ is the (v,w)-Futaki invariant of the smooth central fibre (X_0, α) introduced in Definition 8. In particular if $(X, \alpha, \mathbb{T})$ is (v,w)-semistable on smooth test configurations, then

$$\int_X \mathrm{Scal}_{\mathrm{v}}(\omega)\omega^{[n]} = c_{(\mathrm{v},\mathrm{w})}(\alpha)\int_X \mathrm{w}(m_\omega)\omega^{[n]} \ \textit{and}\ \mathcal{F}^{\alpha}_{\mathrm{v},\mathrm{w}} \equiv 0.$$

Proof. We just adapt the arguments from [39] to our weighted setting. From Definition 5 we have

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau) &= -\int_{X_1}\dot{\phi}_\tau\big(\mathrm{Scal}_{\mathrm{v}}(\omega_\tau) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\tau)\big)\omega_\tau^{[n]},\\ &= -\int_{X_\tau}\rho(\tau^{-1})^*\dot{\phi}_\tau\big(\mathrm{Scal}_{\mathrm{v}}(\Omega_\tau) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_{\Omega_\tau})\big)\Omega^{[n]},\end{aligned} \tag{4.20}$$

where $t = -\log|\tau|$, $\dot{\phi}_\tau = \frac{d\phi_\tau}{dt}$ and ω_τ, ϕ_τ, m_τ are given by (4.7) and (4.3). Note that the flow of the vector field $\mathcal{J}V_\rho$ is $\varphi^t_{\mathcal{J}V_\rho} = \rho(e^{-t})$ where $\mathcal{J}$ denotes the complex structure of $\mathcal{X}$. Let h^ρ be a Hamiltonian function of V_ρ with respect to Ω. We have $\frac{d}{dt}\rho(\tau)^*\Omega = -dd^c(\rho(\tau)^*h^\rho)$. On the other hand, using (4.7) we get $\frac{d}{dt}(\rho(\tau)^*\Omega)_{|X_1} = dd^c\dot{\phi}_\tau$. By taking the restriction on X_1 of the first equality and comparing to the secon, we get

$$h^\rho_{|X_\tau} = -\rho(\tau^{-1})^*\dot{\phi}_\tau + a(\tau), \tag{4.21}$$

where $a(\tau) \in \mathbb{R}$ is a constant depending on $\tau \in \mathbb{C}^*$. By (4.21) and Lemma 3, we have

$$a(\tau) = \frac{1}{\mathrm{Vol}(X,\alpha)}\Big(\int_{X_\tau} h^\rho\Omega^{[n]} + \frac{d\mathcal{E}_1(\phi_\tau)}{dt}\Big).$$

Using that $\pi : \mathcal{X} \to \mathbb{P}^1$ is a smooth submersion and Lemma 24, we get

$$\lim_{t\to\infty} a(\tau) = \frac{1}{\mathrm{Vol}(X,\alpha)}\Big(\int_{X_0} h^\rho\Omega^{[n]} + \mathrm{Vol}(\mathcal{X},\mathcal{A})\Big). \tag{4.22}$$

Substituting (4.21) in (4.20), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau) = &\int_{X_\tau}\big(\mathrm{Scal}_{\mathrm{v}}(\Omega_\tau) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_{\Omega_\tau})\big)h^\rho\Omega^{[n]}\\ &- a(\tau)\int_{X_\tau}\big(\mathrm{Scal}_{\mathrm{v}}(\Omega_\tau) - c_{\mathrm{v},\mathrm{w}}([\Omega_\tau])\mathrm{w}(m_{\Omega_\tau})\big)\Omega^{[n]}.\end{aligned} \tag{4.23}$$

Passing to the limit when $t \to \infty$ in (4.23) and using Theorem 11, we obtain

$$
\begin{aligned}
\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X},\mathcal{A}) &= \lim_{t\to\infty}\frac{d}{dt}\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau)\\
&= \int_{X_0}\big(\mathrm{Scal}_{\mathrm{v}}(\Omega_0) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_{\Omega_0})\big)h^\rho\Omega^{[n]}\\
&- \frac{1}{\mathrm{Vol}(X,\alpha)}\Big(\int_{X_0} h^\rho\Omega^{[n]} + \mathrm{Vol}(\mathcal{X},\mathcal{A})\Big)\int_{X_0}\big(\mathrm{Scal}_{\mathrm{v}}(\Omega_0) - c_{\mathrm{v},\mathrm{w}}([\Omega_0])\mathrm{w}(m_{\Omega_0})\big)\Omega^{[n]}\\
&= \int_{X_0}\big(\mathrm{Scal}_{\mathrm{v}}(\Omega_0) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_{\Omega_0})\big)\Big(h^\rho - \frac{1}{\mathrm{Vol}(X,\alpha)}\int_{X_0} h^\rho\Omega^{[n]}\Big)\Omega^{[n]}\\
&- \frac{\mathrm{Vol}(\mathcal{X},\mathcal{A})}{\mathrm{Vol}(X,\alpha)}\int_{X_0}\big(\mathrm{Scal}_{\mathrm{v}}(\Omega_0) - c_{\mathrm{v},\mathrm{w}}([\Omega_0])\mathrm{w}(m_{\Omega_0})\big)\Omega^{[n]}\\
&= \mathcal{F}^\alpha_{\mathrm{v},\mathrm{w}}(V_\rho) - \frac{\mathrm{Vol}(\mathcal{X},\mathcal{A})}{\mathrm{Vol}(X,\alpha)}\int_X\big(\mathrm{Scal}_{\mathrm{v}}(\omega) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\omega)\big)\omega^{[n]}.
\end{aligned}
$$

where $\Omega_0 = \Omega_{|X_0} \in \mathcal{A}_{|X_0}$, and we have used in the last equality that for any $\tau \in \mathbb{C}^\star$ we have

$$
\begin{aligned}
\int_{X_\tau}\mathrm{Scal}_{\mathrm{v}}(\Omega_\tau)\Omega^{[n]} &= \int_{X_1}\mathrm{Scal}_{\mathrm{v}}(\omega_\tau)\omega_\tau^{[n]} = \int_X \mathrm{Scal}_{\mathrm{v}}(\omega)\omega^{[n]},\\
\int_{X_\tau}\mathrm{w}(m_{\Omega_\tau})\Omega^{[n]} &= \int_{X_1}\mathrm{w}(m_{\omega_\tau})\omega_\tau^{[n]} = \int_X \mathrm{w}(m_\omega)\omega^{[n]},
\end{aligned}
$$

see Lemma 2.

For the second statement, as $\int_X\big(\mathrm{Scal}_{\mathrm{v}}(\omega) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\omega)\big)\omega^{[n]} = 0$ by the definition of semi-stability, we consider the product test configurations associated to V and $-V$ for any $V \in \mathfrak{h}_{\mathrm{red}}$, we obtain $\mathcal{F}^\alpha_{\mathrm{v},\mathrm{w}}(V) = -\mathcal{F}^\alpha_{\mathrm{v},\mathrm{w}}(V) \geq 0$ i.e. $\mathcal{F}^\alpha_{\mathrm{v},\mathrm{w}} \equiv 0$. □

Remark 7. In [36], Dervan defines a $\mathbb{T}$-relative Donaldson–Futaki invariant $\mathrm{DF}_{\mathbb{T}}(\mathcal{X},\mathcal{A})$ for a smooth $\mathbb{T}$-compatible Kähler test configuration $\mathcal{X}$ as follows

$$
\mathrm{DF}_{\mathbb{T}}(\mathcal{X},\mathcal{A}) := \mathcal{F}_{1,1}(\mathcal{X},\mathcal{A}) - \sum_{i=1}^{\ell}\frac{\langle h_\rho, h_i\rangle_{X_0}}{\langle h_i, h_i\rangle_{X_0}}\mathcal{F}^\alpha_{1,1}(\xi_i),
$$

where $\boldsymbol{\xi} := (\xi_i)_{i=1,\cdots,\ell}$ is a basis of $\mathfrak{t}$ with corresponding Killing potentials $h_i = f_i(m_\Omega) = \langle m_\Omega, \xi_i\rangle + \lambda_i$, such that $\langle h_i, h_j\rangle_{X_0} = \int_{X_0} h_i h_j \Omega^n = 0$ for $i \neq j$ and $\int_{X_0} h_i\Omega^n = 0$, where the integration on X_0 is defined by $\int_{X_0} := \sum_i m_i \int_{(X_0^{(i)})_{\mathrm{reg}}}$ with $[X_0] = \sum_i m_i X_0^{(i)}$ being the analytic cycle associated to X_0 and $(X_0^{(i)})_{\mathrm{reg}}$ standing for the regular part of the irreducible component $X_0^{(i)}$ of X_0. Using Lemma 22, we have

$$\begin{aligned} &\int_X \mathrm{w}_{\mathrm{ext}}(m_\omega)\omega^{[n]} = \int_{X_1} \mathrm{w}_{\mathrm{ext}}(m_\Omega)\Omega^{[n]} = \int_{X_\tau} \mathrm{w}_{\mathrm{ext}}(m_\Omega)\Omega^{[n]}, \\ &\mathcal{F}_{1,1}(\xi_i) = \langle \mathrm{w}_{\mathrm{ext}}(m_\omega), h_i\rangle_X = \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), f_i(m_\Omega)\rangle_{X_1} = \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), h_i\rangle_{X_\tau}, \end{aligned} \tag{4.24}$$

for any $\tau \in \mathbb{C}^\star \subset \mathbb{P}^1$. As the family $\pi : \mathcal{X} \to \mathbb{P}^1$ is proper and flat, the current of integration along the fibers X_τ is continuous and converges to the integration over the analytic cycle of the central fiber $[X_0]$ (see [14]). Passing to the limit when $\tau \to 0$ in (4.24), we thus obtain $\int_X \mathrm{w}_{\mathrm{ext}}(m_\omega)\omega^{[n]} = \int_{X_0} \mathrm{w}_{\mathrm{ext}}(m_\Omega)\Omega^{[n]}$ and $\mathcal{F}_{1,1}(\xi_i) = \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), h_{(\rho_i,\Omega)}\rangle_{X_0}$. Thus,

$$\mathrm{DF}_{\mathbb{T}}(\mathcal{X},\mathcal{A}) = \mathcal{F}_{1,1}(\mathcal{X},\mathcal{A}) - \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), h_\rho\rangle_{X_0}. \tag{4.25}$$

On the other hand, the $(1, \mathrm{w}_{\mathrm{ext}})$-Futaki invariant of $(\mathcal{X},\mathcal{A})$ is given by

$$\mathcal{F}_{1,\mathrm{w}_{\mathrm{ext}}}(\mathcal{X},\mathcal{A}) = -\int_{\mathcal{X}} \mathrm{Scal}(\Omega)\Omega^{[n+1]} + 2\int_{\mathcal{X}} \pi^\star\omega_{\mathrm{FS}} \wedge \Omega^{[n]} + \int_{\mathcal{X}} \mathrm{w}_{\mathrm{ext}}(m_\Omega)\Omega^{[n+1]}. \tag{4.26}$$

(Recall that $c_{(1,\mathrm{w}_{\mathrm{ext}})}(\alpha) = 1$, see Section 2.2.2). From (4.25) and (4.26), we infer

$$\begin{aligned} \mathcal{F}_{1,\mathrm{w}_{\mathrm{ext}}}(\mathcal{X},\mathcal{A}) - \mathrm{DF}_{\mathbb{T}}(\mathcal{X},\mathcal{A}) =& \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), h_\rho\rangle_{X_0} + \int_{\mathcal{X}} (\mathrm{w}_{\mathrm{ext}}(m_\Omega) - c_{1,1}(\alpha))\Omega^{[n+1]} \\ =& \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), h_\rho\rangle_{X_0} + \lim_{t\to\infty} \frac{d\mathcal{E}_{\mathring{\mathrm{w}}_{\mathrm{ext}}}(\phi_\tau)}{dt} \\ =& \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), h_\rho\rangle_{X_0} + \lim_{t\to\infty}\Big(\int_{X_1} \dot{\phi}_\tau \mathring{\mathrm{w}}_{\mathrm{ext}}(m_\tau)\omega_\tau^{[n]}\Big) \\ =& \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), h_\rho\rangle_{X_0} - \lim_{t\to\infty}\Big(\int_{X_\tau} h_\rho \mathring{\mathrm{w}}_{\mathrm{ext}}(m_\Omega)\Omega^{[n]}\Big) \\ =& \langle \mathrm{w}_{\mathrm{ext}}(m_\Omega), h_\rho\rangle_{X_0} - \int_{X_0} h_\rho \mathring{\mathrm{w}}_{\mathrm{ext}}(m_\Omega)\Omega^{[n]} = 0, \end{aligned}$$

where in the second equality we used Lemma 24 for

$$\mathring{\mathrm{w}}_{\mathrm{ext}} = \mathrm{w}_{\mathrm{ext}} - c_{1,1}(\alpha) = \mathrm{w}_{\mathrm{ext}} - \frac{1}{\mathrm{Vol}(X,\alpha)}\int_{X_\tau} \mathrm{w}_{\mathrm{ext}}(m_\Omega)\Omega^{[n]},$$

for any $\tau \in \mathbb{C}^\star$ and in the fourth equality we used (4.21). It follows that

$$\mathcal{F}_{1,\mathrm{w}_{\mathrm{ext}}}(\mathcal{X},\mathcal{A}) = \mathrm{DF}_{\mathbb{T}}(\mathcal{X},\mathcal{A}).$$

◇

4.2 Algebraic definition of a (v, w)-Donaldson-Futaki invariant

4.2.1 The (v, w)-Donaldson-Futaki invariant of a smooth polarized variety.

Let (X, L) be a smooth polarized projective variety endowed with a torus action $\mathbb{T} \subset \mathrm{Aut}(X, L)$ with corresponding polytope $\mathrm{P}_L \subset \mathfrak{t}^*$ as in Section 3.1. For any $\mathbb{C}^*$-action ρ commuting with $\mathbb{T}$ and a family $\boldsymbol{\xi}$ of $\mathbb{S}^1$-generators of $\mathbb{T}$, we consider the weight

$$W_{\mathrm{v}}^{(k)}(\boldsymbol{\xi}, \rho) := \mathrm{tr}\left(\mathrm{v}\big(k^{-1}\boldsymbol{A}_{\boldsymbol{\xi}}^{(k)}\big) \cdot k^{-1}A_{\rho}^{(k)}\right),$$

where $A_{\rho}^{(k)}$ is the induced infinitisimal action of ρ on $\mathcal{H}_k$ and v is a smooth weight function on P_L. By Lemma 17, $W_{\mathrm{w}}(\boldsymbol{\xi}, \rho)$ admits an asymptotic expansion

$$W_{\mathrm{v}}^{(k)}(\boldsymbol{\xi}, \rho) = a_{\mathrm{v}}^{(0)}(\boldsymbol{\xi}, \rho)k^n + a_{\mathrm{v}}^{(1)}(\boldsymbol{\xi}, \rho)k^{n-1} + \mathcal{O}(k^{n-2}).$$

Thus we obtain a quantized version of the (v, w)-Futaki invariant of $(X, 2\pi c_1(L))$:

Corollary 8. *The* (v, w)*-Futaki invariant introduced in Definition 8 with respect to the Kähler class $\alpha := 2\pi c_1(L)$ satisfies*

$$\frac{1}{4(2\pi)^n}\mathcal{F}_{\mathrm{v,w}}^{\alpha}(V_\rho) = a_{\mathrm{v}}^{(1)}(\boldsymbol{\xi}, \rho) - \frac{c_{\mathrm{v,w}}(L)}{4}a_{\mathrm{w}}^{(0)}(\boldsymbol{\xi}, \rho),$$

where V_ρ is the generator of the $\mathbb{S}^1_\rho$-action on X, and $c_{\mathrm{v,w}}(L)$ is the (v, w)*-slope of $(X, 2\pi c_1(L))$ defined in* (2.6).

4.2.2 The (v, w)-Donaldson-Futaki invariant of a polarized test configuration

Following [42], we consider a (possibly singular) polarized test configuration of exponent $r \in \mathbb{N}$, compatible with $(X, L, \mathbb{T})$, defined as follows:

Definition 20. A $\mathbb{T}$*-compatible polarized test configuration* $(\mathcal{X}, \mathcal{L})$ of exponent $r \in \mathbb{N}$ associated to the smooth polarized variety (X, L) is a normal polarized variety $(\mathcal{X}, \mathcal{L}, \hat{\mathbb{T}})$ endowed with a torus $\hat{\mathbb{T}} \subset \mathrm{Aut}(\mathcal{X}, \mathcal{L})$ and

- a flat morphism $\pi : \mathcal{X} \to \mathbb{P}^1$ such that the torus action $\hat{\mathbb{T}}$ on $\mathcal{X}$ preserves each fiber $X_\tau := \pi^{-1}(\tau)$, and $(X_1, \mathcal{L}_{|X_1}, \hat{\mathbb{T}})$ is equivariantly isomorphic to $(X, L^r, \mathbb{T})$;

- a $\mathbb{C}^\star$-action ρ on $\mathcal{X}$ commuting with $\hat{\mathbb{T}}$ and covering the usual $\mathbb{C}^\star$-action on $\mathbb{P}^1$;
- an isomorphism

$$\lambda : (X \times (\mathbb{P}^1 \setminus \{0\}), L^r \otimes \mathcal{O}_{\mathbb{P}^1}(r)) \cong (\mathcal{X} \setminus X_0, \mathcal{L}), \tag{4.27}$$

  which is equivariant with respect to the actions of $\mathbb{G} := \hat{\mathbb{T}} \times \mathbb{S}^1_\rho$ on $\mathcal{X} \setminus X_0$ and the action of $\mathbb{T} \times \mathbb{S}^1$ on $X \times \left(\mathbb{P}^1 \setminus \{0\}\right)$.

To simplify the discussion, we shall assume in the sequel that $r = 1$ and that L is a very ample polarization on X.

By the consideration in Section 4.2.1, for each $\tau \neq 0$, $(X_\tau, \mathcal{L}_{|X_\tau}, \hat{\mathbb{T}})$ gives rise to a momentum polytope $\mathrm{P}_\tau \subset \mathfrak{t}^*$. Using the biholomorphism (4.27), we know that $(X_\tau, \mathcal{L}_{|X_\tau}, \hat{\mathbb{T}})$ and $(X_1, \mathcal{L}_{|X_1}, \hat{\mathbb{T}})$ are equivariantly isomorphic polarized varieties, and thus $\mathrm{P}_\tau = \mathrm{P}_1 = \mathrm{P}$ for all $\tau \neq 0$.

For any $\tau \in \mathbb{P}^1$, following Section 4.2.1, we let $\boldsymbol{A}^{(k)}_{\boldsymbol{\xi}}(\tau) := (A^{(k)}_{\xi_1}(\tau), \dots, A^{(k)}_{\xi_\ell}(\tau))$ be infinitisimal generators of the $\mathbb{S}^1$-actions on $\mathcal{H}_k(\tau) := H^0(X_\tau, \mathcal{L}^k_{|X_\tau})$, induced by the $\mathbb{S}^1$-generators $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_\ell)$ for the $\hat{\mathbb{T}}$-action, on the fiber $(X_\tau, \mathcal{L}_{|X_\tau})$. We claim that the spectrum of the operators $A^{(k)}_{\xi_j}(\tau)$ is independent of $\tau \in \mathbb{P}^1$, and is contained in P. To see this, we can use the observation from [42, Sect. 2.3] which associates to any $\mathbb{T}$-compatible polarized test configuration $(\mathcal{X}, \mathcal{L}, \hat{\mathbb{T}})$ a continuous family $\mathcal{V}_k(\tau) \subset \mathrm{Sym}^k(\mathbb{C}^{N+1})$ of m-planes in the Grassmanian $\mathrm{Gr}_m(\mathrm{Sym}^k(\mathbb{C}^{N+1}))$, where Sym^k denotes the vector space of symmetric homogeneous polynomials in $N+1$ complex variables. In this picture, $(X_\tau, \mathcal{L}_{|X_\tau})$ is seen as a polarized subvariety of $(\mathbb{P}^N, \mathcal{O}(1))$, and the space of sections $\mathcal{H}_k(\tau) := H^0(X_\tau, (\mathcal{L}_{|X_\tau})^k)$ is identified to $\mathrm{Sym}^k(\mathbb{C}^{N+1})/\mathcal{V}_k(\tau)$. We can further assume that the action of $\hat{\mathbb{T}}$ on $(X_\tau, \mathcal{L}_{|X_\tau})$ comes from the restriction to X_τ of a subtorus of $\hat{\mathbb{T}} \subset \mathrm{SL}(N+1, \mathbb{C})$, and thus $\hat{\mathbb{T}}$ also acts on $\mathrm{Sym}^k(\mathbb{C}^{N+1})$; furthermore, writing $\widehat{\boldsymbol{A}}^{(k)}_{\boldsymbol{\xi}} := (\hat{A}^{(k)}_{\xi_1}, \cdots \hat{A}^{(k)}_{\xi_\ell})$, where $\hat{A}^{(k)}_{\xi_j}$ is the infinitisimal generator of the circle action $\mathbb{S}^1_{\xi_j}$ on $\mathrm{Sym}^k(\mathbb{C}^{N+1})$, the operators

$$\hat{A}^{(k)}_{\xi_j} : \mathrm{Sym}^k(\mathbb{C}^{N+1}) \to \mathrm{Sym}^k(\mathbb{C}^{N+1}),$$

must preserve the m-planes $\mathcal{V}_k(\tau)$ (as the action preserves each X_τ viewed as the subspace of common zeroes of elements in $\mathcal{V}_k(\tau)$), and thus

$$A^{(k)}_{\xi_j}(\tau) : \mathrm{Sym}^k(\mathbb{C}^{N+1})/\mathcal{V}_k(\tau) \to \mathrm{Sym}^k(\mathbb{C}^{N+1})/\mathcal{V}_k(\tau)$$

are the linear maps induced by $\widehat{\boldsymbol{A}}^{(k)}_{\boldsymbol{\xi}}$ on the quotient spaces $\mathcal{H}_k(\tau)$. Introducing a $\hat{\mathbb{T}}$-invariant Hermitian product on $\mathrm{Sym}^k(\mathbb{C}^{N+1})$, we thus obtain a continuous $\hat{A}^{(k)}_{\xi_j}$-invariant decomposition

$$\mathrm{Sym}^k(\mathbb{C}^{N+1}) = \mathcal{V}_k(\tau) \oplus \mathcal{V}_k^{\perp}(\tau),$$

and the spectrum of $A^{(k)}_{\xi_i}(\tau)$ is nothing but the spectrum of $\hat{A}^{(k)}_{\xi_i}$ restricted to $\mathcal{V}_k^{\perp}(\tau)$. Using that $\mathcal{V}_k^{\perp}(\tau)$ vary continuously in the Gramsannian, we conclude that the spectrum of $\hat{A}^{(k)}_{\xi_j}$ restricted to $\mathcal{V}_k^{\perp}(\tau)$ is constant. It is contained in P by Lemma 15.

It follows that for any $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R})$, we can define $\mathrm{v}(k^{-1}\boldsymbol{A}^{(k)}_{\boldsymbol{\xi}}(0))$, where $\boldsymbol{A}^{(k)}_{\boldsymbol{\xi}}(0) = (A^{(k)}_{\xi_1}(0), \cdots, A^{(k)}_{\xi_\ell}(0))$ denote the the generators of circle actions corresponding to the central fibre $(X_0, \mathcal{L}_{|X_0}, \hat{\mathbb{T}})$. Thus, for $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R})$ we can consider the following v-weight

$$W^{(k)}_{\mathrm{v}}(\boldsymbol{\xi}, \rho) := \mathrm{tr}\left(\mathrm{v}\big(k^{-1}\boldsymbol{A}^{(k)}_{\boldsymbol{\xi}}(0)\big) \cdot k^{-1}A^{(k)}_{\rho}\right). \tag{4.28}$$

Definition 21. Let $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$ and $\mathrm{w} \in C^\infty(\mathrm{P}, \mathbb{R})$, and suppose that we have the following asymptotic expansions on the central fiber (X_0, L_0)

$$\begin{aligned} W^{(k)}_{\mathrm{w}}(\boldsymbol{\xi}, \rho) =& a^{(0)}_{\mathrm{w}}(\boldsymbol{\xi}, \rho)k^n + \mathcal{O}(k^{n-1}), \\ W^{(k)}_{\mathrm{v}}(\boldsymbol{\xi}, \rho) =& a^{(0)}_{\mathrm{v}}(\boldsymbol{\xi}, \rho)k^n + a^{(1)}_{\mathrm{v}}(\boldsymbol{\xi}, \rho)k^{n-1} + \mathcal{O}(k^{n-2}). \end{aligned} \tag{4.29}$$

Then we define the (v, w)-*Donaldson-Futaki* invariant of the normal $\mathbb{T}$-compatible polarized test configuration $(\mathcal{X}, \mathcal{L})$ to be the number

$$\mathrm{DF}_{\mathrm{v},\mathrm{w}}(\mathcal{X}, \mathcal{L}) := a^{(1)}_{\mathrm{v}}(\boldsymbol{\xi}, \rho) - \frac{c_{\mathrm{v},\mathrm{w}}(L)}{4} a^{(0)}_{\mathrm{w}}(\boldsymbol{\xi}, \rho), \tag{4.30}$$

where $c_{\mathrm{v},\mathrm{w}}(L)$ is the (v, w)-slope of $(X, 2\pi c_1(L))$ given by (2.6).

Using Corollary 8, we have the following

Corollary 9. *If $(\mathcal{X},\mathcal{L})$ is a $\mathbb{T}$-compatible polarized test configuration with smooth central fiber, then the expansions* (4.29) *hold, and*

$$\begin{aligned}(2\pi)^n W_{\mathrm{w}}^{(k)}(\boldsymbol{\xi},\rho) =& k^n\int_{X_0} h_\rho \mathrm{w}(m_{\Omega_0})\Omega_0^{[n]} + \mathcal{O}(k^{n-1}),\\ (2\pi)^n W_{\mathrm{v}}^{(k)}(\boldsymbol{\xi},\rho) =& k^n\int_{X_0} h_\rho \mathrm{v}(m_{\Omega_0})\Omega_0^{[n]} + \frac{k^{n-1}}{4}\int_{X_0} h_\rho \mathrm{Scal}_{\mathrm{v}}(\Omega_0)\Omega_0^{[n]} + \mathcal{O}(k^{n-2}),\end{aligned}$$

where h_ρ is the Ω-Hamiltonian of the generator V_ρ of the action $\mathbb{S}^1_\rho$ on X_0 with respect to a $\mathbb{G}$ invariant Kähler metric $\Omega \in 2\pi c_1(\mathcal{L})$ and $\Omega_0 := \Omega_{|X_0}$. In particular, the (v,w)*-Donaldson-Futaki invariant* (4.30) *of $(\mathcal{X},\mathcal{L})$ is given by*

$$\mathrm{DF}_{\mathrm{v},\mathrm{w}}(\mathcal{X},\mathcal{L}) = \frac{1}{4(2\pi)^n}\mathcal{F}^{\alpha}_{\mathrm{v},\mathrm{w}}(V_\rho),$$

where $\mathcal{F}^{\alpha}_{\mathrm{v},\mathrm{w}}(V_\rho)$ the Futaki invariant of the class $\alpha := 2\pi c_1(L)$, introduced in Definition 8.

We deduce from Corollary 9 and Proposition 9

Corollary 10. *If $(\mathcal{X},\mathcal{L})$ is a smooth $\mathbb{T}$-compatible polarized test configuartion such that $\pi : \mathcal{X} \to \mathbb{P}^1$ is a smooth submersion, then*

$$\mathrm{DF}_{\mathrm{v},\mathrm{w}}(\mathcal{X},\mathcal{L}) = \frac{1}{4(2\pi)^n}\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X},2\pi c_1(\mathcal{L})),$$

where $\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X},2\pi c_1(\mathcal{L}))$ is the (v,w)*-Futaki invariant of the $\mathbb{T}$-compatible Kähler test configuration $(\mathcal{X},2\pi c_1(\mathcal{L}))$ introduced in Definition 18.*

4.3 The (v,w)-Futaki invariant of a toric test configurations

In this section we consider the special case when X is a smooth toric variety i.e. $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ with $\dim_{\mathbb{R}}\mathbb{T} = \dim_{\mathbb{C}} X = n$. Let $\omega \in \alpha$ be a fixed $\mathbb{T}$-invariant Kähler form, $m_\omega : X \to \mathfrak{t}^*$ a corresponding momentum map, and $\mathrm{P} = m_\omega(X)$ the corresponding momentum polytope. By Delzant Theorem [34], (X,α) can be recovered from the *labelled integral Delzant polytope* $(\mathrm{P},\mathbf{L})$ where $\mathbf{L} = (L_j)_{j=1,d}$ is the collection of non-negative defining affine-linear functions for P, with dL_j being primitive elements of the lattice Λ of circle subgroups of $\mathbb{T}$. We denote by P^0 the interior of P and by $X^0 := m_\omega^{-1}(\mathrm{P}^0)$ the dense open set of X of points with principle $\mathbb{T}$ orbits. Let us consider the momentum/angle coordinates $(p,t) \in \mathrm{P}^0 \times \mathbb{T}$ with respect to the Kähler metric (g, J, ω). By a result of Guillemin (see [55])

$$\begin{aligned} g &= \langle dp, \boldsymbol{G}^u, dp\rangle + \langle dt, \boldsymbol{H}^u, dt\rangle, \\ Jdt &= -\langle \boldsymbol{G}^u, dp\rangle, \\ \omega &= \langle dp \wedge dt\rangle, \end{aligned} \tag{4.31}$$

on X^0, where u is a smooth, strictly convex function called the *symplectic potantial* of (ω, J), $\boldsymbol{G}^u : \mathrm{P}^0 \to S^2\mathfrak{t}$ is the Hessian of u, $\boldsymbol{H}^u : \mathrm{P}^0 \to S^2\mathfrak{t}^*$ is its point-wise inverse and $\langle \cdot, \cdot, \cdot \rangle$ denote the contraction $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^* \to \mathbb{R}$ or the dual one. Conversely if u is a strictly convex smooth function on P^0, (4.31) defines a Kähler structure on X^0 which extends to a global $\mathbb{T}$-invariant Kähler structure on X iff u satisfies the boundary conditions of Abreu (see [1]). We denote by $\mathcal{S}(\mathrm{P}, \mathbf{L})$ the set of smooth strictly convex functions on P^0 satisfying these boundary conditions. For $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$, we have the following expression for the scalar curvature of (g, J) (see [2]),

$$\mathrm{Scal}(g) = -\sum_{i,j=1}^{n} \boldsymbol{H}^u_{ij,ij},$$

where $\boldsymbol{H}^u = (\boldsymbol{H}^u_{ij})$ in a basis of $\mathfrak{t}$. Let $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$. By the calculations in [9, Section 3], the following expression for the v-scalar curvature of (g, J) is straightforward

$$\mathrm{Scal}_{\mathrm{v}}(g) = -\sum_{i,j=1}^{n} \left(\mathrm{v}\boldsymbol{H}^u_{ij}\right)_{,ij}. \tag{4.32}$$

We recall that by the maximality of $\mathbb{T}$, any $\mathbb{T}$-invariant Killing potential of (4.31) is the pull-back by m_ω of an affine-linear function on P.

Lemma 28. *Let* $\mathrm{v} \in C^\infty(\mathrm{P}, \mathbb{R}_{>0})$ *and* $\mathrm{w} \in C^\infty(\mathrm{P}, \mathbb{R})$. *For any affine-lirear function* f *on* P, *the* (v, w)*-Futaki invariant corresponding to the* $\mathbb{T}$*-invariant Hamiltonian Killing vector field* $\xi := df$ *is given by*

$$(2\pi)^{-n}\mathcal{F}^\alpha_{\mathrm{v},\mathrm{w}}(\xi) = 2\int_{\partial \mathrm{P}} f\mathrm{v}d\sigma - c_{(\mathrm{v},\mathrm{w})}(\alpha)\int_{\mathrm{P}} f\mathrm{w}dp, \tag{4.33}$$

where dp *is a Lebesgue measure on* $\mathfrak{t}^*$, $d\sigma$ *is the induced measure on each face* $F_i \subset \partial\mathrm{P}$ *by letting* $dL_i \wedge d\sigma = -dp$ *and the constant* $c_{(\mathrm{v},\mathrm{w})}(\alpha)$ *is given by*

$$c_{(\mathrm{v},\mathrm{w})}(\alpha) = 2\left(\frac{\int_{\partial\mathrm{P}} \mathrm{v}d\sigma}{\int_{\mathrm{P}} \mathrm{w}dp}\right). \tag{4.34}$$

Proof. Let $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ and (g, J) be the corresponding ω-compatible Kähler structure X given by (4.31). The (v, w)-Futaki invariant of the Kähler class $\alpha = [\omega]$ is given by

$$\mathcal{F}^{\alpha}_{\mathrm{v},\mathrm{w}}(\xi) = \int_X \mathrm{Scal}_{\mathrm{v}}(g) f(m_\omega)\omega^{[n]} - c_{(\mathrm{v},\mathrm{w})}(\alpha)\int_X f(m_\omega)\mathrm{w}(m_\omega)\omega^{[n]},$$

where f is an affine linear function on $\mathfrak{t}^*$ with $\xi = df \in \mathfrak{t}$. In the momentum-action coordinates $(p, t) \in \mathrm{P}^0 \times \mathbb{T}$ we have $\omega^{[n]} = \langle dp \wedge dt\rangle^{[n]} = dp_1 \wedge dt_1 \wedge \cdots \wedge dp_n \wedge dt_n$. By [9, Lemma 2], for any $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ and any smooth functions φ, ψ on $\mathfrak{t}^*$ we have

$$\int_{\mathrm{P}} \Big(\sum_{i,j=1}^{n} \left(\psi \boldsymbol{H}^u_{ij}\right)_{,ij}\Big)\varphi dp = \int_{\mathrm{P}} \Big(\sum_{i,j=1}^{n} (\psi \boldsymbol{H}^u_{ij})\varphi_{,ij}\Big) dp - 2\int_{\partial \mathrm{P}} \varphi\psi d\sigma. \tag{4.35}$$

Then, using (4.32) together with (4.35), we obtain

$$\begin{aligned}(2\pi)^{-n}\mathcal{F}^{\alpha}_{\mathrm{v},\mathrm{w}}(\xi) =& -\int_{\mathrm{P}} \Big(\sum_{i,j=1}^{n} (\mathrm{v}\boldsymbol{H}^u_{ij})_{,ij}\Big) f dp - c_{(\mathrm{v},\mathrm{w})}(\alpha)\int_{\mathrm{P}} f\mathrm{w} dp \\ =& 2\int_{\partial \mathrm{P}} f\mathrm{v} d\sigma - c_{(\mathrm{v},\mathrm{w})}(\alpha)\int_{\mathrm{P}} f\mathrm{w} dp.\end{aligned}$$

Similarly we deduce (4.34). □

For any continuous function $f \in C^0(\mathrm{P}, \mathbb{R})$ we define

$$\mathcal{F}^{\mathrm{P}}_{\mathrm{v},\mathrm{w}}(f) := 2\int_{\partial \mathrm{P}} f\mathrm{v} d\sigma - c_{(\mathrm{v},\mathrm{w})}(\alpha)\int_{\mathrm{P}} f\mathrm{w} dp. \tag{4.36}$$

Using again [9, Lemma 2] we obtain

$$(2\pi)^{-n}\int_X (\mathrm{Scal}_{\mathrm{v}}(g_u) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\omega))f\omega^{[n]} = \mathcal{F}^{\mathrm{P}}_{\mathrm{v},\mathrm{w}}(f) - \int_{\mathrm{P}} \left(\sum_{i,j=1}^{n} \boldsymbol{H}_{ij} f_{,ij}\right)\mathrm{v} dp, \tag{4.37}$$

for any $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ and $f \in C^\infty(\mathrm{P}, \mathbb{R})$. It follows that

Lemma 29. [9, 42] *If there exist $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ such that the corresponding ω-compatible Kähler structure (g, J) solves $\mathrm{Scal}_{\mathrm{v}}(g) = c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathrm{w}(m_\omega)$, then $\mathcal{F}^{\mathrm{P}}_{\mathrm{v},\mathrm{w}}(f) \geq 0$ for any smooth convex function f on* P.

4.3.1 Toric test configuration

We start by recalling the construction of toric test configurations introduced by Donaldson in [42, Section 4]. Let (X, L) be a smooth polarized toric manifold with integral momentum polytope $\mathrm{P} \subset \mathfrak{t}^* \cong \mathbb{R}^n$ (with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$) and

$$f := \max(f_1, \cdots, f_r), \tag{4.38}$$

a convex piece-wise affine-linear function with integer coefficients, i.e. we assume that each f_j in (4.38) is an affine-linear function $f_j(p) := \langle v_j, p\rangle + \lambda_j$ with $v_j \in \mathbb{Z}^n$ and $\lambda_j \in \mathbb{Z}$. We also assume that the polytope Q defined by

$$\mathrm{Q} = \{(p, p') \in \mathrm{P} \times \mathbb{R} \,:\, 0 \leq p' \leq R - f(p)\}, \tag{4.39}$$

has integral vertices in $\mathbb{Z}^{n+1}$, where R is an integer such that $f \leq R$ on P. By [42, Proposition 4.1.1] there exist an $(n+1)$-dimensional projective toric variety $(\mathcal{X}_{\mathrm{Q}}, \mathbb{G})$ and a polarization $\mathcal{L}_{\mathrm{Q}} \to \mathcal{X}_{\mathrm{Q}}$ corresponding to the labelled integral Delzant polytope $\mathrm{Q} \subset \mathbb{R}^{n+1}$ and the lattice $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$. In general, $\mathcal{X}_{\mathrm{Q}}$ is a compact toric orbifold (see [69]), but $\mathcal{X}_{\mathrm{Q}}$ can be smooth for a suitable choice of f. There is an embedding $\iota : X \hookrightarrow \mathcal{X}_{\mathrm{Q}}$ such that $\iota(X)$ is the pre-image of the face $\mathrm{P} = \mathrm{Q} \cap \left(\mathbb{R}^n \times \{0\}\right)$ of Q, and the restriction of $\mathcal{L}_{\mathrm{Q}}$ to $\iota(X)$ is isomorphic to L. Notice that by the Delzant Theorem [34, 69] the stabilizer of $\iota(X) \subset \mathcal{X}_{\mathrm{Q}}$ in $\mathbb{G}$ is $\mathbb{S}^1_\rho = \mathbb{S}^1_{(n+1)}$, where $\mathbb{S}^1_{(n+1)}$ is the $(n+1)$-th factor of $\mathbb{G} = \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$ so that $\mathbb{G}/\mathbb{S}^1_\rho$ is identified with the torus action $\mathbb{T} = \mathbb{R}^n/2\pi\mathbb{Z}^n$ on X. Furthermore, Donaldson shows in [42] that there exist a $\mathbb{C}^*$-equivariant map $\pi : \mathcal{X}_{\mathrm{Q}} \to \mathbb{P}^1$ such that $(\mathcal{X}_{\mathrm{Q}}, \mathbb{S}^1_\rho, \mathcal{L}_{\mathrm{Q}})$ is a $\mathbb{T}$-compatible polarized test configuration. We consider the Futaki-invariant $\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}_{\mathrm{Q}}, 2\pi c_1(\mathcal{L}_{\mathrm{Q}}))$ given by (4.4) corresponding to $(\mathcal{X}_{\mathrm{Q}}, 2\pi c_1(\mathcal{L}_{\mathrm{Q}}))$, and notice that it makes sense even when $\mathcal{X}_{\mathrm{Q}}$ is just an orbifold.

Proposition 10. *Let $f = \max(f_1, \cdots, f_r)$ be a convex piece-wise linear function on* P, *with integer coefficients and $\mathcal{X}_{\mathrm{Q}}$ the toric test configuration constructed as above. Then the* (v,w)*-Futaki invariant* (4.4) *of* $(\mathcal{X}_{\mathrm{Q}}, 2\pi c_1(\mathcal{L}_{\mathrm{Q}}))$ *is given by*

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}_{\mathrm{Q}}, 2\pi c_1(\mathcal{L}_{\mathrm{Q}})) = (2\pi)^{n+1}\mathcal{F}^{\mathrm{P}}_{\mathrm{v},\mathrm{w}}(f), \tag{4.40}$$

where $\mathcal{F}^{\mathrm{P}}_{\mathrm{v},\mathrm{w}}(f)$ is the integral defined in (4.36). *Furthermore, the* (v,w)*-Donaldson-Futaki invariant* (4.30) *corresponding to* $(\mathcal{X}_{\mathrm{Q}}, \mathcal{L}_{\mathrm{Q}})$ *is well-defined, and is given by*

$$\mathrm{DF}_{\mathrm{v},\mathrm{w}}(\mathcal{X}_{\mathrm{Q}}, \mathcal{L}_{\mathrm{Q}}) = 4\mathcal{F}^{\mathrm{P}}_{\mathrm{v},\mathrm{w}}(f). \tag{4.41}$$

Proof. We start by proving the first claim (4.40). Let $\Omega \in 2\pi c_1(\mathcal{L}_{\mathrm{Q}})$ be a $\mathbb{G}$-invariant Kähler form on $\mathcal{X}_{\mathrm{Q}}$ and $\omega \in 2\pi c_1(L)$ be the induced $\mathbb{T}$-invariant Kähler form on $\iota(X) \subset \mathcal{X}_{\mathrm{Q}}$. We have by Remark 6 2

$$\begin{aligned}\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}, 2\pi c_1(\mathcal{L}_{\mathrm{Q}})) = &-\int_{\mathcal{X}} \left(\mathrm{Scal}_{\mathrm{v}}(\Omega) - c_{(\mathrm{v},\mathrm{w})}(2\pi c_1(L))\mathrm{w}(m_\Omega)\right)\Omega^{[n+1]} \\ &+ (8\pi)\int_X \mathrm{v}(m_\omega)\omega^n.\end{aligned} \tag{4.42}$$

Let $(p, p', t, t') \in \mathrm{Q} \times \mathbb{T} \times \mathbb{S}^1_\rho$ be the momentum/angular coordinates on $\mathcal{X}^0_{\mathrm{Q}}$ such that $(p, t) \in \mathrm{P} \times \mathbb{T}$ are the momentum/angular coordinates on X^0. Then,

$$(8\pi)\int_X \mathrm{v}(m_\omega)\omega^n = 4(2\pi)^{n+1}\int_{\mathrm{P}} \mathrm{v}(p)dp. \tag{4.43}$$

and

$$\int_{\mathcal{X}_{\mathrm{Q}}} \mathrm{w}(m_\Omega)\Omega^{[n+1]} = (2\pi)^{n+1}\int_{\mathrm{Q}} \mathrm{w}(p)dp \wedge dp' = (2\pi)^{n+1}\int_{\mathrm{P}} \mathrm{w}(p)(R - f(p))dp. \tag{4.44}$$

For the remaining term in (4.42), using (4.37) we have

$$\begin{aligned}(2\pi)^{-(n+1)}\int_{\mathcal{X}_{\mathrm{Q}}} \mathrm{Scal}_{\mathrm{v}}(\Omega)\Omega^{[n+1]} &= 2\int_{\partial \mathrm{Q}} \mathrm{v}d\sigma_{\mathrm{Q}} \\ &=2\int_{\mathrm{P}} \mathrm{v}dp + 2\int_{(R-f)(\mathrm{P})} \mathrm{v}d\mu_{(R-f)(\mathrm{P})} + 2\int_{\partial \mathrm{P}} (R-f)\mathrm{v}d\sigma_{\mathrm{P}} \\ &=4\int_{\mathrm{P}} \mathrm{v}dp + 2\int_{\partial \mathrm{P}} (R-f)\mathrm{v}d\sigma_{\mathrm{P}},\end{aligned} \tag{4.45}$$

where the measure $d\mu_{(R-f)(\mathrm{P})}$ is defined by $df \wedge d\mu_{(R-f)(\mathrm{P})} = dp \wedge dp'$. Substituting (4.43)–(4.45) into (4.42) yields

$$\begin{aligned}(2\pi)^{-(n+1)}\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X}_{\mathrm{Q}}, 2\pi c_1(\mathcal{L}_{\mathrm{Q}})) &= -2\int_{\partial \mathrm{P}} (R-f)\mathrm{v}d\sigma_{\mathrm{P}} + c_{\mathrm{v},\mathrm{w}}(\alpha)\int_{\mathrm{P}} (R-f)\mathrm{w}dp \\ &=\mathcal{F}^{\mathrm{P}}_{\mathrm{v},\mathrm{w}}(f).\end{aligned}$$

Now we give the proof of the second claim (4.41). The central fiber X_0 is the reduced divisor on $\mathcal{X}_{\mathrm{Q}}$ associated to the preimage of the union of facets of Q corresponding to the graph of $R - f$. By a well-known fact in toric geometry (see e.g. [42]) the set of weights for the complexified torus $\mathbb{G}^c$ on $H^0(\mathcal{X}, \mathcal{L}^k_{\mathrm{Q}})$ is $k\mathrm{Q} \cap \mathbb{Z}^{n+1}$. It thus follows that the weights for the $\mathbb{C}^\star_\rho$-action on $H^0(X_0, L^k_0)$ are $k(R-f)(k\mathrm{P}) \cap \mathbb{Z}$. We conclude that

$$W^{(k)}_{\mathrm{v}}(\boldsymbol{\xi}, \rho) = \sum_{\lambda \in k\mathrm{P}\cap\mathbb{Z}^n} (R-f)\left(\frac{\lambda}{k}\right)\mathrm{v}\left(\frac{\lambda}{k}\right),$$

where $W_{\mathrm{v}}^{(k)}(\boldsymbol{\xi},\rho)$ is the v-weight defined by (4.28). By [56,99], for any smooth function Φ on $\mathfrak{t}^*$ and k large enough we have

$$\sum_{\lambda\in k\mathrm{P}\cap\mathbb{Z}^n}\Phi\Big(\frac{\lambda}{k}\Big)=k^n\int_{\mathrm{P}}\Phi dp+\frac{k^{n-2}}{2}\int_{\partial\mathrm{P}}\Phi d\sigma_{\mathrm{P}}+\mathcal{O}(k^{n-2}).$$

Taking $\Phi:=(R-f)\mathrm{v}$ and using the above formula for any affine-linear piece of Φ, we get

$$W_{\mathrm{v}}^{(k)}(\boldsymbol{\xi},\rho)=k^n\int_{\mathrm{P}}(R-f)\mathrm{v}dp+\frac{k^{n-2}}{2}\int_{\partial\mathrm{P}}(R-f)\mathrm{v}d\sigma_{\mathrm{P}}+\mathcal{O}(k^{n-2}).$$

Analogously, for $W_{\mathrm{w}}^{(k)}(\boldsymbol{\xi},\rho)$ we obtain

$$W_{\mathrm{w}}^{(k)}(\boldsymbol{\xi},\rho)=k^n\int_{\mathrm{P}}(R-f)\mathrm{w}dp+\mathcal{O}(k^{n-1}).$$

Using (4.30), it follows that

$$\mathrm{DF}_{\mathrm{v},\mathrm{w}}(\mathcal{X}_{\mathrm{Q}},\mathcal{L}_{\mathrm{Q}})=4\mathcal{F}_{\mathrm{v},\mathrm{w}}^{\mathrm{P}}(f).$$

□

Remark 8. Instead of a convex piece-wise affine-linear function f with integer coefficients we can take a convex piece-wise affine-linear functions with rational differentials, i.e. assuming that each f_j in (4.38) is of the form with $f_j(p)=\langle v_j,p\rangle+\lambda_j$ with $v_j\in\mathbb{Q}^n$. The polytope Q such a function defines is not longer with rational vertices, but still defines a toric Kähler orbifold $(\mathcal{X}_{\mathrm{Q}},\mathcal{A}_{\mathrm{Q}})$, see [69]. This gives rise to a toric Kähler test configuration compatible with $\mathbb{T}$ and the formula (4.40) in Proposition 10 computes the corresponding (v,w)-Futaki invariant of $(\mathcal{X}_{\mathrm{Q}},\mathcal{A}_{\mathrm{Q}})$. ◇

CHAPTER V

APPLICATIONS

5.1 Existence of cKEM metrics and the automorphism group

Let (X, α) be a compact Kähler manifold with Kähler class α and $\xi \in \mathfrak{h}_{\text{red}}$ is a real holomorphic vector generating a torus $\mathbb{T}_\xi \subset \text{Aut}_{\text{red}}(X)$. For any $\mathbb{T}_\xi$-invariant Kähler metric $\omega \in \alpha$, the vector field is Hamiltonian with respect to ω with ω-Hamiltonian function $f_{(\xi,\omega,a)}$ normalized by $\int_X f_{(\xi,\omega,a)}\omega^{[n]} = a$ where $a > 0$ is positive constant. One can always choose the constant $a > 0$ such that for $\mathbb{T}_\xi$-invariant Kähler metric $\omega \in \alpha$ the ω-Hamiltonian function $f_{(\xi,\omega,a)} > 0$ is positive. In the setting of Section 2.2.4, $f_{(\xi,\omega,a)} = \langle \xi, m_\omega \rangle + c$ for a fixed positive affine-linear function $\langle \xi, m_\omega \rangle + c$ over P_α, where P_α is a momentum polytope associated to $(\mathbb{T}_\xi, \alpha)$ as in Lemma 1.

As explained in the Introduction (see (1.2)), we say that $\mathbb{T}_\xi$-invariant Kähler metric $\omega \in \alpha$ is a Kähler metric conformal to a Einstein-Maxwell metric (cKEM) if the scalar curvature of the conformal metric $f^{-2}_{(\xi,\omega,a)} g_\omega$ is constant i.e.

$$\text{Scal}(f^{-2}_{(\xi,\omega,a)} g_\omega) = \text{const},$$

which is also equivalent to the Einstein-Maxwell equations (1.1). As we have noticed in Section 2.2.4, the cKEM metrics are (v, w)-cscK metrics for $\mathrm{v}_{\xi,a}, \mathrm{w}_{\xi,a} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R})$ such that

$$\mathrm{v}_{\xi,a}(m_\omega) = f^{-2m+1}_{(\xi,\omega,a)} \text{ and } \mathrm{w}_{\xi,a}(m_\omega) = f^{-2m-1}_{(\xi,\omega,a)}, \tag{5.1}$$

where $m_\omega : X \to \mathfrak{t}^*_\xi$ is the ω-momentum map with momentum image $m_\omega(X) = \mathrm{P}_\alpha$.

Similarly to the cases of Kähler-Einstein and cscK metrics [73,78], Theorem 1 places an obstruction for X to admit a weighted cscK metric in terms of the centraliser $\mathrm{Aut}_0^{\xi}(X)$ of $\mathbb{T}_\xi$ in $\mathrm{Aut}_0(X)$. In particular this result applies to cKEM metrics. By [9, Theorem 5] and by Corollary 4, any cKEM metric on the toric complex surfaces $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbb{CP}^1$ must be given either by the Calabi Ansatz [60, 65, 66] or by the hyperbolic ambitoric ansatz [4] (a Riemannian analogue of the Plebanski-Damianski explicit solutions [81]). In practice, however, the algorithm of [9, Theorem 5] allowing one to decide whether or not a given Kähler class, a quasi-periodic holomorphic vector field ξ and a constant $a > 0$ there exists a compatible cKEM metric is of considerable complexity, see [52]. The case $\mathbb{CP}^1 \times \mathbb{CP}^1$ has been successfully resolved by [9,65] (see also [52]) whereas the case of $\mathbb{F}_n$ and ξ being tangent to the fibers is settled in [51]. The possibility of other choices of ξ and $a > 0$ on $\mathbb{F}_n$ is open.

The following result completes the classification of cKEM metrics started in [9,52,65].

Corollary 11. *Any conformally-Kähler, Einstein–Maxwell metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$, must be toric, and if it is not a product of Fubini-Study metrics on each factor, it must be homothetically isometric to one of the metrics constructed in* [65].

Proof. From Corollary 4 any cKEM metric g on $\mathbb{CP}^1 \times \mathbb{CP}^1$ must be toric. In this case [9, Proposition 6] yields that the metric g must be either a product of Fubini-Study metrics or one of the metrics found in [65]. □

We illustrate our theory with a non existence result.

Corollary 12. *Let $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)_E) \to \mathbb{F}_n$ where $E = (\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbb{CP}^1$ and $\mathbb{F}_n = \mathbb{P}(E)$ is the n-th Hirzebruch complex surface. Denote by ξ the generator of the $\mathbb{S}^1$-action on X corresponding to diagonal multiplications on the $\mathcal{O}_E(1)$-factor. Then X admits no cKEM metrics.*

Proof. We have the following exact sequence (see [6, Proposition 1.3]):

$$0 \to \mathfrak{h}_B(X) \to \mathfrak{h}(X) \to \mathfrak{h}(B) \to 0$$

where $B = \mathbb{F}_n$ and $\mathfrak{h}_B(X)$ denote the Lie algebra of holomorphic vector fields on X which are tangent to the fibers of π. The proof of [6, Proposition 1.3] also shows that

$$0 \to \mathfrak{h}^{\xi}_B(X) \to \mathfrak{h}^{\xi}(X) \to \mathfrak{h}(B) \to 0$$

where $\mathfrak{h}^{\xi}_B(X) = \mathrm{span}_{\mathbb{C}}\{\xi, J\xi\}$ is the abelian sub-algebra generated by the vector fields ξ, $J\xi$ and $\mathfrak{h}^{\xi}(X)$ is the centraliser of ξ in the Lie algebra $\mathfrak{h}(X)$. If X admits a Kähler metric (ω, g) conformal to an Einstein-Maxwell metric with conformal factor $f_{(\xi,\omega,a)} > 0$, then $\mathfrak{h}^{\xi}(X)$ must be reductive by Theorem 1. As $\mathfrak{h}^{\xi}_B(X)$ is in the center of $\mathfrak{h}^{\xi}(X)$, it would follow that $\mathfrak{h}(B)$ is reductive, which is not the case for $B = \mathbb{F}_n$ (see e.g. [18]). It follows that X admits no cKEM metrics. $\square$

5.2 The YTD correspondence of $\mathbb{P}^1$-bundles

We start with the case 4 from the Introduction. Following [8], we consider $X = V \times_{\mathbb{T}} K \xrightarrow{\pi} B$ be the total space of a fibre-bundle associated to a principle $\mathbb{T}$-bundle $K \to B$ over the product $B = \prod_{j=1}^{N}(B_j, \omega_j, g_j)$ of compact cscK manifolds (B_j, ω_j, g_j) of complex dimension d_j, satisfying the Hodge condition $[\omega_j/2\pi] \in H^2(B_j, \mathbb{Z})$, and a compact 2ℓ-dimensional toric Kähler manifold $(V, \omega_V, g_V, J_V, \mathbb{T})$ corresponding to a labelled Delzant polytope $(\mathrm{P}, \mathbf{L})$ in $\mathfrak{t}^*$. We assume that K is endowed with a connection 1-form $\boldsymbol{\theta} \in \Omega^1(K, \mathfrak{t})$ satisfying

$$d\boldsymbol{\theta} = \sum_{j=1}^{N} \xi_j \otimes \omega_j,\ \xi_j \in \mathfrak{t},\ j = 1, \cdots, N.$$

and that the toric Kähler metric (g_V, ω_V, J_V) on V is given by (4.31) for a symplectic potential $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ where the space of symplectic potentials $\mathcal{S}(\mathrm{P}, \mathbf{L})$ is introduced in Section 4.3. As shown in [8], X admits a bundle-adapted Kähler metric (g, ω) which, on the open dense subset $X^0 = K \times \mathrm{P}^0 \subset X$, takes the form

$$\begin{aligned} g &= \sum_{j=1}^{N} \Big(\langle \xi_j, p\rangle + c_j\Big)\pi^* g_j + \langle dp, \boldsymbol{G}^u, dp\rangle + \langle \boldsymbol{\theta}, \boldsymbol{H}^u, \boldsymbol{\theta}\rangle, \\ \omega &= \sum_{j=1}^{N} \Big(\langle \xi_j, p\rangle + c_j\Big)\pi^* \omega_j + \langle dp \wedge \boldsymbol{\theta}\rangle, \end{aligned} \tag{5.2}$$

where $p \in \mathrm{P}^0$ and c_j are real constants such that $(\langle \xi_j, p\rangle + c_j) > 0$ on P. Such Kähler metrics, parametrized by $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ and the real constants c_j, are referred to in [8] as given by *the generalized Calabi ansatz* in reference to the well-known construction of Calabi [22] of extremal Kähler metrics on $\mathbb{P}^1$-bundles.

We notice that the Kähler manifold (X, ω, g) is invariant under the $\mathbb{T}$-action with momentum map identified with $p \in \mathrm{P}$. Furthermore, it is shown in [8, (7)] that the scalar curvature of (5.2) is given by

$$\begin{aligned}\mathrm{Scal}(g) &= \sum_{j=1}^{N} \frac{\mathrm{Scal}_j}{\langle \xi_j, p\rangle + c_j} - \frac{1}{\mathrm{u}(p)} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial p_r \partial p_s}\Big(\mathrm{u}(p) \boldsymbol{H}^u_{rs}\Big)\\ &= \sum_{j=1}^{N} \frac{\mathrm{Scal}_j}{\langle \xi_j, p\rangle + c_j} + \frac{1}{\mathrm{u}(p)}\mathrm{Scal}_{\mathrm{u}}(g_V),\end{aligned}$$

where we have put $\mathrm{u}(p) := \prod_{j=1}^{N}(\langle \xi_j, p\rangle + c_j)^{d_j}$ and we have used (4.32) for passing from the first line to the second. Similarly, by [8, (12)], the g-Laplacian of (the pull-back to X) of a smooth function $f(p)$ on P is given by

$$\Delta_g f = -\frac{1}{\mathrm{u}(p)} \sum_{r,s=1}^{\ell} \frac{\partial}{\partial p_r}\Big(\mathrm{u}(p) \frac{\partial f}{\partial p_s} \boldsymbol{H}^u_{rs}\Big).$$

Using the above formulae, we check by a direct computation that for any positive smooth function v on P we have

$$\mathrm{Scal}_{\mathrm{v}}(g) = \mathrm{v}(p)\Big(\sum_{j=1}^{N} \frac{\mathrm{Scal}_j}{\langle \xi_j, p\rangle + c_j}\Big) + \frac{1}{\mathrm{u}(p)}\mathrm{Scal}_{\mathrm{uv}}(g_V) \tag{5.3}$$

Using that the volume form of (5.2) is

$$\omega^{[n]} = \mathrm{u}(p)\Big(\bigwedge_{j=1}^{N} \omega_j^{[d_j]}\Big) \wedge \langle dp \wedge \boldsymbol{\theta}\rangle^{[\ell]},$$

and the integration by parts formula (4.35), we compute that the (v, w)-Futaki invariant on X acts on a vector field $\xi \in \mathfrak{t}$ by

$$\begin{aligned}\frac{\mathcal{F}^{[\omega]}_{\mathrm{v},\mathrm{w}}(\xi)}{(2\pi)^{\ell}\Big(\prod_{j=1}^{N} \mathrm{Vol}(B_j, [\omega_j])\Big)} =& 2\int_{\partial \mathrm{P}} f \mathrm{vu} d\sigma + \int_{\mathrm{P}} \Big(\sum_{j=1}^{N} \frac{\mathrm{Scal}_j}{\langle \xi_j, p\rangle + c_j}\Big) f\mathrm{vu}dp\\ &- c_{\mathrm{v},\mathrm{w}}([\omega]) \int_{\mathrm{P}} f\mathrm{wu}dp,\end{aligned} \tag{5.4}$$

where $f = \langle \xi, p\rangle + \lambda$ is a Killing potential of ξ.

As in Section 4.3.1, we can construct a $\mathbb{T}$-compatible smooth Kähler test configuration associated to X, defined by a convex piece-wise linear function $f = \max(f_1, \cdots, f_r)$, on $\mathfrak{t}^*$ such that the polytope $\mathrm{Q} \subset \mathbb{R}^{\ell+1}$ given by (4.39) is Delzant with respect to the the lattice $\mathbb{Z}^{\ell+1}$. Denote by $(\mathcal{V}_{\mathrm{Q}}, \mathcal{A}_{\mathrm{Q}})$ the corresponding smooth toric variety, and by $\mathcal{K} = K \times \mathbb{S}^1_{(\ell+1)} \to B$ the principal $\mathbb{T}^{\ell+1}$-bundle over B with trivial $(\ell+1)$-factor, and let $\mathcal{X} = \mathcal{V} \times_{\mathbb{T}^{\ell+1}} \mathcal{K} \to B$ be the resulting $\mathcal{V}$-bundle over B. We can now consider a Kähler form Ω on $\mathcal{X}$ obtained by the generalized Calabi ansatz (5.2); as the connection 1-form on $\mathcal{K}$ has a curvature $\sum_{j=1}^N \xi_j \otimes \omega_j$ with $\xi_j \in \mathfrak{t} = \mathrm{Lie}(\mathbb{T}^\ell) \subset \mathrm{Lie}(\mathbb{T}^{\ell+1})$, Ω induces on the pre-image $X \subset \mathcal{X}$ of the facet $\mathrm{P} \subset \mathrm{Q}$ a Kähler form ω given by (5.2) with the same affine linear functions $(\langle \xi_j, p\rangle + c_j)$. A similar computation to (5.4), performed on the total space $(\mathcal{X}_{\mathrm{Q}}, \Omega)$ by using Definition 18 (see also the proof of Lemma 28 above) leads to the expression (5.4) for the (v, w)-Futaki invariant associated to $(\mathcal{X}_{\mathrm{Q}}, \mathcal{A}_{\mathrm{Q}})$ with f being the piece-wise linear convex function defining Q.

Let us now suppose that $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \xrightarrow{\pi} B$ with B as above, where $\mathcal{O}$ stands for the trivial holomorphic line bundle over B and $\mathcal{L}$ is a holomorphic line bundle of the form $\mathcal{L} = \bigotimes_{j=1}^N \mathcal{L}_j$ for $\mathcal{L}_j$ being the pull-back to B of a holomorphic line bundle over B_j with $c_1(\mathcal{L}_j) = \xi_j[\omega_j/2\pi]$, $\xi_j \in \mathbb{Z}$. This is the so-called *admissible setting* (without blow-downs) of [7], pioneered in [22] and studied in many works. In our setting above, such an X is a $\mathbb{P}^1$-bundle obtained from the principle $\mathbb{S}^1$-bundle over B associated to $\mathcal{L}^{-1}$. We can take $\mathrm{P} = [-1, 1] \subset \mathbb{R}$, and suppose that $\mathrm{v}(z) > 0$ and $\mathrm{w}(z)$ are smooth functions defined over $[-1, 1]$. A Kähler metric (ω, g) on X of the form (5.2) can be equivalently written as

$$
\begin{aligned}
g &= \sum_{j=1}^N (\xi_j z + c_j)\pi^* g_j + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2 \\
\omega &= \sum_{j=1}^N (\xi_j z + c_j)\pi^* \omega_j + dz \wedge \theta, \ d\theta = \sum_{j=1}^N \xi_j \pi^* \omega_j,
\end{aligned}
\tag{5.5}
$$

for positive affine-linear functions $\xi_j z + c_j$ on $[-1, 1]$. This is the more familiar Calabi ansatz, written in terms of the *profile function* $\Theta(z)$ (see e.g. [57]) which must be smooth on $[-1,1]$ and satisfy

$$\Theta(\pm 1) = 0, \quad \Theta'(\pm 1) = \mp 2, \tag{5.6}$$

and

$$\Theta(z) > 0 \quad \text{on} \quad (-1,1), \tag{5.7}$$

for (5.5) to define a smooth Kähler metric on X. We let $\mathrm{u}(z) = \prod_{j=1}^{N}(\xi_j z + c_j)^{d_j}$ be the corresponding polynomial in z.

We now take Q be the chopped rectangle with base P, corresponding to the convex piece-wise affine linear function $f_{z_0}(z) = \max(z + 1 - z_0, 1)$ where $z_0 \in (-1,1)$ is a given point. We can construct as above an $\mathbb{S}^1$-compatible Kähler test configuration $(\mathcal{X}_{\mathrm{Q}}, \mathcal{A}_{\mathrm{Q}})$ associated to $(X, [\omega], \mathbb{S}^1)$. It is not difficult to see that the complex manifold $\mathcal{X}_{\mathrm{Q}}$ is the degenaration to the normal cone with respect to the infinity section $S_\infty \subset X$, see [7, 83] but the Kähler class $\mathcal{A}_{\mathrm{Q}}$ on $\mathcal{X}_{\mathrm{Q}}$ defines a polarization only for rational values of z_0. Formula (5.4) shows that the (v, w)-Futaki invariant of $(\mathcal{X}_{\mathrm{Q}}, \mathcal{A}_{\mathrm{Q}})$ is a positive multiple of the quantity

$$\begin{aligned} F(z_0) &:= 2\Big(f_{z_0}(1)\mathrm{v}(1)\mathrm{u}(1) - f_{z_0}(-1)\mathrm{v}(-1)\mathrm{u}(-1)\Big) \\ &+ \int_{-1}^{1} f_{z_0}(z)\Big(\mathrm{v}(z)\mathrm{u}(z)\Big(\sum_{j=1}^{N} \frac{\mathrm{Scal}_j}{\xi_j z + c_j}\Big) - c_{\mathrm{v},\mathrm{w}}([\omega])\mathrm{w}(z)\mathrm{u}(z)\Big)dz. \end{aligned} \tag{5.8}$$

Let us now assume that there exists a smooth function $\Theta(z)$ on $[-1,1]$, which satisfies (5.6) and

$$(\mathrm{vu}\Theta)''(z) = \mathrm{v}(z)\mathrm{u}(z)\Big(\sum_{j=1}^{N} \frac{\mathrm{Scal}_j}{\xi_j z + c_j}\Big) - c_{\mathrm{v},\mathrm{w}}([\omega])\mathrm{w}(z)\mathrm{u}(z). \tag{5.9}$$

Substituting in the RHS of (5.8) and integrating by parts over the intervals $[-1, z_0]$ and $[z_0, 1]$ gives

$$F(z_0) = \mathrm{v}(z_0)\mathrm{u}(z_0)\Theta(z_0). \tag{5.10}$$

As $\mathrm{v}(z)$ and $\mathrm{u}(z)$ are positive functions on $[-1,1]$, we conclude that if $(X, [\omega], \mathbb{S}^1)$ is (v, w)-K-stable on smooth $\mathbb{S}^1$-compatible Kähler test configurations with reduced central fibre, then $\Theta(z)$ must also satisfy (5.7). By the formula (5.3), the corresponding Kähler metric (5.5) will be then (v, w)-cscK.

The existence of a solution of (5.9) satisfying (5.6) is in general overdetermined. Following [10], in the case when $\mathrm{w}(z) > 0$ on $[-1, 1]$ one can resolve the over-determinacy by letting the constant $c_{\mathrm{v},\mathrm{w}}([\omega]) = 1$ and introducing an affine-linear function $\mathrm{w}_{\mathrm{ext}}(z) = A_1 z + A_2$, such that

$$(\mathrm{vu}\Theta)''(z) = \mathrm{v}(z)\mathrm{u}(z)\Big(\sum_{j=1}^{N} \frac{\mathrm{Scal}_j}{\xi_j z + c_j}\Big) - \mathrm{w}(z)\mathrm{w}_{\mathrm{ext}}(z)\mathrm{u}(z) \tag{5.11}$$

admits a unique solution $\Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}(z)$ satisfying (5.6): the coefficients A_1 and A_2, as well as the two constants of integration in (5.9), are then uniquely determined from the four boundary conditions in (5.6). Furthermore, a straightforward generalization of [10, Lemma 2.4] shows that $\mathrm{w}_{\mathrm{ext}}(z)$ corresponds to the affine-linear function introduced in Section 3.2, i.e. $(\mathrm{v}, \mathrm{ww}_{\mathrm{ext}})$-cscK metrics are (v, w)-extremal. Combined with Theorem 2, this allow us to obtain the following generalization of [10, Theorem 3].

Theorem 12. *Let $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to B$ be a projective $\mathbb{P}^1$-bundle as above, endowed with the $\mathbb{S}^1$-action by multiplication on $\mathcal{O}$, and $\alpha = [\omega/2\pi]$ be the Kähler class of a Kähler metric in the form* (5.5)*. We let $\mathrm{P} = [-1, 1]$ be the momentum polytope of $(X, \alpha, \mathbb{S}^1)$, v, w be smooth positive functions on $[-1, 1]$ and $\Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}(z)$ the unique solution of* (5.11) *satisfying* (5.6)*. Then,*

- *If $(X, \alpha, \mathbb{S}^1)$ is $(\mathrm{v}, \mathrm{ww}_{\mathrm{ext}})$-K-stable on $\mathbb{S}^1$-compatible smooth Kähler test configurations with reduced central fibre, then $\Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}(z) > 0$ on $(-1, 1)$ and α admits a (v, w)-extremal Kähler metric of the form* (5.5) *with $\Theta = \Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}$.*

- *If $(X, \alpha, \mathbb{S}^1)$ admits a (v, w)-extremal Kähler metric, then $(X, \alpha, \mathbb{S}^1)$ is $(\mathrm{v}, \mathrm{ww}_{\mathrm{ext}})$-K-semistable on $\mathbb{S}^1$-compatible smooth Kähler test configurations with reduced central fiber and $\Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}(z) \geq 0$.*

Proof. The first part follows from the identity (5.10) which shows that $\Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}$ must satisfy both (5.6) and (5.7). The second part follows from formula (5.10) and Theorem 4, if the constants $(c_1, \ldots, c_N)$ in (5.5) are rational as in this case the corresponding Kähler class α is rational. To treat the case when $(c_1, \ldots, c_N)$ are not necessarily rational, we can use Theorem 2 below (with fixed v, w and varying the constants c_j). Accordingly, for any rational constants $(\tilde{c}_1, \ldots, \tilde{c}_N)$ sufficiently close to $(c_1, \ldots, c_N)$ the corresponding Kähler class $\tilde{\alpha}$ will admit a (v, w) extremal Kähler metric, and hence the corresponding function $\widetilde{\Theta}^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}(z)$ will be non-negative on $(-1, 1)$ by virtue of Theorem 4. As $\Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}(z)$ depends smoothly on $(c_1, \ldots, c_N)$, it follows that $\Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}(z) \geq 0$ too. □

Remark 9. (i) We expect that Theorem 2 can be improved by showing that the existence of (v, w)-cscK metric in α implies (v, w)-K-stability, not only (v, w)-K-semi-stability. Accordingly, we expect Theorem 12 to be improved to a complete Yau–Tian–Donaldson type correspondence between $(\mathrm{v}, \mathrm{w}\mathrm{w}_{\mathrm{ext}})$-K-stable and (v, w)-extremal Kähler classes on X of the form (5.5), in which either notion corresponds to the positivity condition (5.7) for $\Theta^{\mathrm{v},\mathrm{w}}_{\mathrm{ext}}(z)$.

(ii) In [10], the analogous statement of Theorem 12 is achieved by considering polarized test configuration $(\mathcal{X}_{\mathrm{Q}}, \mathcal{L}_{\mathrm{Q}})$ as above (corresponding to rational values of z_0), and computing the relative version of the algebraic (v, w)-Donaldson–Futaki invariant $\mathrm{DF}_{\mathrm{v},\mathrm{w}}(\mathcal{X}_{\mathrm{Q}}, \mathcal{L}_{\mathrm{Q}})$. This provides a yet another instance where the differential-geometric definition coincides with the algebraic definition of the (v, w)-Futaki invariant. ◇

5.3 The conformally Kähler, Einstein–Maxwell metrics on ruled surfaces

In this section, we give the proof of the Corollary 2 from the Introduction.

5.3.1 The Calabi construction of cKEM metrics on ruled surfaces

Let $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to C$ be a geometrically ruled complex surface over a compact complex curve C of genus $\mathbf{g} \geq 2$. Following [60], cKEM metrics can be constructed by using the Calabi ansatz.

Let (g_C, ω_C) be a Kähler metric on C with constant scalar curvature $4(1 - \mathbf{g})$, where $\ell = \deg(\mathcal{L}) > 0$ is the degree of $\mathcal{L}$. We denote by θ the connection 1-form on the principal $\mathbb{S}^1$-bundle P over C, with curvature $d\theta = \ell\omega_C$. Notice that P can be identified with the unitary bundle of $(\mathcal{L}^*, h^*)$ over C, where h^* is the Hermitian metric with Chern curvature $-\ell\omega_C$; viewing equivalently X as a compactification at infinity of $\mathcal{L}^* \to C$ (i.e. $X = \mathbb{P}(\mathcal{L}^* \oplus \mathcal{O})$). We have a class of Kähler metrics on X given by the Calabi ansatz

$$g = \ell(z+\kappa)g_C + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2, \; \omega = \ell(z+\kappa)\omega_C + dz \wedge \theta, \tag{5.12}$$

where: $z \in [-1,1]$ is a momentum variable for the $\mathbb{S}^1$-action on $\mathcal{L}^*$, $\Theta(z)$ is the profile function satisfying the first order boundary conditions (5.6) and the positivity condition (5.7). Here $\kappa > 1$ is a real constant which parametrizes the Kähler class

$$\alpha_\kappa = [\omega] = 2\pi\Big(c_1(\mathcal{O}(2)_{\mathbb{P}(\mathcal{O}\oplus\mathcal{L})}) + (1+\kappa)\ell[\omega_C]\Big).$$

Notice that for the ruled surfaces we consider $H^2(X,\mathbb{R}) \cong \mathbb{R}^2$, so that any Kähler class on X can be written as $\lambda\alpha_\kappa$ for some $\lambda > 0$ and $\kappa > 1$, see [49]. Furthermore, α_κ is homothetic to a Hodge class if and only if $\kappa \in (1, +\infty) \cap \mathbb{Q}$.

For any $|c| > 1$, $f = |z + c|$ is a positive Killing potential with respect to (5.12) which corresponds up to sign to the Killing vector field ξ generating the $\mathbb{S}^1$-action on $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{L})$ by multiplications of the first factor $\mathcal{O}$. The main results of [60] can be summarized as follows

Proposition 11. [60] *Let $X = \mathbb{P}(\mathcal{O}\oplus\mathcal{L}) \to C$ be a ruled complex surface as above and $\mathcal{F}_{\xi,\kappa,b} := \mathcal{F}^{\alpha_\kappa}_{\mathrm{v}_{\xi,b},\mathrm{w}_{\xi,b}}$ the $(\mathrm{v}_{\xi,b}, \mathrm{w}_{\xi,b})$-Futaki invariant of the Kähler class α_κ (see 8), where $(\mathrm{v}_{\xi,b}, \mathrm{w}_{\xi,b})$ are given by* (5.1).

- *For any $\kappa > 1$, the Futaki invariant $\mathcal{F}_{\xi,\kappa,b}$ vanishes if and only if b satisfies*

$$\kappa = \frac{1+b^2}{2b}. \tag{5.13}$$

  We denote by $b_\kappa > 1$ the unique solution of (5.13) *satisfying $|b| > 1$.*

- *There exits a polynomial $P_\kappa(z)$ of degree ≤ 4 such that $\Theta(z) = P_\kappa(z)/(z+\kappa)$ satisfies the first order boundary conditions* (5.6) *and, on any open subset when $\Theta(z) > 0$, the metric* (5.12) *is conformal to a Einstein-Maxwell metric with conformal factor $(z + b_\kappa)^{-2}$.*

- *There exists $\kappa_0(X) \in (1, +\infty)$ such that*

  (a) *for each $\kappa \in (\kappa_0(X), +\infty)$ the corresponding polynomial $P_\kappa(z) > 0$ on $(-1,1)$, i.e. α_κ admits a Kähler metric of the form* (5.12) *with $\Theta(z) = P_\kappa(z)/(z+\kappa)$, such that $(z+b_\kappa)^{-2}g$ is cKEM;*

  (b) *for each $\kappa \in (1, \kappa_0(X))$ the corresponding polynomial $P_\kappa(z)$ is negative somewhere on $(-1,1)$;*

  (c) *for $\kappa = \kappa_0(X)$ the corresponding polynomial $P_\kappa(z)$ is non-negative and has a zero with multiplicity* 2 *on $(-1,1)$.*

5.3.2 Proof of Corollary 2

There are no cscK metrics on X (see e.g. [11]), so that we are looking for strictly conformally Kähler, Einstein–Maxwell metrics. As in our case $\mathrm{Aut}_{\mathrm{red}}(X,J) = \mathbb{C}^*$ (see e.g. [11]), the Killing vector field ξ must be a multiple of the vector field generating rotations on the factor $\mathcal{O}$. As the theory is invariant under homothety of the Killing potential, without loss we assume that this multiple is ± 1. Finally, as $H^2(X,\mathbb{R}) = \mathbb{R}^2$, by rescaling the Kähler class we can also assume $\alpha_\kappa, \kappa > 1$. For a Kähler metric $\omega \in \alpha_\kappa$ of the form (5.12), the Killing potential of ξ is $|z+b|$ with $|b| > 1$. The necessary condition $\mathcal{F}_{\kappa,\xi,b} = 0$ then forces us to consider $b = b_\kappa$, see Proposition 11. The existence of conformally Kähler, Einstein–Maxwell metrics for $\kappa \in (\kappa_0(X), \infty)$ and conformal factor $(z+b_\kappa)^{-2}$ follows from the statement in (a) of Proposition 11.

We are left to show non-existence for $\kappa \in (1, \kappa_0(X)]$. Again, by Proposition 11, we have to take $b = b_\kappa > 1$.

Consider first the case $\kappa \in (1, \kappa_0(X))$. If κ is rational, the result follows from Theorem 12. Otherwise, if $\kappa \in (1, \kappa_0(X)) \setminus \mathbb{Q}$, we suppose for contradiction that α_κ admits a Kähler metric g_κ such that $(z+b_\kappa)^{-2}g$ is Einstein-Maxwell. By Theorem 2, the same will hold for all $(\kappa', b_{\kappa'})$ on the rational curve (5.13) which are sufficiently close to (κ, b_κ), in particular for all rational pairs $(\kappa', b_{\kappa'})$ close to (κ, b_κ), a contradiction. Finally, consider

$\kappa = \kappa_0(X) = \kappa_0$, $b_{\kappa_0} = b_0$. Again, suppose for contradiction that α_{κ_0} admits a Kähler metric g_0 such that $(z+b_0)^{-2}g_0$ is Einstein-Maxwell. We use again Theorem 2 to deduce that this holds also for all (κ, b_κ) near (κ_0, b_0) and we can find again *rational* valued (κ, b_κ) arbitrarily close to (κ_0, b_0) with $\kappa < \kappa_0$, and still admitting a Kähler metric g_κ such that $\mathrm{Scal}((z+b_\kappa)^{-2}g)$ is a Killing potential, a contradiction.

CHAPTER VI

CONCLUSIONS AND FUTURE DIRECTIONS

In this thesis we have defined weighted cscK metrics on Kähler manifolds with symplectic and variational interpretations. We have seen that the problem of finding a weighted cscK metric in a Kähler class, englobes a number of natural problems in Kähler geometry. We have defined a new notion of weighted K-stability for $(X, \alpha, \mathbb{T}, \mathrm{P}, \mathrm{v}, \mathrm{w})$, consisting of a Kähler manifold X with Kähler class α, a Hamiltonian torus action $\mathbb{T}$ with momentum polytope P and two smooth weight functions v, w on P. A Yau-Tian-Donaldson conjecture relating this stability notion with the existence of weighted cscK metrics has been stated and the direction existence implies stability has been proven in the special case of projective varieties.

This thesis starts a programme, the one of studying weighted cscK metrics, and sets many open questions that will be considered in future work. Possible future directions are:

1. Showing that the weighted Mabuchi energy is convex along weak geodesics in the space of Kähler potentials. Consider the question of uniqueness of weighted cscK metrics modulo the action of isometries.

2. Showing the uniqueness of weighted cscK metrics on projective varieties using the momentum map picture for weighted balanced metrics.

3. Develop a gluing construction of weighted cscK metrics on the blow up of a weighted cscK manifold.

4. Proving that the weighted Dondaldson-Futaki invariant can be expressed as an equivariant intersection number on the total space of a test configuration and showing that it agrees with the slope of the weighted Mabuchi energy.

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