UNIVERSITÉ DU QUÉBEC À MONTRÉAL

ON SOME PARISIAN RUIN PROBLEMS FOR LEVY INSURANCE RISK MODELS

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QUELQUES PROBLÈMES DE RUINE PARISIENNE POUR LES MODÈLES DE RISQUE DE LÉVY

THÈSE

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CONTENTS

LIST OF TABLES

LIST OF FIGURES

RÉSUMÉ

La théorie de la ruine est le cadre de la théorie de risque qui s'intéresse à l'analyse et la modélisation de la situation financière d'une compagnie d'assurance. Sous la théorie de la ruine classique, le processus de risque modélisant l'évolution de la richesse d'une compagnie d'assurance est observé d'une manière continue. La ruine est ainsi déclarée dès que le processus de surplus devient négatif. Récemment, une nouvelle définition de la ruine appelée ruine parisienne a été étudiée pour les processus de Lévy spectralement négatifs. Dans ce cas, la compagnie d'assurance n'est pas immédiatement liquidée lorsqu'elle est en défaut et la ruine survient lorsque le processus de risque reste dans la zone rouge (sous 0) pour une période de temps (consécutive) supérieure à un délai prédéterminé.

Dans cette thèse, nous étudions les problèmes de ruine parisienne pour les processus de risque de Lévy. Le Chapitre 1 est consacré à quelques rappels sur les processus de Lévy spectralement négatifs, les fonctions d'échelle ainsi qu'un aperçu de la littérature académique récente sur la théorie de la ruine classique et parisienne. Motivé par l'article de Loeffen et al. [60], au Chapitre 2, nous généralisons les résultats de cet article en considérant un processus de Lévy réfracté, introduit par Kyprianou et Loeffen [43], comme processus sous-jacent. Des expressions explicites de la probabilité de ruine parisienne et des transformées de Laplace sont obtenues. Quelques exemples sont également présentés. A la fin de ce chapitre, la distribution de Gerber-Shiu à la ruine parisienne pour le processus de Lévy réfracté est obtenue comme une extension du résultat de Baurdoux et al. [8].

Au Chapitre 3, nous unifions les deux types de ruine parisienne en un seul type de ruine appelée *ruine parisienne mixte*. Dans ce cas, la ruine est déclarée la première fois qu'une excursion dans la zone rouge dure plus longtemps qu'un délai avec composantes déterministe et stochastique. Pour cette ruine parisienne, nous identifions la distribution conjointe du temps de ruine et du déficit à la ruine, généralisant ainsi de nombreux résultats précédemment obtenus.

Au Chapitre 4, nous examinons une ruine parisienne sous un modèle avec un schéma d'observation hybride récemment introduit par Li et al. [49]. Dans ce modèle, le processus de surplus est observé d'une manière discrète en des temps d'arrivée d'un processus de Poisson jusqu'à ce que le surplus devienne négatif. À ce moment, le processus est observé d'une manière continue et un délai de recouvrement est accordé à la compagnie d'assurance. Nous améliorons les résultats obtenus et nous calculons d'autres identités.

Au Chapitre 5, nous proposons une mesure de risque de type VaR basée sur la probabilité de ruine parisienne cumulée, introduite par Guérin et Renaud [33]. Nous dérivons quelques propriétés de cette mesure et nous la comparons aux mesures de risque de Trufin et al. [73] et de Loisel et Trufin [62].

Dans le dernier chapitre, des pistes de recherches potentielles sont présentées.

Mots clés : Ruine classique, Ruine parisienne, Processus de risque, Fonction d'échelle, Processus de Lévy réfracté.

ABSTRACT

Ruin theory, as a part of risk theory, is concerned with the analysis of risk processes also known as *insurance risk processes*. Under the classical ruin theory, the risk process is monitored continuously and, as soon as the surplus enters the *red zone* (below 0), ruin is declared and the company ceases its operations. One of the most important risk measure that plays an important role in quantifying risk is the probability of ruin. Recently, an exotic type of ruin, called Parisian ruin, was studied for Lévy insurance risk processes. In this case, the insurance company is not immediately liquidated when it defaults since ruin occurs when the surplus process stays below 0 for a consecutive period of time greater than a pre-specified delay.

In this thesis, we study Parisian ruin problems for Lévy insurance risk processes. The first chapter is dedicated to some background material on spectrally negative Lévy processes and scale functions and to an overview of recent literature on classical and Parisian ruins. Motivated by the paper of Loeffen et al. [58], in Chapter 2, we generalize their results by considering a refracted Lévy process, introduced by Kyprianou and Loeffen [43], as the underlying process. Explicit expressions for the Parisian ruin probability and Laplace transforms are presented. A few examples are also considered. At the end of this chapter, the Gerber-Shiu distribution at Parisian ruin for a refracted Lévy Process is obtained as an extension of the results of Baurdoux et al. [8].

In Chapter 3, we unify two types of Parisian ruin into one called *mixed Parisian ruin*. In this case, ruin is declared the first time an excursion into the red zone lasts longer than an implementation delay with a deterministic and a stochastic component. For this Parisian ruin with mixed delays, we identify the joint distribution of the time of ruin and the deficit at ruin, thus providing generalizations of many previously obtained results in the existing literature.

In the subsequent chapter, we examine a Parisian ruin under a hybrid observation scheme model, as introduced by Li et al. [49]. In that model, the surplus process is monitored discretely at Poisson arrival times until a negative surplus is observed. Then, a fixed grace period is granted to the insurance company and the surplus process is monitored continuously during this grace period. We improve the result originally obtained and we compute other fluctuation identities using a different approach.

In Chapter 5, we propose a VaR-type risk measure, based on cumulative Parisian ruin, as introduced by Guérin and Renaud [33]. We derive some properties of this risk measure and we compare it to the risk measures of Trufin et al. [73] and of Loisel and Trufin [62].

The final chapter discusses some potential directions that future research could take.

Keywords : Classical Ruin, Parisian Ruin, Lévy Insurance Risk Processes, Scale Functions, Refracted Lévy Process.

CHAPTER I

INTRODUCTION

1.1 A Literature review

Ruin theory is concerned with the study of the behaviour of an insurance company's surplus process. The investigation of classical ruin quantities has a long history that started with the well-known Cramér-Lundberg risk reserve process first proposed by Lundberg [63] and further developed by Cramér [15]. The Cramér-Lundberg process is the basic process in risk theory and it reflects the behavior of the insurance business (e.g., number of claims, total claim payments, premium income). The first ruin-based risk measure of interest studied under this model was the probability of ruin which is the probability that the surplus process enters the *red zone* (drops below 0). For this model, an upper bound, called the *Lundberg bound*, was first derived for the infinite-time ruin probability (but not for all claim distributions) using either the martingale theory or an inductive approach (see Dickson [21]). Later, a solution to the integral differential equation satisfied by the survival probability of ruin leads to the well known *Pollaczeck–Khinchine formula* which states that the probability of ruin is equal to the tail distribution function of a compound geometric random variable (see Beekman [9]). In most cases, it is not possible to exactly compute the probability of ruin. Hence, numerical methods, such as the Panjer algorithm [65], De Vylder approximation

[74] and the Beekman-Bowers approximation in [10] have been used to obtain approximations of ruin probabilities.

Contrary to the infinite time ruin probability, the derivation of explicit expressions for the finite-time ruin probability has been and remains a long-standing problem in classical risk theory. The latter can be obtained from the distribution of the time of ruin using either a probabilistic approach, as in Seal [69], based on the skip-free upward and strong Markov property, or an analytical approach using the Lagrange's implicit theorem to invert the Laplace transform of the time to ruin (see Dickson and Willmot [22]). In the particular case of the Cramér-Lundberg model with exponential claims, an explicit expression for the distribution of the time of ruin was derived by Drekic [24]. However, in the literature of Brownian risk processes, many results such as the distribution of the time of ruin, the distribution of the occupation time or the distribution of the area in the red are already known; see Akahori [1], Karatzas et al. [36], Perman and Wellener [67]. A more general quantity of interest in ruin theory is the *Gerber-Shiu expected discounted penalty function* introduced by Gerber and Shiu [30] for the Cramér-Lundberg process, which connects the time of ruin, the surplus before ruin and the deficit at ruin. For example, in a particular case, Lin and Willmot [53], studied the joint and marginal moments of the time of ruin, the surplus before the time of ruin, and the deficit at the time of ruin by solving defective renewal equations. The solutions are expressed in terms of compound geometric tails. A more general treatment of Gerber-Shiu functionals for spectrally negative Lévy processes can be found in Doney and Kyprianou [23], Biffis and Kyprianou [12] and Biffis and Morales [13].

The extension to more general Lévy insurance risk process follows using almost identical arguments as in the case of a Cramér-Lundberg process and, for this class of processes, fluctuation identities involve scale functions (see, e.g., Emery [26], Bertoin [11], Kyprianou [40]). However, difficulties arise whenever the risk process has paths of unbounded variation and, in this case, the passage from the bounded to unbounded variation case can be done using approximation techniques. The first technique, known as ϵ-*approximation*, consists of a spatial shift of the sample paths of the underlying process which leads to a bound that converges to the desired quantity as in Landriault et al. [45], Dassios and Wu [20] and Loeffen et al. [58]. The second approach consists of introducing a sequence of bounded variation processes, $(X_n)_{n\geq 1}$, which converge to the unbounded variation process X as n goes to infinity. Hence, the scale function corresponding to X_n converges to the one of X, which is also true for the quantity of interest (see Bertoin [11], Loeffen et al. [60], Guérin and Renaud [32]). A technique based on excursion theory, away from zero, of strong Markov processes has also been used by Baurdoux et al. [8] and by Kyprianou et al. [44] for refracted Lévy processes. Recently, a new technique based on discrete Poisson observations was proposed as a unified approach to the bounded or unbounded variation cases. This type of approximation is a bridge between periodic and continuous observations since it allows one to recover results in the continuous case; see for example, Albrecher et al. [5], Albrecher and Ivanovs [4] and Albrecher et al. [2].

In the last few years, Parisian ruin theory has attracted considerable attention. In Parisian-type ruin models, the insurance company is not immediately liquidated when it defaults: a grace period is granted before liquidation. More precisely, Parisian ruin occurs if the time spent below a pre-determined critical level is longer than the implementation delay, also called the *clock* (the Parisian ruin time is compared to the classical ruin time in Figure 1.1). The idea of Parisian ruin is inspired from Parisian options which are barrier options for which the knock-in or knock-out is activated when price of the underlying asset price process has spent a given period of time beyond a fixed barrier. The concept of Parisian ruin is of practical importance since, under the U.S. bankruptcy code, a reorganization is proposed to the firm when it defaults instead of immediate liquidation (see François and Morellec [28], Galai et al. [29] and Li et al. [48]).

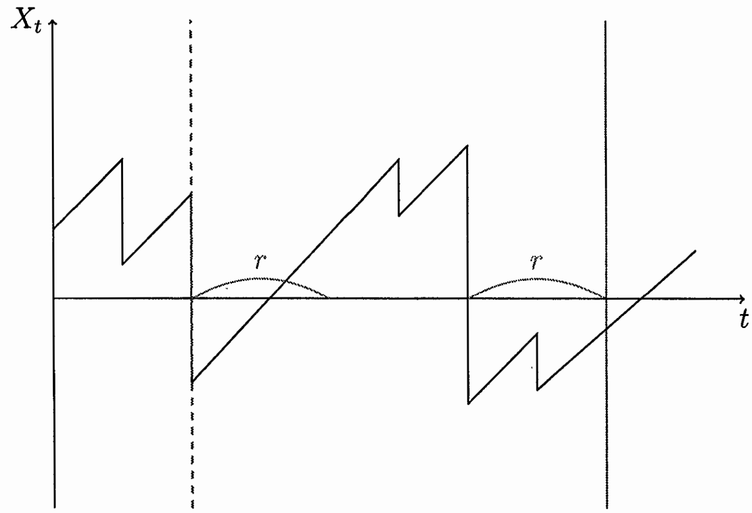

Figure 1.1 A sample path of a Cramér-Lundberg process. The time of classical ruin is shown in dashed red and the Parisian ruin time is shown in blue and r is the delay.

Two types of Parisian ruin have been considered: with deterministic delays or with stochastic delays. In the first case, the Parisian ruin probability was first computed by Dassios and Wu [19] but only for the Brownian motion risk process. The same authors studied in [20] the Laplace transform of the Parisian ruin time and the Parisian ruin probability for the special case of the Cramér-Lundberg process with exponential claims. In a more general setup, Czarna and Palmowski [17] generalized these results for spectrally negative Lévy processes. Their analysis is split into two cases, depending on whether the process is of bounded or unbounded variation. The same problem was studied in Loeffen et al. [58], but the clever use of some identities, especially Kendall's identity that provides the distribution of the first upward crossing of a specific level (see Kendall [37] or Borovkov and

Burq [14]), helps to greatly simplify the expression and unifies the bounded and unbounded variation cases.

When each implementation delay is modelled by an exponentially distributed random variable e_q, as considered in Landriault et al. [46], a copy of e_q is assigned to each excursion below 0 of the risk process. They also considered Parisian ruin with Erlang-distributed implementation delays (see also Albrecher and Ivanovs [4]). Also, for this type of ruin, an expression for the probability of Parisian ruin for the refracted process was given by Renaud [68] using a relationship between occupation times and this type of Parisian ruin. As an extension of the results in [46], Baurdoux et al. [8] derived the Gerber–Shiu distribution at Parisian ruin with exponential implementation delays. Recently, more definitions of Parisian ruin have been proposed. Cumulative Parisian ruin has been proposed in [32]; in that case, the *race* is between a single deterministic clock and the sum of the excursions below the critical level. Moreover, in [18], Parisian ruin with an ultimate bankruptcy level which is in the line with the principle of limited liability. This type of ruin occurs if either the process goes below a predetermined negative level or if Parisian ruin with deterministic delays occurs.

Recently, Poisson observation problems have attracted considerable attention; see Albrecher et al. [5], Albrecher and Ivanovs [4], Li et al. [49], Li et al. [52], among others. In this case, the risk process is monitored discretely at arrival epochs of an independent Poisson process, which can be interpreted as the observation times of the regulatory body (see Figure 4.1). The Poisson observation technique was first used to compute the joint Laplace transform of occupation times over disjoint intervals for spectrally negative Lévy processes (see e.g., Li and Palmowski [47] , Li et al. [52]). Additionally, by considering a risk process observed at discrete Poisson arrival times, we get around the problem caused by the unbounded variation case and we recover classical results (in the continuous monitoring case) when the Poisson observation rate goes to infinity. More interestingly, from the memoryless property of the exponential distribution, Parisian ruin with random delays corresponds to the first time the surplus process is observed below 0 at a Poisson arrival time. In the same vein as that illustrated in Albrecher et al. [5], Li et al. [49] proposed a new approach to study Parisian ruin for spectrally negative Lévy process. They introduce the idea of Parisian ruin under a hybrid observation scheme. More specifically, when the risk process is positive, it is monitored discretely at Poisson arrival times until a negative surplus is observed. Then, the process is observed continuously and a grace period is granted for the insurance company to recover to a solvable level.

Another generalization of models with Parisian ruin with exponential delays is the *omega risk model*, which was introduced by Albrecher et al. [3]. In this model, bankruptcy does not necessarily happen when the surplus is below 0. Instead, it happens at a rate function w that is a decreasing function of the level of negative surplus. Gerber et al. [31] and Albrecher and Lautscham [6] studied this model when the risk process is a Brownian motion with drift or when it is a compound Poisson risk model with exponential claim sizes. Recently, Li and Palmowski [47] studied the Laplace transforms of occupation times weighted by w and expressions are given in terms of the w-scale functions. They recovered previous results on occupation times; see [45] and [60]. These results were extended in Li and Zhou [50] for the refracted Lévy process.

1.2 Lévy insurance risk models

In this section, we give some background on spectrally negative Lévy processes. Let $X = \{X_t,\ t \geq 0\}$ be a Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We will use the following standard notation: the law of X when starting from $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

We refer readers to Kyprianou [42] for further discussion and references related to the analysis of Lévy Insurance Risk Models.

Definition 1. A stochastic process X is a Lévy process if it satisfies the following properties:

- $X_0 = 0$ a.s. ;
- For $0 \le s \le t$, the increment $X_t - X_s$ is independent of the process $\{X_u,\ u \le s\}$ and has the same distribution as X_{t-s} ;
- The paths of X are right continuous with left limits.

By the Lévy-Khintchine formula, the characteristic exponent is given as follows : for $\theta \in \mathbb{R}$,

$$\Psi(\theta) = i\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - \mathrm{e}^{-i\theta z} + i\theta z \mathbf{1}_{|z|<1}\right) \Pi(\mathrm{d}z), \tag{1.1}$$

for $\gamma \in \mathbb{R}$ and $\sigma \ge 0$, and where Π is a σ-finite measure on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

The measure Π is called the *Lévy measure* of X and the parameter σ the Gaussian coefficient. The triplet (γ, σ, Π) is called the *Lévy triplet* and it characterizes the process X. From the Lévy-Itô decomposition, there exists a probability space in which a Lévy process with characteristic exponent given in (1.1), can be expressed as follows

$$X = X^1 + X^2 + X^3,$$

where

- X^1 is a Brownian motion with drift;

- X^2 is a compound Poisson process with Poisson intensity rate $\Pi(\mathbb{R}\setminus(-1,1))$ and the jumps are independent and identically distributed with common distribution $\Pi(\mathrm{d}z)/\Pi(\mathbb{R}\setminus(-1,1))$;

- X^3 is a square integrable martingale with an almost surely countable number of jumps over a finite time interval and jumps magnitude less than 1.

We say that X is a *spectrally negative Lévy process* if it has no positive jumps, i.e. if $\Pi(0,\infty)=0$ (we exclude the case that X has monotone paths). As the Lévy process X has no positive jumps, its Laplace transform exists: for all $\theta, t \geq 0$,

$$\mathbb{E}\left[\mathrm{e}^{\theta X_t}\right] = \mathrm{e}^{t\psi(\theta)},$$

where ψ is the *Laplace exponent*. Then, by the Lévy-Khintchine formula, we have, for $\theta \in \mathbb{R}$,

$$\psi(\theta) = -\Psi(-i\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{0}\left(\mathrm{e}^{\theta z} - 1 - \theta z \mathbf{1}_{\{z>-1\}}\right)\Pi(\mathrm{d}z). \qquad (1.2)$$

The Laplace exponent ψ is infinitely differentiable, strictly convex and

$$\lim_{\theta\to\infty}\psi(\theta) = \infty,$$

whilst $\psi(0) = 0$.

A spectrally negative Lévy process might have paths of bounded or unbounded variation. In the first case, that is when $\int_0^1 z\Pi(\mathrm{d}z) < \infty$ and $\sigma = 0$, we can write

$$X_t = ct - S_t,$$

where $c = \gamma + \int_0^1 z\Pi(\mathrm{d}z) > 0$ is the *drift* of X and where $S = \{S_t, t \geq 0\}$ is a driftless subordinator which is a Lévy process with non-decreasing paths (e.g. a Gamma process or a compound Poisson process with positive jumps).

The right-inverse of ψ is a function $\Phi\colon [0,\infty) \to [0,\infty)$ defined by $\Phi(p) = \sup\{\theta \geq 0\colon \psi(\theta) = p\}$ such that

$$\psi\left(\Phi(p)\right) = p, \quad p \geq 0.$$

We will write $\Phi = \Phi(p)$ when $p = 0$. Note that we have $\Phi(p) = 0$ if and only if $p = 0$ and $\psi'(0+) \geq 0$. The last condition is known as the *net profit condition* and it ensures that X drifts to infinity (i.e. $\lim_{t\to\infty} X_t = \infty$). Otherwise, ruin occurs with probability one.

In ruin theory, spectrally negative Lévy processes are also called *Lévy insurance risk processes.*

1.3 Scale functions

In the the study of spectrally negative Lévy processes, we often want to express ruin-related quantities in terms of scale functions.

Definition 2. For $q \geq 0$, the q-scale function of X is defined as the unique, continuous and increasing function on $[0, \infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta y} W^{(q)}(y)\mathrm{d}y = \frac{1}{\psi_q(\theta)}, \quad \text{for } \theta > \Phi(q), \tag{1.3}$$

where

$$\psi_q(\theta) = \psi(\theta) - q, \tag{1.4}$$

and such that $W^{(q)}(x) = 0$ for $x < 0$.

We will write $W = W^{(0)}$ when $q = 0$. Unfortunately, explicit expressions of scale function are not always known. However, the inversion of the Laplace transform (1.3) can be done numerically as in [71].

Here are some examples for which the scale functions can be computed explicitly.

Example 3. *When X is a Cramér-Lundberg risk process with exponentially distributed claims, then*

$$X_t - X_0 = ct - \sum_{i=1}^{N_t} C_i \quad ,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity $\eta > 0$, and where

$\{C_1, C_2, \dots\}$ *are independent and exponentially distributed random variables with parameter* α. *The Poisson process and the random variables are mutually independent. In this case, the Laplace exponent of* X *is given by*

$$\psi(\theta) = c\theta + \eta\left(\frac{\alpha}{\theta+\alpha} - 1\right), \quad \textit{for } \theta > -\alpha.$$

Then, for $x \geq 0$, *we have*

$$W^{(q)}(x) = \frac{\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{\mathrm{e}^{-\zeta_1 x}}{\psi'(-\zeta_1)},$$

where

$$\begin{aligned} \zeta_1 &= \frac{1}{2c}\left(\sqrt{(\eta+q-\alpha c)^2 + 4c\alpha q} - (\eta+q-\alpha c)\right), \\ \Phi(q) &= \frac{1}{2c}\left(\sqrt{(\eta+q-\alpha c)^2 + 4c\alpha q} + (\eta+q-\alpha c)\right). \end{aligned}$$

Example 4. *If* X *is a Brownian risk process, i.e. if*

$$X_t - X_0 = ct + \sigma B_t,$$

where $B = \{B_t, t \geq 0\}$ *is a standard Brownian motion, then the Laplace exponent of* X *is given by*

$$\psi(\theta) = c\theta + \frac{1}{2}\sigma^2\theta^2.$$

Then, for $x \geq 0$, *we have*

$$W^{(q)}(x) = \frac{1}{\sqrt{c^2+2q\sigma^2}}\left(\mathrm{e}^{(\sqrt{c^2+2q\sigma^2}-c)x/\sigma^2} - \mathrm{e}^{-(\sqrt{c^2+2q\sigma^2}+c)x/\sigma^2}\right),$$

Example 5. *When* X *is a jump diffusion process, then*

$$X_t - X_0 = ct - \sum_{i=1}^{N_t} C_i + \sigma B_t,$$

where $\{C_1, C_2, \dots\}$ *are independent of* N *and* B, *the Laplace exponent of* X *is given by*

$$\psi(\theta) = \frac{\sigma^2}{2}\theta^2 + c\theta + \eta\left(\frac{\alpha}{\theta+\alpha} - 1\right), \quad \textit{for } \theta > -\alpha.$$

Then, for $x \geq 0$, we have

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{-\zeta_1 x}}{\psi'(-\zeta_1)} + \frac{e^{-\zeta_2 x}}{\psi'(-\zeta_2)},$$

where $\Phi(q)$, ζ_1 and ζ_2 are the solutions of the equation $\psi(\theta) = q$ such that $-\zeta_2 < -\alpha < -\zeta_1 < 0 < \Phi(q)$.

Example 6. *If X is a α-stable risk process with $\alpha \in (1, 2)$, i.e. if*

$$X_t - X_0 = ct + Z_t,$$

where $Z = \{Z_t, t \geq 0\}$ is a spectrally negative α-stable process, then the Laplace exponent of X is given by $\psi(\theta) = \theta^\alpha$. Then, for $x \geq 0$, we have

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_{\alpha,1}(qx^\alpha),$$

where $E_{\alpha,1}(x) = \sum_{n\geq 0} \frac{x^n}{\Gamma(1+\alpha n)}$ is the Mittag-Leffler function, $E'_{\alpha,1}$ its derivative and Γ is the Gamma function.

The Laplace exponent of X under the change of measure

$$\left.\frac{d\mathbb{P}^c_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = e^{c(X_t - x) - \psi(c)t},$$

for c such that $\mathbb{E}_x\left[e^{cX_1}\right] < \infty$, is given by

$$\psi^c(\theta) = \psi(\theta + c) - \psi(c), \quad \text{for } \theta \geq -c, \tag{1.5}$$

and the corresponding scale function $W^{(q)}_c$ is given by

$$W^{(q)}(x) = e^{cx} W_c^{(q-\psi(c))}(x). \tag{1.6}$$

The q-scale function $W^{(q)}$ is differentiable except for at most countably many points. Moreover, $W^{(q)}$ is continuously differentiable if X has paths of unbounded variation or if the tail of the Lévy measure is continuous, and it is twice continuously differentiable on $(0, \infty)$ if $\sigma > 0$. The initial values of $W^{(q)}$ and $W^{(q)\prime}$ are given by

$$W^{(q)}(0+) = \begin{cases} 1/c & \text{when } \sigma = 0 \text{ and } \int_0^1 z\Pi(\mathrm{d}z) < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

$$W^{(q)\prime}(0+) = \begin{cases} 2/\sigma^2 & \text{when } \sigma > 0, \\ (\Pi(0,\infty) + q)/c^2 & \text{when } \sigma = 0 \text{ and } \Pi(0,\infty) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

Using the notation introduced by Albrecher et al. [5], we define another scale function $Z_q(x,\theta)$ by

$$Z_q(x,\theta) = \mathrm{e}^{\theta x}\left(1 - \psi_q(\theta)\int_0^x \mathrm{e}^{-\theta y} W^{(q)}(y)\mathrm{d}y\right), \tag{1.7}$$

for $x \geq 0$, and by $Z_q(x,\theta) = \mathrm{e}^{\theta x}$, for $x < 0$. We will write $Z = Z_0$ when $q = 0$. The scale function $Z_q(x,\theta)$ was first defined by Avram et al. [7] as the Esscher transform of the following scale function $Z^{(q)}(x)$ given by

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y, \quad x \in \mathbb{R}, \tag{1.8}$$

also known as a *second* scale function. Of course, for $\theta = 0$, $Z_q(x,\theta) = Z^{(q)}(x)$.

We can also re-write the scale function $Z_q(x,\theta)$ as follows :

$$Z_q(x,\theta) = \psi_q(\theta)\int_0^\infty \mathrm{e}^{-\theta y} W_q(x+y)\mathrm{d}y, \quad x \geq 0,\ \theta \geq \Phi(q). \tag{1.9}$$

1.4 Delayed scale functions

In the the study of Parisian ruin problems, identities can be expressed in terms of delayed scale functions. Inspired by [59], we define the (r,s)-*delayed* p-*scale function of* X by

$$\Lambda^{(p)}(x;r,s) = \int_0^\infty \mathcal{W}_z^{(p+s,-s)}(x+z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z), \tag{1.10}$$

where the auxiliary function $\mathcal{W}$ (also known as the *second-generation scale function*) is given by

$$\mathcal{W}_a^{(p,s)}(x) = W^{(p+s)}(x) - s\int_0^a W^{(p+s)}(x-y)\,W^{(p)}(y)\,\mathrm{d}y, \tag{1.11}$$

for $p, p+s \geq 0$ and $a, x \in \mathbb{R}$. As obtained in [60], we have

$$(s-p)\int_0^x W^{(p)}(x-y)W^{(s)}(y)\mathrm{d}y = W^{(s)}(x) - W^{(p)}(x). \tag{1.12}$$

Thus, using (1.12), $\mathcal{W}_a^{(p,s)}(x)$ can also be written as follows:

$$\mathcal{W}_a^{(p,s)}(x) = W^{(p)}(x) + s\int_a^x W^{(p+s)}(x-y)W^{(p)}(y)\mathrm{d}y, \tag{1.13}$$

and then we can rewrite $\Lambda^{(p)}(x;r,s)$ in the form

$$\Lambda^{(p)}(x;r,s) = \int_0^\infty \mathcal{W}_x^{(p,s)}(x+z)\,\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z). \tag{1.14}$$

When $s=0$, we recover the function $\Lambda^{(p)}$ defined by Loeffen et al. [59]. To be more precise, when $s=0$, we have $\Lambda^{(p)}(x;r,0) = \Lambda^{(p)}(x,r)$, where

$$\Lambda^{(p)}(x,r) = \int_0^\infty W^{(p)}(x+z)\,\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z).$$

We write $\Lambda = \Lambda^{(0)}$. Here is another connection between these two functions. We can re-write $\Lambda^{(p)}(x;r,s)$ using (1.11)

$$\Lambda^{(p)}(x;r,s) = \Lambda^{(p+s)}(x,r) - s\int_0^x \Lambda^{(p+s)}(y,r)\,W^{(p)}(x-y)\,\mathrm{d}y. \tag{1.15}$$

In Chapter 3, we will see that the delayed scale function in (1.10) plays a similar rôle, as the one played by the scale functions $W^{(p)}$ and Z_p in the classical and Parisian fluctuation identities respectively, for the mixed-type Parisian ruin.

1.5 Classical ruin exit problems

In this section, we present some of the existing fluctuation identities for spectrally negative Lévy processes in terms of their scale functions.

First, recall the definitions of standard first-passage stopping times : for $b \in \mathbb{R}$,

$$\tau_b^- = \inf\{t > 0 \colon X_t < b\} \quad \text{and} \quad \tau_b^+ = \inf\{t > 0 \colon X_t > b\},$$

with the convention $\inf \emptyset = \infty$. If $a \leq x \leq b$ and $q, \lambda \geq 0$, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^- + \lambda X_{\tau_a^-}} \mathbf{1}_{\{\tau_a^- < \tau_b^+\}}\right] = Z_q(x-a,\lambda) - \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} Z_q(b-a,\lambda), \tag{1.16}$$

and also

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \tau_a^-\}}\right] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}. \tag{1.17}$$

Letting $a \to -\infty$ in (1.17) and using (1.6), we have

$$\lim_{a\to-\infty} \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} = \mathrm{e}^{\Phi(q)(x-b)}, \tag{1.18}$$

we then obtain the well-known expression for the Laplace transform of the first passage time above b

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \infty\}}\right] = \mathrm{e}^{\Phi(q)(x-b)}. \tag{1.19}$$

Moreover, the *classical* probability of ruin is given by

$$\mathbb{P}_x\left(\tau_0^- < \infty\right) = 1 - \left(\mathbb{E}\left[X_1\right]\right)_+ W(x), \tag{1.20}$$

where $(x)_+ = \max(x, 0)$.

The next identities can be found in numerous references, but first proved in [43] and [60] (and also indirectly proved in [45]). For any $p, q \geq 0$ and $a \leq x \leq b$, we have

$$\begin{aligned}\mathbb{E}_x&\left[\mathrm{e}^{-q\tau_a^-} W^{(p)}\left(X_{\tau_a^-} + z\right) \mathbf{1}_{\{\tau_a^- < \tau_b^+\}}\right] \\ &= \mathcal{W}_{a+z}^{(p,q-p)}(x+z) - \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} \mathcal{W}_{a+z}^{(p,q-p)}(b+z),\end{aligned} \tag{1.21}$$

and when $b \to \infty$, we obtain

$$\begin{aligned}\mathbb{E}_x&\left[\mathrm{e}^{-q\tau_a^-} W^{(p)}\left(X_{\tau_a^-} + z\right) \mathbf{1}_{\{\tau_a^- < \infty\}}\right] \\ &= \mathcal{W}_{a+z}^{(p,q-p)}(x+z) - W^{(q)}(x-a)\, Z_p(z+a, \Phi(q)).\end{aligned} \tag{1.22}$$

These identities (and also similar ones involving the scale function Z_q) constitute a major turning point in terms of simplifying many results by getting rid of the Lévy measure in many expressions of quantities involving spectrally negative Lévy processes. Recently, a generalization of Equation (1.21) and (1.22) was derived by Kuang and Zhou [38] in order to compute n-dimensional Laplace transforms of occupation times over n-disjoint subintervals. Also, Renaud [68] derived its analogues for the refracted Lévy process.

We also recall the following expression for the $q-$potential measure of X killed on exiting $(-\infty, a]$: for $a \in \mathbb{R}$ and $x, y \leq a$, we have

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_a^+\right)\mathrm{d}t = \left(\mathrm{e}^{\Phi(q)(x-a)}W^{(q)}\left(a-y\right) - W^{(q)}\left(x-y\right)\right)\mathrm{d}y. \tag{1.23}$$

Finally, here is Kendall's identity (see [11, Corollary VII.3]): on $(0,\infty)\times(0,\infty)$, we have

$$r\mathbb{P}(\tau_z^+ \in \mathrm{d}r)\mathrm{d}z = z\mathbb{P}(X_r \in \mathrm{d}z)\mathrm{d}r.$$

1.6 Parisian ruin exit problems

In this section, we present some of the existing fluctuation identities with delays for a standard Lévy insurance risk process X.

1.6.1 Parisian ruin with deterministic delays

The time of Parisian ruin, with a fixed delay $r > 0$, has been studied in [58]: it is defined as

$$\kappa_r = \inf\left\{t > 0 \colon t - g_t > r\right\}, \tag{1.24}$$

where $g_t = \sup\left\{0 \leq s \leq t \colon X_s \geq 0\right\}$ is the last time before t when the process was non-negative.

Dassios and Wu [20] obtained the probability of Parisian ruin for the special case of a Cramér-Lundberg process with exponential claims. Czarna and Palmowski [17] later generalized this result for spectrally negative Lévy processes but the final expression is very explicit and is given in the next theorem.

Theorem 7 (Theorem 1 in [16]). *Parisian ruin probability for a spectrally negative Lévy risk process equals: for $x \in \mathbb{R}$ and $r, \theta > 0$*

$$\begin{aligned}\mathbb{P}_x(\kappa_r < \infty) &= \mathbb{P}_x\left(\tau_0^- < \infty\right)\mathbb{P}(\kappa_r < \infty) \\ &\quad + (1 - \mathbb{P}(\kappa_r < \infty))\int_0^\infty \mathbb{P}_x\left(\tau_0^+ > r\right)\mathbb{P}_x\left(\tau_0^- < \infty, -X_{\tau_0^-} \in \mathrm{d}z\right), \quad (1.25)\end{aligned}$$

where

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\theta r}\left(\int_0^\infty \mathbb{P}\left(\tau_z^+ > r\right)\mathbb{P}_x\left(\tau_0^- < \infty, -X_{\tau_0^-} \in \mathrm{d}z\right)\right)\mathrm{d}r \\ &\qquad = \frac{1 - \psi'(0^+)W(x)}{\theta} - \frac{\mathrm{e}^{\Phi(\theta)x}}{\theta}\left(Z_{\Phi(\theta)}^{(-\theta)}(x) + \frac{\theta}{\Phi(-\theta)}W_{\Phi(\theta)}^{(-\theta)}(x)\right). \quad (1.26)\end{aligned}$$

In order to compute the probability of Parisian ruin starting at $x = 0$, Czarna and Palmowski [17] splitted the results into two cases, for processes of bounded and unbounded variation.

Theorem 8 (Theorem 2 in [16]). *If X is a process of bounded variation, then*

$$\mathbb{P}(\kappa_r < \infty) = \frac{\int_0^\infty \mathbb{P}(\tau_z^+ > r)\mathbb{P}\left(\tau_0^- < \infty, -X_{\tau_0^-} \in \mathrm{d}z\right)}{\mathbb{P}\left(\tau_0^- = \infty\right) + \int_0^\infty \mathbb{P}(\tau_z^+ > r)\mathbb{P}\left(\tau_0^- < \infty, -X_{\tau_0^-} \in \mathrm{d}z\right)}, \quad (1.27)$$

where

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\theta r}\left(\int_0^\infty \mathbb{P}\left(\tau_z^+ > r\right)\mathbb{P}\left(\tau_0^- < \infty, -X_{\tau_0^-} \in \mathrm{d}z\right)\right)\mathrm{d}r \\ &\qquad = \frac{1}{\theta c}\left(1 - \mathrm{e}^{-\Phi(\theta)z}\right)\Pi_{\hat{X}}(z)\mathrm{d}z, \quad (1.28)\end{aligned}$$

and $\hat{X}$ is the dual process of X and $\Pi_{\hat{X}}$ its Lévy measure.

If X is a process of unbounded variation, then

$$\mathbb{P}(\kappa_r < \infty) = \lim_{b\to\infty}\lim_{\epsilon\to 0}\frac{\mathbb{P}_\epsilon\left(\tau_0^- < b\right) - p(b,r)}{1 - p(b,r)}, \quad (1.29)$$

where

$$p(b,r) = \mathbb{P}\left(\tau_\epsilon^+ \le r\right)\mathbb{P}_\epsilon\left(\tau_0^- < b, X_{\tau_0^-} = 0\right) \\ + \int_0^\infty \mathbb{P}\left(\tau_{z+\epsilon}^+ \le r\right)\mathbb{P}_\epsilon\left(\tau_0^- < b, -X_{\tau_0^-} \in \mathrm{d}z\right). \quad (1.30)$$

Later, Loeffen et al. [58] derived a new expression for $\mathbb{P}_x(\kappa_r < \infty)$ and unified the bounded and unbounded variation cases using the ϵ-*approximation* technique. More specifically, they derived the following new identities that lead to an explicit inverse Laplace transform of (1.28). The key ingredient of the proof is Kendall's identity.

Lemma 9. *For $r, \theta > 0$, we have*

$$\Lambda(0,r) = 1, \quad (1.31)$$

$$\int_0^\infty \mathrm{e}^{-\theta r}\int_y^\infty \frac{z}{r}\mathbb{P}\left(X_r \in \mathrm{d}z\right)\mathrm{d}r = \frac{1}{\Phi(\theta)}\mathrm{e}^{-\Phi(\theta)y}, \quad y \ge 0, \quad (1.32)$$

$$\mathbb{P}_x\left(\tau_0^+ \le r\right) = \Lambda(x,r) \quad x \le 0, \quad (1.33)$$

$$\mathbb{E}_x\left[\mathrm{e}^{\Phi(\theta)X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^- < \infty\}}\right] = \frac{\theta}{\Phi(\theta)}\int_0^\infty \mathrm{e}^{-\Phi(\theta)y}W'(x+y)\mathrm{d}y. \quad (1.34)$$

Using those identities, the main result in [58] is a very nice and compact expression for the probability of Parisian ruin.

Theorem 10 (Theorem 1 in [58]). *For $r > 0$ and $x \in \mathbb{R}$, we have*

$$\mathbb{P}_x\left(\kappa_r < \infty\right) = 1 - (\mathbb{E}[X_1])_+ \frac{\Lambda(x,r)}{\int_0^\infty (z/r)\mathbb{P}\left(X_r \in \mathrm{d}z\right)}. \quad (1.35)$$

Remark 11. Applying the Initial Value Theorem to Equation (1.35) (to the numerator and the denominator separately), one can recover the expression of the classical ruin (5.2) as $r \to 0$.

Remark 12. Instead of looking to invert the Laplace transform (1.28) using again the technique of taking Laplace transform with respect to the delay r, we can compute the right-hand side of Equation (1.25) using a probabilistic approach by combining (1.34) with (1.22) and Tonelli's Theorem to get

$$\mathbb{E}_x\left[\Lambda(X_{\tau_0^-},r)\mathbf{1}_{\{\tau_0^-<\infty\}}\right]=\int_0^\infty (W(x+z)-W(x))\,\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z). \tag{1.36}$$

Recently, Loeffen et al. [59] proposed a new analytical approach for computing some Parisian ruin identities based on the work of Loeffen [57]. They derived the joint Laplace transform of the Parisian ruin time κ_r and the level of the deficit X_{κ_r}.

Theorem 13 (Theorem 3.1 in [59]). *For $p,\lambda\geq 0$, $b,r>0$ and $x\leq b$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_r+\lambda X_{\kappa_r}}\mathbf{1}_{\{\kappa_r<\tau_b^+\}}\right]=\mathcal{F}^{(p,\lambda)}(x,r)-\frac{\Lambda^{(p)}(x,r)}{\Lambda^{(p)}(b,r)}\mathcal{F}^{(p,\lambda)}(b,r),$$

where

$$\mathcal{F}^{(p,\lambda)}(x,r)\ =\ \mathrm{e}^{\psi_p(\lambda)r}\left(Z_p(x,\lambda)-\psi_p(\lambda)\int_0^r \mathrm{e}^{-\psi(\lambda)u}\Lambda^{(p)}(x;u)\,\mathrm{d}u\right).$$

The auxiliary function $\mathcal{F}^{(p,\lambda)}(x,r)$ was not defined in [59] but greatly helps to reduce the final expression (it will also be generalized in Chapter 3). The Parisian two-sided exit problem is also given by

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r\}}\right]=\frac{\Lambda^{(p)}(x,r)}{\Lambda^{(p)}(b,r)}. \tag{1.37}$$

Remark 14. It is worthwhile to mention that the last result was first derived in [55] (see Theorem 21 for $\delta=0$ in Chapter 3).

1.6.2 Parisian ruin with exponential delays

The time of Parisian ruin with exponential delays is defined as

$$\kappa^q=\inf\left\{t>0\mid t-g_t>\mathrm{e}_q^{g_t}\right\},$$

where e_q^{gt} is exponentially distributed with rate $q > 0$. More precisely, a copy of e_q is assigned to each excursion below level 0. The probability of Parisian ruin with exponential delays was first computed in [45] through the relation between the occupation times and this type of Parisian ruin (see Remark 18).

Theorem 15. *For $q > 0$ and $x \in \mathbb{R}$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{\{X_s \le 0\}}\mathrm{d}s}\right] = 1 - (\mathbb{E}[X_1])_+ \frac{\Phi(q)}{q}\mathcal{H}^{(q,-q)}(x), \tag{1.38}$$

where the function $\mathcal{H}^{(p,q)}(x)$ was defined by Loeffen et al. [60] *and given by*

$$\mathcal{H}^{(p,q)}(y) = \mathrm{e}^{\Phi(p)y}\left(1 + q\int_0^y \mathrm{e}^{-\Phi(p)z}W^{(p+q)}(z)\mathrm{d}z\right), \tag{1.39}$$

where $p \ge 0$ and $q \in \mathbb{R}$ such that $p + q \ge 0$ and $y \in \mathbb{R}$.

There is a one-to-one correspondence between $Z_q(x,\theta)$ and $\mathcal{H}$ given by

$$\mathcal{H}^{(p,q)}(x) = Z_{p+q}(x, \Phi(p)). \tag{1.40}$$

Thus, Equation (1.38) can be re-written in terms of the scale function Z and the probability of Parisian ruin is given by

$$\mathbb{P}_x(\kappa^q < \infty) = 1 - (\mathbb{E}[X_1])_+ \frac{\Phi(q)}{q} Z(x, \Phi(q)). \tag{1.41}$$

More general identities were later obtained. For example, for $x \le b$ and $p, q \ge 0$, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+ < \kappa^q\}}\right] = \frac{Z_p(x, \Phi(p+q))}{Z_p(b, \Phi(p+q))}, \tag{1.42}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-p\kappa^q}\mathbf{1}_{\{\kappa^q < \tau_b^+\}}\right] = \frac{q}{p+q}\left(Z^{(p)}(x) - \frac{Z_p(x, \Phi(p+q))}{Z_p(b, \Phi(p+q))}Z^{(p)}(b)\right), \tag{1.43}$$

where the first one is taken from [5] but was first proved in [60] (see Remark 18). The second identity is taken from [8] (where a slightly different notation is used).

Also, the Gerber-Shiu distribution at Parisian ruin was computed in [8]. Their expressions can be re-written in terms of the second generation scale function $\mathcal{W}$ as follows : for θ, a, $b \geq 0$, $x \in [-a, b)$ and $y \in [-a, 0)$, we have

$$\begin{aligned}\mathbb{E}_x &\left[\mathrm{e}^{-\theta\kappa^q}, X_{\kappa^q} \in \mathrm{d}y, \kappa^q < \tau_b^+ \wedge \tau_{-a}^-\right] \\ &= q\left[\mathcal{W}_b^{(\theta,q)}(b-y)\frac{\mathcal{W}_x^{(\theta,q)}(a+x)}{\mathcal{W}_b^{(\theta,q)}(a+b)} - \mathcal{W}_x^{(\theta,q)}(x-y)\right]\mathrm{d}y,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_x &\left[\mathrm{e}^{-\theta\kappa^q}, X_{\kappa^q} \in \mathrm{d}y, \kappa^q < \tau_b^+\right] \\ &= q\left[\frac{Z_\theta(x, \Phi(\theta+q))}{Z_\theta(b, \Phi(\theta+q))}\mathcal{W}_b^{(\theta,q)}(b-y) - \mathcal{W}_x^{(\theta,q)}(x-y)\right]\mathrm{d}y, \quad (1.44)\end{aligned}$$

where $y \leq 0$ and $x \leq b$. When $b \to \infty$, we obtain

$$\begin{aligned}\mathbb{E}_x &\left[\mathrm{e}^{-\theta\kappa^q}, X_{\kappa^q} \in \mathrm{d}y, \kappa^q < \infty\right] \\ &= \left(\left(\Phi(\theta+q) - \Phi(\theta)\right) Z_\theta(x, \Phi(\theta+q))\, Z_{\theta+q}(-y, \Phi(\theta)) - q\mathcal{W}_x^{(\theta,q)}(x-y)\right)\mathrm{d}y. \\ &\qquad\qquad (1.45)\end{aligned}$$

Remark 16. The last Gerber-Shiu function agrees with the result in Equation (19) in Guérin and Renaud [32]. The connexion is made through the following potential measure discounted by the occupation time over the half line $(-\infty, 0)$

$$\mathbb{E}_x\left[\mathrm{e}^{-\theta\kappa_q}, X_{\kappa^q} \in \mathrm{d}y, \kappa^q < \infty\right] = \frac{q}{\theta}\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^{\mathrm{e}_\theta}\mathbf{1}_{(-\infty,0)}(X_s)\mathrm{d}s}, X_{\mathrm{e}_\theta} \in \mathrm{d}y\right].$$

This result was also obtained by Li et al. [52] using the Poisson observation technique and they also extended it to the joint Laplace transform of both semi-intervals $(0, \infty)$ and $(-\infty, 0)$.

Remark 17. At first sight, one might think that the last identities can be recovered from those in Subsection 1.6.1 only by taking Laplace transforms with respect to r. However, this is not the case. It is due to the fact that a copy of e_q is assigned to each excursion below zero of the risk process X, not a single one for the whole trajectory.

Remark 18. By the memoryless property of the exponential distribution, Parisian ruin with exponential delays corresponds to the first time the process X is observed to be negative at Poisson arrival times, that is

$$T_0^- = \min\{T_i > 0\colon X_{T_i} < 0,\ i \in \mathbb{N}\}, \tag{1.46}$$

where T_i are the arrival times of an independent Poisson process of rate $q > 0$. We recall that $T_0 = 0$ and the inter-arrival times $T_i - T_{i-1}$, for $i \geq 1$, are independent and exponentially distributed with parameter λ.

Then, we have the following equalities :

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa^q\}}\right] &= \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^-\}}\right] = \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}, N\left(B\right) = 0, \tau_b^+ < \infty\right] \\ &= \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+ - q\int_0^{\tau_b^+} \mathbf{1}_{\{X_t<0\}}\mathrm{d}t}\mathbf{1}_{\{\tau_b^+<\infty\}}\right],\end{aligned}$$

where N is an independent Poisson random measure with intensity $q\mathrm{d}t$ and the set B is given by

$$B = \left\{t \in [0, \tau_b^+),\ X_t < 0\right\}.$$

See Albrecher el al. [5] and Loeffen et al. [60] for more details.

1.7 Organization of the thesis and contributions

The rest of the thesis is organized as follows. In Chapter 2, we investigate Parisian ruin for a Lévy surplus process with an adaptive premium rate, namely a refracted Lévy process. The latter has been used to build models with a threshold strategy at a fixed level b which is the strategy where no dividends are paid out when the process is below b and dividends are paid out, at rate δ, when the process is above b. In Kyprianou and Loeffen [43], the refracted Lévy process was introduced and many fluctuation identities, including the probability of ruin, have been derived. Our main contribution is a generalization of Theorem (10) for the probability of

Parisian ruin of a standard Lévy insurance risk process. More general Parisian boundary-crossing problems with a deterministic implementation delay are also considered. Despite the more general setup considered here, our main result is as compact and has a similar structure. This Chapter constitutes a paper that has already appeared in *Insurance : Mathematics and Economics* and has been written with Irmina Czarna and Jean-François Renaud. At the end of this Chapter, the Gerber-Shiu distribution at Parisian ruin with stochastic delays for a refracted Lévy Process is obtained as an extension of the results of Baurdoux et al. [8]. This section is not a part of the paper.

Chapter 3 contains the paper *A unified approach to ruin probabilities with delays for spectrally negative Lévy processes*, written with Jean-François Renaud, which is still under review. In this Chapter, we unify two approaches for the definition of Parisian ruin, defined in Subsections 1.6.1 and 1.6.2, by considering a Parisian ruin with *mixed delays*. For this type of Parisian ruin, we identify the joint distribution of the time of ruin and the deficit at ruin, therefore providing generalizations of many results previously obtained in the literature, such as in [8] and [59] for the case of exponential delays and that of deterministic delays, respectively.

Chapter 4 contains the paper *A note on Parisian ruin under a hybrid observation scheme* that has already appeared in *Statistics & Probability Letters* and for which I am the sole author (see [54]). This paper is still under review. In this paper, we study the concept of Parisian ruin under a hybrid observation scheme, as introduced by Li et al. [49]. Under this model, the process is observed at Poisson arrival times whenever the business is financially healthy and it is continuously observed when it enters a period of financial distress (below 0). The Parisian ruin is then declared when the duration of this period is greater than a fixed delay. We improve the result originally obtained in [49] and we compute other fluctuation identities.

In Chapter 5, we investigate a VaR-type risk measure introduced by Guérin and Renaud [33] and which is based on cumulative Parisian ruin. We derive some properties of this risk measure and we compare it to the risk measures of Trufin et al. [73] and Loisel and Trufin [62]. This Chapter constitutes the manuscript *A VaR-type risk measure derived from cumulative Parisian ruin* that has appeared in the special issue *"Risk, Ruin and Survival: Decision Making in Insurance and Finance"* of *Risks* journal written with Jean-François Renaud (see [56]).

The final Chapter is the conclusion and it discusses some potential directions for future research.

CHAPTER II

PARISIAN RUIN FOR A REFRACTED LÉVY PROCESS

2.1 Introduction

In this Chapter, we study the Parisian ruin for a Lévy process with adaptive premium known as the *refracted Lévy process.* More precisely, when the company is in financial distress, that is when its surplus is below the critical level, the premium is increased; and when its surplus leaves that *red zone* then the premium is brought back to its regular level. Therefore, we will use a refracted Lévy process as our surplus process. Note that we could also interpret this change in the premium rate as a way to invest (for R&D, modernization, etc.): if the surplus of the company is in a good financial situation, i.e. above the *critical level*, then it invests at rate δ; otherwise it does not. However, for the rest of this Chapter, we will use the previous interpretation.

In general, fluctuation identities for refracted Lévy processes can be tedious compared to their classical counterparts because scale functions of two different Lévy risk processes are involved (see [43]). Therefore, our main contribution is a surprisingly compact expression for the probability of Parisian ruin for a refracted Lévy risk process (see Equation (2.13) below), in the spirit of the one in Equation (1.35) for a standard Lévy risk process. Our formula also provides information on how the *refraction parameter* affects this probability while displaying the impact of the *delay parameter*. Moreover, we analyze more general Parisian boundary-crossing problems for the refracted Lévy process which have not been studied previously, even for a standard Lévy risk process. As a consequence, when the *refraction parameter* it set to zero, new identities for the classical Lévy setup are obtained.

The rest of the Chapter is organized as follows. In Section 2.2, we present our model in more details. The main results are presented in Section 2.3, while Section 2.4 presents a few examples. Section 2.5 is devoted to the proofs of the main results as well as (new) technical lemmas.

2.2 The model and background material

As mentioned in the introduction, we are interested in a surplus process U whose dynamics change by adding a fixed linear drift (premium) whenever it is below the critical level b, a region also called the *red zone* (see Figure 2.1). Without loss of generality, we will choose b to be 0.

In our model, Y is the surplus process during *regular business periods* (above zero), while X is the surplus process, with an additional rate of premium δ, for *critical business periods* (below zero). More precisely, let Y be a Lévy insurance risk process (see the definition in section 1.2) modelling the dynamic of the surplus U above 0. Below 0, our surplus process U evolves as $X = \{X_t = Y_t + \delta t, t \geq 0\}$. Clearly, X is also a Lévy insurance risk process; in fact, X and Y share many properties except for those affected by the value of the linear part of the Lévy process.

In other words, our surplus process is given by the solution $U = \{U_t, t \geq 0\}$ to the following stochastic differential equation: for $\delta \geq 0$,

$$\mathrm{d}U_t = \mathrm{d}Y_t + \delta \mathbf{1}_{\{U_t<0\}} \mathrm{d}t, \quad t \geq 0. \tag{2.1}$$

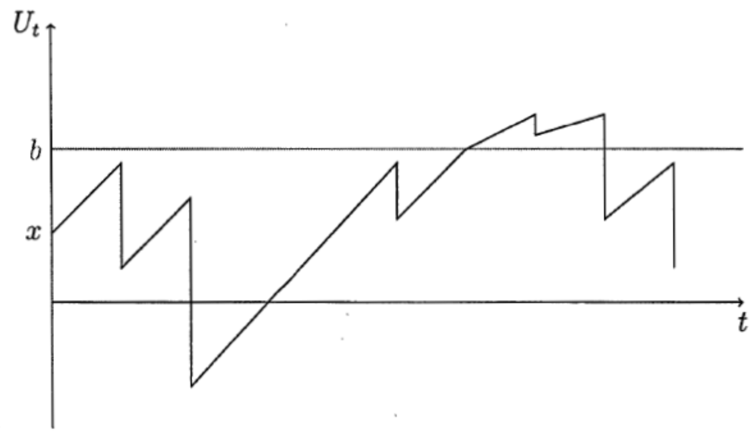

Figure 2.1 A sample path of the refracted process U when X is a Cramér-Lundberg process.

Above 0, our surplus process U evolves as $Y = \{Y_t = X_t - \delta t, t \geq 0\}$ which is also a Lévy insurance risk process (if it doesn't have monotone paths): its linear part is given by $\gamma - \delta$ but it has the same Gaussian coefficient σ and Lévy measure Π as X. In fact, X and Y share many properties. Note that we could have specified Y first and then define $X = \{X_t = Y_t + \delta t, t \geq 0\}$. The two approaches are equivalent. The Laplace exponent of Y is given by

$$\lambda \mapsto \psi(\lambda) - \delta\lambda,$$

where $\psi(\lambda)$ is defined in (1.2), with right-inverse $\varphi(q) = \sup\{\lambda \geq 0 \mid \psi(\lambda) - \delta\lambda = q\}$. Then, for each $q \geq 0$, we define its scale functions $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ as in Equations (1.3) and (1.8):

$$\int_0^\infty \mathrm{e}^{-\lambda y} \mathbb{W}^{(q)}(y) \mathrm{d}y = \frac{1}{\psi(\lambda) - \delta\lambda - q}, \quad \text{for } \lambda > \varphi(q)$$

and

$$\mathbb{Z}^{(q)}(x) = 1 + q \int_0^x \mathbb{W}^{(q)}(y) \mathrm{d}y, \quad x \in \mathbb{R}.$$

2.2.1 Refracted Lévy processes

Recall from Equation (2.1), that our surplus process $U = \{U_t, t \geq 0\}$ is equivalently the solution to

$$\mathrm{d}U_t = \mathrm{d}Y_t + \delta \mathbf{1}_{\{U_t<0\}} \mathrm{d}t, \quad t \geq 0,$$

or

$$\mathrm{d}U_t = \mathrm{d}X_t - \delta \mathbf{1}_{\{U_t>0\}} \mathrm{d}t, \quad t \geq 0,$$

where $\delta \geq 0$ is the refraction parameter. The second stochastic differential equation is the one used in [43]. It was proved in that article that such a process exists and that it is a skip-free upward strong Markov process.

For technical reasons, we need to assume that if X (and also Y) has paths of bounded variation then

$$0 \leq \delta < c = \gamma + \int_{(0,1)} z \Pi(\mathrm{d}z). \tag{2.2}$$

Since in this case, X may be written as $X_t = ct - S_t$, the condition in Equation (2.2) amounts to making sure Y has a strictly positive linear drift.

In [43], many fluctuation identities, including the probability of ruin for U, have been derived using *scale functions* for U: for $q \geq 0$ and for $x \in \mathbb{R}$, set

$$w^{(q)}(x;z) = W^{(q)}(x-z) + \delta \mathbf{1}_{\{x \geq 0\}} \int_0^x \mathbb{W}^{(q)}(x-y) W^{(q)\prime}(y-z) \mathrm{d}y. \tag{2.3}$$

Note that when $x < 0$, we have

$$w^{(q)}(x;z) = W^{(q)}(x-z).$$

For $q = 0$, we will write $w^{(0)}(x;z) = w(x;z)$. See [42] for more details.

2.2.2 Classical ruin and exit problems for the refracted Lévy process

Here is a collection of known fluctuation identities for the spectrally negative Lévy processes X and Y, as well as for the refracted Lévy process U. See [42] for more details. First, for real numbers a and b, we define the following first-passage stopping times:

$$\begin{aligned}
\tau_a^- &= \inf\{t > 0 \colon X_t < a\} \quad \text{and} \quad \tau_b^+ = \inf\{t > 0 \colon X_t \geq b\} \\
\nu_a^- &= \inf\{t > 0 \colon Y_t < a\} \quad \text{and} \quad \nu_b^+ = \inf\{t > 0 \colon Y_t \geq b\} \\
\kappa_a^- &= \inf\{t > 0 \colon U_t < a\} \quad \text{and} \quad \kappa_b^+ = \inf\{t > 0 \colon U_t \geq b\},
\end{aligned}$$

with the convention $\inf \emptyset = \infty$. For $a \leq 0 \leq b$ and $q \geq 0$, if $a \leq x \leq b$ then we have

$$\mathbb{E}_x \left[\mathrm{e}^{-q\kappa_b^+} \mathbf{1}_{\{\kappa_b^+ < \kappa_a^-\}} \right] = \frac{w^{(q)}(x; a)}{w^{(q)}(b; a)},$$

from which we can deduce that

$$\mathbb{E}_x \left[\mathrm{e}^{-q\kappa_b^+} \mathbf{1}_{\{\kappa_b^+ < \infty\}} \right] = \frac{\mathrm{e}^{\Phi(q)x} + \delta\Phi(q) \mathbf{1}_{\{x \geq 0\}} \int_0^x \mathrm{e}^{\Phi(q)y} \mathbb{W}^{(q)}(x-y)\mathrm{d}y}{\mathrm{e}^{\Phi(q)b} + \delta\Phi(q) \int_0^b \mathrm{e}^{\Phi(q)y} \mathbb{W}^{(q)}(b-y)\mathrm{d}y}. \tag{2.4}$$

See Theorem 5 in [43]. For $x \leq b$, we also have

$$\mathbb{E}_x \left[\mathrm{e}^{-q\nu_b^+} \mathbf{1}_{\{\nu_b^+ < \nu_0^-\}} \right] = \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(b)}, \tag{2.5}$$

and

$$\mathbb{E}_x \left[\mathrm{e}^{-q\nu_0^-} \mathbf{1}_{\{\nu_0^- < \nu_b^+\}} \right] = \mathbb{Z}^{(q)}(x) - \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(b)} \mathbb{Z}^{(q)}(b). \tag{2.6}$$

Moreover, the *classical* probability of ruin, associated with the processes Y and U, is given by

$$\mathbb{P}_x \left(\nu_0^- < \infty \right) = 1 - \left(\mathbb{E}\left[X_1\right] - \delta \right)_+ \mathbb{W}(x) \tag{2.7}$$

and

$$\mathbb{P}_x \left(\kappa_0^- < \infty \right) = 1 - \frac{\left(\mathbb{E}\left[X_1\right] - \delta \right)_+}{1 - \delta W(0)} w(x; 0). \tag{2.8}$$

Of course, the expressions in Equations (2.7) and (2.8) should be equal because $\{Y_t, t < \nu_0^-\}$ and $\{U_t, t < \kappa_0^-\}$ have the same distribution with respect to $\mathbb{P}_x$ when $x > 0$. Using Equation (2.11), we can see that this is the case.

Finally, since the Laplace exponent of Y is given by $\lambda \mapsto \psi(\lambda) - \delta\lambda$, then for $x, \theta > 0$ we have

$$\mathbb{E}_x\left[\mathrm{e}^{\theta Y_{\nu_0^-}}\mathbf{1}_{\{\nu_0^- < \infty\}}\right] = \mathrm{e}^{\theta x} - (\psi(\theta) - \delta\theta)\,\mathrm{e}^{\theta x}\int_0^x \mathrm{e}^{-\theta z}\mathbb{W}(z)\mathrm{d}z - \frac{\psi(\theta) - \delta\theta}{\theta}\mathbb{W}(x). \tag{2.9}$$

We conclude this section with definitions of auxiliary functions. For the sake of compactness, we define for $p, p+q \geq 0$ and $a, x \in \mathbb{R}$

$$\begin{aligned}\mathcal{W}_{a,\delta}^{(p,q)}(x) &= \mathbb{W}^{(p)}(x) - \delta W^{(p+q)}(0)\mathbb{W}^{(p)}(x) \\ &\quad + \int_a^x \left(qW^{(p+q)}(x-y) - \delta W^{(p+q)\prime}(x-y)\right)\mathbb{W}^{(p)}(y)\,\mathrm{d}y \\ &= W^{(p+q)}(x) - \int_0^a \left(qW^{(p+q)}(x-y) - \delta W^{(p+q)\prime}(x-y)\right)\mathbb{W}^{(p)}(y)\,\mathrm{d}y,\end{aligned} \tag{2.10}$$

where the second expression for $\mathcal{W}_{a,\delta}^{(p,q)}$ in Equation (2.10) follows from identity (2.11).

The second identity follows from the following useful identity taken from [68]: for $p, q \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}&(q-p)\int_0^x \mathbb{W}^{(p)}(x-y)W^{(q)}(y)\mathrm{d}y \\ &\quad = W^{(q)}(x) - \mathbb{W}^{(p)}(x) + \delta\left(W^{(q)}(0)\mathbb{W}^{(p)}(x) + \int_0^x \mathbb{W}^{(p)}(x-y)W^{(q)\prime}(y)\mathrm{d}y\right).\end{aligned} \tag{2.11}$$

Finally, we set

$$\mathcal{H}_\delta^{(p,q)}(x) = \mathrm{e}^{\varphi(p)x}\left(1 + (q - \delta\varphi(p))\int_0^x \mathrm{e}^{-\varphi(p)y}W^{(p+q)}(y)\mathrm{d}y\right). \tag{2.12}$$

If no refraction is considered, Equations (2.11) and (2.12) are the analogues of (1.12) and (1.39) respectively, i.e., $\mathcal{H}_0^{(p,q)} = \mathcal{H}^{(p,q)}$ and $\mathcal{W}_{a,0}^{(p,q)} = \mathcal{W}_a^{(p,q)}$.

2.3 Main results

Following the definition for a standard Lévy insurance risk process, we define the time of Parisian ruin, with delay $r > 0$, for the refracted Lévy insurance risk process U by

$$\kappa_r^U = \inf\left\{t > 0 \colon t - g_t^U > r\right\},$$

where $g_t^U = \sup\{0 \leq s \leq t \colon U_s \geq 0\}$. Our main objective is to obtain an expression for the corresponding probability of Parisian ruin that has a similar structure as the one in Equation (1.35).

Theorem 19. *For $x \in \mathbb{R}$ and $r > 0$*

$$\mathbb{P}_x\left(\kappa_r^U < \infty\right) = 1 - (\mathbb{E}[X_1] - \delta)_+ \frac{\int_0^\infty w(x; -z) z \mathbb{P}(X_r \in \mathrm{d}z)}{\int_0^\infty z \mathbb{P}(X_r \in \mathrm{d}z) - \delta r}. \tag{2.13}$$

For classical ruin and Parisian ruin for a standard SNLP, if the *net profit condition* is not verified then (Parisian) ruin occurs almost surely. In the last result, if $\mathbb{E}[X_1] \leq \delta$, then the probability of Parisian ruin for U is equal to 1. This is because asking for $\mathbb{E}[Y_1] = \mathbb{E}[X_1] - \delta > 0$ is the same as the net profit condition in this model, namely for the surplus process U. Also, it should be clear that, if we set $\delta = 0$ in the above result, then we recover Equation (1.35).

Remark 20. Using identities from Section 2.5, we can also re-write the result in Equation (2.13) as follows:

$$\mathbb{P}_x\left(\kappa_r^U < \infty\right) = 1 - (\mathbb{E}[X_1] - \delta)_+ \frac{\int_0^\infty w(x; -z) z \mathbb{P}(X_r \in \mathrm{d}z)}{\int_0^\infty (1 - \delta W(z)) z \mathbb{P}(X_r \in \mathrm{d}z)}.$$

2.3.1 Other results

Using some of the results/lemmas in Section 2.5, it is possible to obtain other fluctuation identities for U involving the time of Parisian ruin. For example, the discounted probability of U reaching level a before being Parisian ruined and the Laplace transform of the time of Parisian ruin time can also be computed.

Theorem 21. *For any $a \geq 0$, $x \leq a$ and $q \geq 0$, we have*

(i)

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q(\kappa_r^U - r)}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right] &= \mathbb{Z}^{(q)}(x)\\ &+\int_0^\infty \left(w^{(q)}(x;-z)\,\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right] - \mathcal{W}_{x,\delta}^{(q,-q)}(x+z)\right)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right),\end{aligned}$$

where

$$\begin{aligned}&\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right]\\ &= 1-\frac{\mathbb{Z}^{(q)}(a)+\int_0^\infty\left(w^{(q)}(a;-z)-\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\right)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}{\int_0^\infty w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}\\ &= \frac{\int_0^\infty \mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}{\int_0^\infty w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}-\frac{\mathbb{Z}^{(q)}(a)}{\int_0^\infty w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}\end{aligned}$$

(ii)

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q(\kappa_r^U - r)}\mathbf{1}_{\{\kappa_r^U<\infty\}}\right] &= \mathbb{Z}^{(q)}.(x)\\ &+\int_0^\infty \left(w^{(q)}(x;-z)\,\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\infty\}}\right] - \mathcal{W}_{x,\delta}^{(q,-q)}(x+z)\right)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right),\end{aligned}$$

where

$$\mathbb{E}\left[\mathrm{e}^{-q(\kappa_r^U-r)}\mathbf{1}_{\{\kappa_r^U<\infty\}}\right]=\frac{\int_0^\infty \mathcal{H}_\delta^{(q,-q)}(z)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)-\frac{q}{\varphi(q)}-\delta}{\int_0^\infty \mathcal{H}_\delta^{(q,0)}(z)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)-\delta\mathrm{e}^{qr}},$$

(iii)

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right]=\frac{\int_0^\infty w^{(q)}(x;-z)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}{\int_0^\infty w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}. \tag{2.14}$$

Remark 22. If we set $\delta=0$, we obtain the same quantities by replacing φ, $w^{(q)}$, $\mathcal{H}_\delta^{(q,-q)}$ and $\mathcal{W}_\delta^{(q,-q)}$ by Φ, $W^{(q)}$, $\mathcal{H}^{(q,-q)}$ and $\mathcal{W}^{(q,-q)}$ respectively.

2.4 Examples

We now present four models in which we can compute the probability of Parisian ruin given in Theorem 19. The task amounts to finding processes X and Y for which both the distribution and the scale function are known. First, we will look at the two classical models: the Cramér-Lundberg model with exponential claims and the Brownian risk model. Then, we will move toward more sophisticated surplus processes, namely a stable risk process and a jump-diffusion risk process with phase-type claims.

2.4.1 Cramér-Lundberg processes with exponential claims

When X and Y are a Cramér-Lundberg risk processes with exponentially distributed claims, then they are given by

$$X_t - X_0 = ct - \sum_{i=1}^{N_t} C_i \quad \text{and} \quad Y_t - Y_0 = (c-\delta)t - \sum_{i=1}^{N_t} C_i,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity $\eta > 0$, and where $\{C_1, C_2, \dots\}$ are independent and exponentially distributed random variables with parameter α. The Poisson process and the random variables are mutually independent. In this case, the net profit condition is given by $\mathbb{E}[Y_1] = c - \delta - \eta/\alpha \geq 0$. Then, for $x \geq 0$, we have

$$\begin{aligned}
W(x) &= \frac{1}{c-\eta/\alpha}\left(1 - \frac{\eta}{c\alpha} e^{\left(\frac{\eta}{c}-\alpha\right)x}\right), \\
\mathbb{W}(x) &= \frac{1}{c-\delta-\eta/\alpha}\left(1 - \frac{\eta}{(c-\delta)\alpha} e^{\left(\frac{\eta}{c-\delta}-\alpha\right)x}\right), \\
w(x;-z) &= \frac{1}{c-\eta/\alpha}\left(1 - \frac{\eta}{c\alpha} e^{\left(\frac{\eta}{c}-\alpha\right)(x+z)}\right) + \frac{K(x,\delta,\alpha,\eta,c)}{(c-\delta-\eta/\alpha)c} e^{\left(\frac{\eta}{c}-\alpha\right)z},
\end{aligned}$$

where

$$K(x,\delta,\alpha,\eta,c) := \delta\eta\left(\frac{1}{\eta - c\alpha}\left(e^{\left(\frac{\eta}{c}-\alpha\right)x} - 1\right) - \frac{1}{\delta\alpha}\left(1 - e^{\frac{-\eta\delta}{c(c-\delta)}x}\right) e^{\left(\frac{\eta}{c-\delta}-\alpha\right)x}\right).$$

As noted in [58], we have

$$\mathbb{P}\left(\sum_{i=1}^{N_r} C_i \in \mathrm{d}y\right) = \sum_{k=0}^{\infty} \mathbb{P}\left(\sum_{i=0}^{k} C_i \in \mathrm{d}y\right) \mathbb{P}(N_r = k)$$
$$= \mathrm{e}^{-\eta r}\left(\delta_0(\mathrm{d}y) + \mathrm{e}^{-\alpha y}\sum_{m=0}^{\infty} \frac{(\alpha\eta r)^{m+1}}{m!(m+1)!} y^m \mathrm{d}y\right),$$

where $\delta_0(\mathrm{d}y)$ is a Dirac mass at 0, and consequently

$$\int_0^{\infty} z\mathbb{P}(X_r \in \mathrm{d}z)$$
$$= \int_0^{cr} z\mathrm{e}^{-\eta r}\left(\delta_0(cr - \mathrm{d}z) + \mathrm{e}^{-\alpha(cr-z)}\sum_{m=0}^{\infty} \frac{(\alpha\eta r)^{m+1}}{m!(m+1)!}(cr - z)^m \mathrm{d}z\right)$$
$$= \mathrm{e}^{-\eta r}\left(cr + \sum_{m=0}^{\infty} \frac{(\eta r)^{m+1}}{m!(m+1)!}\left[cr\Gamma(m+1, cr\alpha) - \frac{1}{\alpha}\Gamma(m+2, cr\alpha)\right]\right)$$

where $\Gamma(a, x) = \int_0^x \mathrm{e}^{-t} t^{a-1} \mathrm{d}t$ is the incomplete gamma function, and

$$\frac{\eta}{c\alpha}\int_0^{\infty} \mathrm{e}^{(\frac{\eta}{c}-\alpha)z} z\mathbb{P}(X_r \in \mathrm{d}z) = \int_0^{\infty} z\mathbb{P}(X_r \in \mathrm{d}z) - (c - \eta/\alpha)r.$$

Putting all the pieces together with the main result of Theorem 19, we obtain the following expression for the probability of Parisian ruin:

$$\mathbb{P}_x(\kappa_r^U < \infty) = 1 - \left(1 - \frac{\delta}{c - \eta/\alpha}\right)\left(1 - \mathrm{e}^{(\frac{\eta}{c}-\alpha)x}\right)$$
$$-\left(1 - \frac{\delta}{c - \eta/\alpha}\right)\frac{\delta r - \mathrm{e}^{(\frac{\eta}{c}-\alpha)x}\left(\delta r - (c - \eta/\alpha)r\right)}{\mathrm{e}^{-\eta r}\left(cr + \sum_{m=0}^{\infty} \frac{(\eta r)^{m+1}}{m!(m+1)!}\left[cr\Gamma(m+1, cr\alpha) - \frac{1}{\alpha}\Gamma(m+2, cr\alpha)\right]\right) - \delta r}$$
$$- \frac{\alpha}{\eta}K(x, \delta, \alpha, \eta, c)$$
$$\times\left(1 + \frac{\delta r - (c - \eta/\alpha)r}{\mathrm{e}^{-\eta r}\left(cr + \sum_{m=0}^{\infty} \frac{(\eta r)^{m+1}}{m!(m+1)!}\left[cr\Gamma(m+1, cr\alpha) - \frac{1}{\alpha}\Gamma(m+2, cr\alpha)\right]\right) - \delta r}\right).$$

The following two tables provide a sensitivity analysis for the probability of Parisian ruin in a refracted Cramér-Lundberg model (with exponential claims) with respect to the refraction parameter δ and the Parisian delay parameter r.

The value of the initial level $U_0 = x$ is also varying. Note that, in this example, we used the notation c for the linear part of X (below 0) and $c - \delta$ for the linear part of Y (above 0). In other words, during *regular business periods*, the drift is given by $c - \delta$. Consequently, in Table 2.1, we have fixed the value of $c - \delta$ (above 0) and looked at the effect of a change in value of δ, the refraction parameter, on the probability of Parisian ruin. Note that, when δ increases, then the value of c (below 0) also increases to keep $c - \delta$ constant. As expected, the larger the value of δ, the smaller the probability of Parisian ruin. In Table 2.2, we have fixed all parameters except for the Parisian delay parameter r. As expected, the larger the value of the delay r, i.e. the larger the *grace period*, the smaller the probability of Parisian ruin.

Table 2.1 Impact of the refraction parameter δ on the probability of Parisian ruin in a refracted Cramér-Lundberg model

x	$\delta = 0$	$\delta = 1$	$\delta = 3$	$\delta = 5$
1	2.87232×10^{-1}	1.85087×10^{-1}	5.57333×10^{-2}	1.22663×10^{-2}
5	1.4747×10^{-1}	9.50271×10^{-2}	2.86144×10^{-2}	6.29775×10^{-3}
10	6.409021×10^{-2}	4.12986×10^{-2}	1.24357×10^{-2}	2.73699×10^{-3}
20	1.21050×10^{-2}	7.80030×10^{-3}	2.348×10^{-3}	5.16951×10^{-4}
30	2.286353×10^{-3}	1.47328×10^{-3}	4.43634×10^{-4}	9.76391×10^{-6}

Parameters: $r = 2$, $c - \delta = 6$ (drift above 0), $\eta = 5$, $\alpha = 1$.

2.4.2 Brownian risk processes

Now, if X and Y are Brownian risk processes, i.e. if

$$X_t - X_0 = ct + \sigma B_t \quad \text{and} \quad Y_t - Y_0 = (c - \delta)t + \sigma B_t,$$

Table 2.2 Impact of the delay parameter r on the probability of Parisian ruin in a refracted Cramér-Lundberg model

x	$r=0$	$r=1$	$r=2$	$r=3$
1	7.05401×10^{-1}	1.72754×10^{-1}	5.57333×10^{-2}	2.06455×10^{-2}
5	3.62165×10^{-1}	8.86951×10^{-2}	2.86144×10^{-2}	1.05997×10^{-2}
10	1.57396×10^{-1}	3.85467×10^{-2}	1.24357×10^{-2}	4.60664×10^{-3}
20	2.97283×10^{-2}	7.28054×10^{-3}	2.34881×10^{-3}	8.70083×10^{-4}
30	5.61495×10^{-3}	1.37511×10^{-3}	4.43634×10^{-4}	1.64337×10^{-4}

Parameters: $\delta=3$, $c=6$ (drift below 0), $c-\delta=3$ (drift above 0), $\eta=5$, $\alpha=1$.

where $B=\{B_t,t\geq 0\}$ is a standard Brownian motion. In this case, the net profit condition is given by $\mathbb{E}[Y_1]=c-\delta\geq 0$. Then, for $x\geq 0$, we have

$$\begin{aligned}
W(x) &= \frac{1}{c}\left(1-\mathrm{e}^{-2\frac{c}{\sigma^2}x}\right),\\
\mathbb{W}(x) &= \frac{1}{c-\delta}\left(1-\mathrm{e}^{-2\frac{c-\delta}{\sigma^2}x}\right),\\
w(x;-z) &= \frac{1}{c}\left(1-\mathrm{e}^{-2\frac{c}{\sigma^2}(x+z)}\right)+M(x,\delta,\sigma,c)\mathrm{e}^{-2\frac{c}{\sigma^2}z},
\end{aligned}$$

where

$$M(x,\delta,\sigma,c):=\frac{\delta}{c-\delta}\left(\frac{1}{c}\left(1-\mathrm{e}^{-2\frac{c}{\sigma^2}x}\right)-\frac{1}{\delta}\left(\mathrm{e}^{-2\frac{c-\delta}{\sigma^2}x}-\mathrm{e}^{-2\frac{c}{\sigma^2}x}\right)\right).$$

Again, as noted in [58], we have

$$\int_0^\infty \mathrm{e}^{-\frac{2c}{\sigma^2}z}z\mathbb{P}(X_r\in\mathrm{d}z)=\int_0^\infty z\mathbb{P}(X_r\in\mathrm{d}z)-cr$$

and consequently

$$\int_0^\infty z\mathbb{P}(X_r\in\mathrm{d}z)=\frac{1}{\sqrt{2\pi\sigma^2 r}}\int_0^\infty z\mathrm{e}^{-\frac{(z-cr)^2}{2\sigma^2 r}}\mathrm{d}z=\frac{\sigma\sqrt{r}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{c^2 r}{2\sigma^2}}+cr\mathcal{N}\left(\frac{c\sqrt{r}}{\sigma}\right).$$

Putting all the pieces together with the main result of Theorem 19, we obtain the following expression for the probability of Parisian ruin:

$$
\begin{aligned}
&\mathbb{P}_x(\kappa_r^U < \infty) \\
&= 1 - \left(\frac{c-\delta}{c}\right) \frac{\left(\frac{\sigma\sqrt{r}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{c^2 r}{2\sigma^2}} + cr\mathcal{N}\left(\frac{c\sqrt{r}}{\sigma}\right)\right)\left(1 - \mathrm{e}^{-\frac{2c}{\sigma^2}x} + cM(x,\delta,\sigma,c)\right)}{\frac{\sigma\sqrt{r}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{c^2 r}{2\sigma^2}} + cr\mathcal{N}\left(\frac{c\sqrt{r}}{\sigma}\right) - \delta r} \\
&\qquad + (c-\delta)\frac{r\left(\mathrm{e}^{-\frac{2c}{\sigma^2}x} - cM(x,\delta,\sigma,c)\right)}{\frac{\sigma\sqrt{r}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{c^2 r}{2\sigma^2}} + cr\mathcal{N}\left(\frac{c\sqrt{r}}{\sigma}\right) - \delta r}.
\end{aligned}
$$

The following two tables provide a sensitivity analysis for the probability of Parisian ruin in a refracted Brownian risk model with respect to the refraction parameter δ and the Parisian delay parameter r. The value of the initial level $U_0 = x$ is also varying. Again in this example we used the notation c for the linear part of X (below 0) and $c-\delta$ for the linear part of Y (above 0). In Table 2.3, we have fixed the value of $c-\delta$ (above 0) and looked at the effect of a change in value of δ, the refraction parameter, on the probability of Parisian ruin. As expected, the larger the value of δ, the smaller the probability of Parisian ruin. In Table 2.4, we have fixed all parameters except for the Parisian delay parameter r. As expected, the larger the value of the delay r, i.e. the larger the *grace period*, the smaller the probability of Parisian ruin.

2.4.3 Jump-diffusion risk processes with phase-type claims

More generally, if we add a Brownian component and if we let the claim distribution be more general, then we consider a Lévy jump-diffusion risk process with phase-type claims:

$$X_t - X_0 = ct + \sigma B_t - \sum_{i=1}^{N_t} C_i \quad \text{and} \quad Y_t - Y_0 = (c-\delta)t + \sigma B_t - \sum_{i=1}^{N_t} C_i,$$

Table 2.3 Impact of the refraction parameter δ on the probability of Parisian ruin in a refracted Brownian risk model

x	$\delta = 0$	$\delta = 1$	$\delta = 3$	$\delta = 4$
1	1.75631×10^{-2}	4.05886×10^{-2}	2.040134×10^{-2}	1.39301×10^{-2}
5	4.62959×10^{-3}	1.06991×10^{-3}	5.37773×10^{-2}	3.67195×10^{-3}
10	8.74418×10^{-4}	2.02079×10^{-3}	1.01572×10^{-3}	6.93542×10^{-4}
20	3.11939×10^{-5}	7.20924×10^{-5}	3.62368×10^{-4}	2.47423×10^{-5}
30	1.11281×10^{-6}	2.57457×10^{-6}	1.29458×10^{-6}	8.83585×10^{-7}

Parameters: $r = 2$, $c - \delta = 6$ (drift above 0), $\sigma = 6$

where $\sigma \geq 0$, $B = \{B_t, t \geq 0\}$ is a standard Brownian motion, $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity $\eta > 0$, and where $\{C_1, C_2, \dots\}$ are independent random variables with common phase-type distribution with the minimal representation $(m, \mathbf{T}, \boldsymbol{\alpha})$, i.e. its cumulative distribution function (cdf) is given by $F(x) = 1 - \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{1}$ and $\mathbf{T}$, is called the *Phase-type generator*, which is an $m \times m$ matrix where $\mathbf{1}$ denotes a column vector of ones. The simplex $\boldsymbol{\alpha} = [\alpha_1, ..., \alpha_m]$ is the initial distribution of a continuous-time Markov chain Y, i.e. $\alpha_i = \mathbb{P}(Y_0 = i)$. All of the aforementioned objects are mutually independent (for details we refer to [25]). The Laplace exponent of X is then clearly given by

$$\psi(\lambda) = c\lambda + \frac{\sigma^2\lambda^2}{2} + \eta\left(\boldsymbol{\alpha}(\lambda\mathbf{I} - \mathbf{T})^{-1}\mathbf{t} - 1\right), \tag{2.15}$$

where $\mathbf{t} = -\mathbf{T}\mathbf{1}$. Let us denote by ρ_j and ζ_i the roots with negative real parts of equations $\lambda \mapsto \psi(\lambda) = 0$ and $\lambda \mapsto \psi(\lambda) - \delta\lambda = 0$, respectively. Since we assume the net profit condition $\mathbb{E}[X_1] > \delta$, from Proposition 5.4 in [39], we have that the ρ_j's and the ζ_i's are distinct roots. Then, from Proposition 2.1 in [25] and

Table 2.4 Impact of the delay parameter r on the probability of Parisian ruin in a refracted Brownian risk model

x	$r=0$	$r=1$	$r=2$	$r=4$
1	8.36506×10^{-1}	8.89538×10^{-2}	2.90834×10^{-2}	5.06685×10^{-3}
5	3.65132×10^{-1}	3.6570×10^{-2}	1.19569×10^{-2}	2.08304×10^{-3}
10	1.26742×10^{-1}	1.20385×10^{-2}	3.93613×10^{-3}	6.85723×10^{-4}
20	1.90869×10^{-2}	1.30459×10^{-3}	4.26551×10^{-4}	7.431054×10^{-4}
30	3.41422×10^{-3}	1.41376×10^{-4}	4.62245×10^{-5}	8.05289×10^{-6}

Parameters: $\delta = 3$, $c = 6$ (drift below 0), $c - \delta = 3$ (drift above 0), $\sigma = 6$

Proposition 5.4 in [39], we can obtain

$$
\begin{aligned}
W(x) &= \frac{1}{\psi'(0)} + \sum_{j \in \mathcal{I}_\rho} A_j \mathrm{e}^{\rho_j x}, \quad W'(x) = \sum_{j \in \mathcal{I}_\rho} \rho_j A_j \mathrm{e}^{\rho_j x}, \\
\mathbb{W}(x) &= \frac{1}{\psi'(0) - \delta} + \sum_{i \in \mathcal{I}_\zeta} B_i \mathrm{e}^{\zeta_i x}, \\
w(x; -z) &= \frac{1}{\psi'(0)} + \sum_{j \in \mathcal{I}_\rho} A_j \mathrm{e}^{\rho_j (x+z)} \\
&\quad + \frac{1}{\psi'(0) - \delta} \sum_{j \in \mathcal{I}_\rho} \rho_j A_j \left(\mathrm{e}^{\rho_j x} - 1\right) \mathrm{e}^{\rho_j z} + \sum_{j \in \mathcal{I}_\rho} \sum_{i \in \mathcal{I}_\zeta} \frac{\mathrm{e}^{\rho_j x} - \mathrm{e}^{\zeta_i x}}{\rho_j - \zeta_i} A_j B_i \mathrm{e}^{\rho_j z},
\end{aligned}
$$

where $A_j = \frac{1}{\psi'(\rho_j)}$ and $B_i = \frac{1}{\psi'(\zeta_i) - \delta}$, and where $\mathcal{I}_\rho$ and $\mathcal{I}_\zeta$ are the sets of indices corresponding to the ρ_j's and the ζ_i's, respectively. Moreover, one can observe that the Laplace exponent in (2.15) and $\psi(\lambda) - \delta\lambda$ are a ratio of two polynomials of degree $m+2$ and m respectively. This is true of course if $\sigma > 0$ and $c > 0$. On the other hand if $\sigma = 0$ and $c > 0$ we obtain ratio of two polynomials of degree $m+1$ and m respectively. Thus if we take $\psi(\lambda) = 0$ and $\psi(\lambda) - \delta\lambda = 0$ we will have $m+2$ or $m+1$ roots depending on whether $\sigma > 0$ or $\sigma = 0$. From [39][Prop. 5.4 (ii)] we know that there are $m+1$ (or m) roots with negative real part. Hence $\mathbf{card}(\mathcal{I}_\zeta) = \mathbf{card}(\mathcal{I}_\rho) = m+1$ (or $\mathbf{card}(\mathcal{I}_\zeta) = \mathbf{card}(\mathcal{I}_\rho) = m$ if

$\sigma = 0$). Moreover,

$$\mathbb{P}(X_r \in \mathrm{d}z) = \mathrm{e}^{-\eta r} \sum_{k=0}^{\infty} \frac{(\eta r)^k}{k!} \int_0^{\infty} F^{*k}(\mathrm{d}y) \mathcal{N}\left((\mathrm{d}z + y - cr)\sigma\sqrt{r}\right),$$

where $\mathcal{N}$ is the cdf of a standard normal random variable, F^{*k} is the k-th convolution of F and for $k = 0$ we understand $F^{*0}(\mathrm{d}y) = \delta_0(\mathrm{d}y)$ to be a Dirac mass at 0. Putting all the pieces together, we obtain an expression for the probability of Parisian ruin.

2.4.4 Stable risk processes

Now, if X and Y are 3/2-stable risk processes, i.e. if

$$X_t - X_0 = ct + Z_t \quad \text{and} \quad Y_t - Y_0 = (c - \delta)t + Z_t,$$

where $Z = \{Z_t, t \geq 0\}$ is a spectrally negative α-stable process with $\alpha = 3/2$. In this case, the Laplace exponent of X is given by $\psi(\lambda) = c\lambda + \lambda^{3/2}$. Then, for $x \geq 0$, we have

$$\begin{aligned} W(x) &= \frac{1 - E_{1/2}(-c\sqrt{x})}{c}, \\ \mathbb{W}(x) &= \frac{1 - E_{1/2}(-(c-\delta)\sqrt{x})}{c - \delta}, \\ w(x; -z) &= \frac{1}{c}\left[1 - E_{1/2}\left(-c\sqrt{x+z}\right)\right] \\ &\quad + \int_0^x \frac{1}{c-\delta}\left[1 - E_{1/2}\left(-(c-\delta)\sqrt{x-y}\right)\right] \\ &\quad \times \left(\frac{1}{\pi\sqrt{x}} - c \cdot E_{1/2}\left(-c\sqrt{y+z}\right)\right) \mathrm{d}y, \end{aligned}$$

where $E_{1/2}$ is the Mittag-Leffler function of order 1/2. Again, as noted in [58], we have

$$\mathbb{P}(Z_r \in \mathrm{d}y) = \mathbb{P}(r^{2/3} Z_1 \in \mathrm{d}y) = \begin{cases} \sqrt{\frac{3}{\pi}} r^{2/3} y^{-1} \mathrm{e}^{-u/2} W_{1/2,1/6}(u)\, \mathrm{d}y & y > 0, \\ -\frac{1}{2\sqrt{3\pi}} r^{2/3} y^{-1} \mathrm{e}^{u/2} W_{-1/2,1/6}(u)\, \mathrm{d}y & y < 0, \end{cases}$$

where $u = \frac{4}{27} r^{9/2} |y|^3$ and $W_{\kappa,\mu}$ is Whittaker's W-function (not to be confused with the 0-scale function of X). Putting all the pieces together with the main result of Theorem 19, we obtain the probability of Parisian ruin.

2.5 Proofs and more

The proofs of our main results are based on technical but important lemmas (provided in the next section), as well as more standard probabilistic decompositions.

2.5.1 Intermediate results

The next lemma is lifted from [58]:

Lemma 23. *For $\theta > q > 0$ and $y \geq 0$,*

$$\int_0^\infty \mathrm{e}^{-\theta r} \int_y^\infty \frac{z}{r} \mathbb{P}(X_r \in \mathrm{d}z) \mathrm{d}r = \frac{1}{\Phi(\theta)} \mathrm{e}^{-\Phi(\theta) y}, \tag{2.16}$$

and

$$\int_0^\infty \mathrm{e}^{-\theta r} \Lambda^{(q)}(-y, r) \mathrm{d}r = \frac{\mathrm{e}^{-\Phi(\theta) y}}{\theta - q}. \tag{2.17}$$

From this first lemma, we can deduce the following two useful identities:

$$\Lambda^{(q)}(0, r) = \mathrm{e}^{qr}, \tag{2.18}$$

and

$$\int_0^\infty \mathrm{e}^{-\theta r} \Lambda(-y, r) \mathrm{d}r = \frac{1}{\theta} \mathrm{e}^{-\Phi(\theta) y}, \quad y \geq 0. \tag{2.19}$$

By (2.17) and Laplace inversion, we obtain, for all $y \leq 0$,

$$\mathbb{E}_y \left[\mathrm{e}^{-q \tau_0^+} \mathbf{1}_{\{\tau_0^+ \leq r\}} \right] = \int_0^\infty \mathrm{e}^{-qr} W^{(q)}(y+z) \frac{z}{r} \mathbb{P}(X_r \in \mathrm{d}z) = \mathrm{e}^{-qr} \Lambda^{(q)}(y, r) \tag{2.20}$$

This identity will be generalized in Equation (2.24). For the proof of our main lemma, which is Lemma 25 below, we will need the following result taken from [57].

Lemma 24. *For all $p, q \geq 0$ and $a \leq x \leq b$,*

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-p\nu_a^-}W^{(q)}(Y_{\nu_a^-})\mathbf{1}_{\{\nu_a^-<\nu_b^+\}}\right]\\&\quad=W^{(q)}(x)-\int_0^{x-a}\Big((q-p)W^{(q)}(x-z)-\delta W^{(q)\prime}(x-z)\Big)\mathbb{W}^{(p)}(z)\mathrm{d}z\\&-\frac{\mathbb{W}^{(p)}(x-a)}{\mathbb{W}^{(p)}(b-a)}\left(W^{(q)}(b)-\int_0^{b-a}\Big((q-p)W^{(q)}(b-z)-\delta W^{(q)\prime}(b-z)\Big)\mathbb{W}^{(p)}(z)\mathrm{d}z\right).\end{aligned} \tag{2.21}$$

Note that another expression for (2.21) can be found in [68, Lemma1]. The following three identities are new and crucial for the proofs of our main results.

Lemma 25. *For $x \in \mathbb{R}$, $q \geq 0$ and $a \geq 0$, we have*

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\Lambda^{(q)}(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]\\&\qquad=\int_0^\infty\left(w^{(q)}\left(x;-z\right)-\frac{\mathbb{W}^{(q)}\left(x\right)}{\mathbb{W}^{(q)}\left(a\right)}w^{(q)}\left(a;-z\right)\right)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right),\end{aligned} \tag{2.22}$$

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\Lambda(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]\\&\quad=\int_0^\infty\left(\mathcal{W}_{x,\delta}^{(q,-q)}\left(x+z\right)-\frac{\mathbb{W}^{(q)}\left(x\right)}{\mathbb{W}^{(q)}\left(a\right)}\mathcal{W}_{a,\delta}^{(q,-q)}\left(a+z\right)\right)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right),\end{aligned} \tag{2.23}$$

and

$$\mathbb{E}_x\left[\Lambda(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\infty\}}\right]=\int_0^\infty\left(w(x;-z)-\mathbb{W}(x)\right)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)+\delta\mathbb{W}(x). \tag{2.24}$$

Proof. By Tonelli's theorem, we have

$$\begin{aligned}&\mathrm{e}^{-qr}\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\Lambda^{(q)}(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]\\&\quad=\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\int_0^\infty\mathrm{e}^{-qr}W^{(q)}\left(Y_{\nu_0^-}+z\right)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]\\&\quad=\int_0^\infty\mathrm{e}^{-qr}\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}W^{(q)}\left(Y_{\nu_0^-}+z\right)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)\\&\quad=\int_0^\infty\mathrm{e}^{-qr}\mathbb{E}_{x+z}\left[\mathrm{e}^{-q\nu_z^-}W^{(q)}\left(Y_{\nu_z^-}\right)\mathbf{1}_{\{\nu_z^-<\nu_{a+z}^+\}}\right]\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z),\end{aligned}$$

where the last line follows by spatial homogeneity of Y. Using identity (2.21) for $p = q$, we have

$$\mathbb{E}_{x+z}\left[\mathrm{e}^{-q\nu_z^-}W^{(q)}\left(Y_{\nu_z^-}\right)\mathbf{1}_{\{\nu_z^-<\nu_{a+z}^+\}}\right] = w^{(q)}(x;-z) - \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}w^{(q)}(a;-z),$$

which proves (2.22). Using again (2.21), Tonelli's theorem and spatial homogeneity of Y, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\Lambda(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\int_0^\infty W\left(Y_{\nu_0^-}+z\right)\frac{z}{r}\mathbb{P}(X_r\in \mathrm{d}z)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right] \\
&= \int_0^\infty \mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}W\left(Y_{\nu_0^-}+z\right)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]\frac{z}{r}\mathbb{P}(X_r\in \mathrm{d}z) \\
&= \int_0^\infty \mathbb{E}_{x+z}\left[\mathrm{e}^{-q\nu_z^-}W\left(Y_{\nu_z^-}\right)\mathbf{1}_{\{\nu_z^-<\nu_{a+z}^+\}}\right]\frac{z}{r}\mathbb{P}(X_r\in \mathrm{d}z) \\
&= \int_0^\infty \left(\mathcal{W}_{x,\delta}^{(q,-q)}(x+z) - \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\right)\frac{z}{r}\mathbb{P}(X_r\in \mathrm{d}z).
\end{aligned}$$

To prove the last identity, we need to compute the following limit

$$\mathbb{E}_x\left[\Lambda(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\infty\}}\right] = \lim_{q\to 0}\lim_{a\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbb{E}_{Y_{\nu_0^-}}\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+\le r\}}\right]\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right].$$

Since

$$\lim_{a\to\infty}\frac{W^{(q)}(z+a)}{\mathbb{W}^{(q)}(a)} = 0 \quad \text{and} \quad \lim_{a\to\infty}\frac{\mathbb{W}^{(q)}(a-y)}{\mathbb{W}^{(q)}(a)} = \mathrm{e}^{-\varphi(q)y}.$$

We obtain using Lebesgue's dominated convergence theorem

$$\begin{aligned}
\lim_{a\to\infty}\frac{w^{(q)}(a;-z)}{\mathbb{W}^{(q)}(a)} &= \delta\int_0^\infty \mathrm{e}^{-\varphi(q)y}W^{(q)\prime}(y+z)\,\mathrm{d}y \\
&= -\delta W^{(q)}(z) + \delta\mathrm{e}^{\varphi(q)z}\left(\frac{1}{\delta} - \varphi(q)\int_0^z \mathrm{e}^{-\varphi(q)y}W^{(q)}(y)\,\mathrm{d}y\right),
\end{aligned}$$

since $\psi(\varphi(q)) - q = \psi(\varphi(q)) - \delta\varphi(q) + \delta\varphi(q) - q = \delta\varphi(q)$. Then

$$\lim_{q\to 0}\lim_{a\to\infty}\frac{w^{(q)}(a,-z)}{W^{(q)}(a)} = -\delta W(z) + 1$$

and the result follows. ■

2.5.2 Proof of Theorem 19

For $x < 0$, using the strong Markov property of U and the fact that it is skip-free upward, we have

$$\mathbb{P}_x\left(\kappa_r^U = \infty\right) = \mathbb{E}_x\left[\mathbb{P}_x\left(\kappa_r^U = \infty \mid \mathcal{F}_{\kappa_0^+}\right)\mathbf{1}_{\{\kappa_0^+<\infty\}}\right] = \mathbb{P}_x\left(\kappa_0^+ \le r\right)\mathbb{P}\left(\kappa_r^U = \infty\right).$$

Since $\left\{X_t, t < \tau_0^+\right\}$ and $\left\{U_t, t < \kappa_0^+\right\}$ have the same distribution with respect to $\mathbb{P}_x$ when $x < 0$, we further have

$$\mathbb{P}_x\left(\kappa_r^U = \infty\right) = \mathbb{P}_x\left(\tau_0^+ \le r\right)\mathbb{P}(\kappa_r^U = \infty). \tag{2.25}$$

For $x \ge 0$, using the strong Markov property of U again, the fact that $\left\{Y_t, t < \nu_0^-\right\}$ and $\left\{U_t, t < \kappa_0^-\right\}$ have the same distribution with respect to $\mathbb{P}_x$ and using (2.25), we get

$$\begin{aligned}\mathbb{P}_x\left(\kappa_r^U = \infty\right) &= \mathbb{P}_x\left(\kappa_0^- = \infty\right) + \mathbb{E}_x\left[\mathbb{P}_x\left(\kappa_r^U = \infty \mid \mathcal{F}_{\kappa_0^-}\right)\mathbf{1}_{\{\kappa_0^-<\infty\}}\right] \\ &= \mathbb{P}_x\left(\kappa_0^- = \infty\right) + \mathbb{E}_x\left[\mathbb{P}_{U_{\kappa_0^-}}\left(\kappa_r^U = \infty\right)\mathbf{1}_{\{\kappa_0^-<\infty\}}\right] \\ &= \mathbb{P}_x\left(\nu_0^- = \infty\right) + \mathbb{P}\left(\kappa_r^U = \infty\right)\mathbb{E}_x\left[\mathbb{P}_{Y_{\nu_0^-}}\left(\tau_0^+ \le r\right)\mathbf{1}_{\{\nu_0^-<\infty\}}\right].\end{aligned} \tag{2.26}$$

Note that this last expression holds for all $x \in \mathbb{R}$. We will first prove the result for $x = 0$. We split this part of the proof into two cases: for processes with paths of bounded variation (BV), and then for processes with paths of unbounded variation (UBV). First, we assume X and Y have paths of BV. Setting $x = 0$ in (2.26) yields

$$\mathbb{P}\left(\kappa_r^U = \infty\right) = \mathbb{P}\left(\nu_0^- = \infty\right) + \mathbb{P}\left(\kappa_r^U = \infty\right)\mathbb{E}\left[\Lambda(Y_{\nu_0^-}, r)\mathbf{1}_{\{\nu_0^-<\infty\}}\right].$$

Solving for $\mathbb{P}\left(\kappa_r^U = \infty\right)$ and using both (2.7) and (2.24), we get

$$\mathbb{P}(\kappa_r^U = \infty) = \frac{(\mathbb{E}[X_1] - \delta)_+}{\int_0^\infty \frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z) - \delta}, \tag{2.27}$$

where we used the fact that $\mathbb{W}(0) > 0$. Now, if X has paths of UBV, we will use the same approximation procedure as in [58]. We denote by $\kappa_{r,\epsilon}^U$ the stopping time describing the first time an excursion, starting when U gets below 0 and ending when U gets back up to ϵ, lasts longer than r. More precisely, for $\epsilon > 0$, define

$$\kappa_{r,\epsilon}^U = \inf\left\{t > r : t - g_{t,\epsilon}^U > r, U_{t-r} < 0\right\},$$

where $g_{t,\epsilon}^U = \sup\{0 \leq s \leq t \colon U_s \geq \epsilon\}$. Clearly, we have $\kappa_{r,\epsilon}^U < \kappa_r^U$ a.s. which implies that $\left\{\kappa_{r,\epsilon}^U = \infty\right\} \subseteq \left\{\kappa_r^U = \infty\right\}$. Then, it can be shown that $\lim_{\epsilon\to 0} \mathbb{P}_\epsilon\left(\kappa_{r,\epsilon}^U = \infty\right) = \mathbb{P}\left(\kappa_r^U = \infty\right)$. Using similar arguments as in the BV case, when $x < 0$, we have

$$\mathbb{P}_x\left(\kappa_{r,\epsilon}^U = \infty\right) = \mathbb{P}_x(\kappa_\epsilon^+ \leq r)\mathbb{P}_\epsilon(\kappa_{r,\epsilon}^U = \infty),$$

and then, when $x \geq 0$, we have

$$\mathbb{P}_x\left(\kappa_{r,\epsilon}^U = \infty\right) = \mathbb{P}_x\left(\nu_0^- = \infty\right) + \mathbb{P}_\epsilon(\kappa_{r,\epsilon}^U = \infty)\mathbb{E}_x\left[\mathbb{P}_{Y_{\nu_0^-}}\left(\kappa_\epsilon^+ \leq r\right)\mathbf{1}_{\{\nu_0^- < \infty\}}\right].$$

Setting $x = \epsilon$ and solving for $\mathbb{P}_\epsilon\left(\kappa_{r,\epsilon}^U = \infty\right)$, we get with the help of (2.7)

$$\mathbb{P}_\epsilon\left(\kappa_{r,\epsilon}^U = \infty\right) = \frac{\left(\mathbb{E}\left[X_1\right] - \delta\right)_+ \mathbb{W}(\epsilon)}{1 - \mathbb{E}_\epsilon\left[\mathbb{P}_{Y_{\nu_0^-}}\left(\kappa_\epsilon^+ \leq r\right)\mathbf{1}_{\{\nu_0^- < \infty\}}\right]}. \tag{2.28}$$

Using (2.4) and then (2.9), we can write

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\theta r}\mathbb{E}_\epsilon\left[\mathbb{P}_{Y_{\nu_0^-}}\left(\kappa_\epsilon^+ \leq r\right)\mathbf{1}_{\{\nu_0^- < \infty\}}\right]\mathrm{d}r \\
&\quad= \mathbb{E}_\epsilon\left[\mathbf{1}_{\{\nu_0^- < \infty\}}\int_0^\infty \mathrm{e}^{-\theta r}\mathbb{P}_{Y_{\nu_0^-}}(\kappa_\epsilon^+ \leq r)\mathrm{d}r\right] \\
&\quad= \frac{1}{\theta}\mathbb{E}_\epsilon\left[\mathbf{1}_{\{\nu_0^- < \infty\}}\mathbb{E}_{Y_{\nu_0^-}}\left[\mathrm{e}^{-\theta\kappa_\epsilon^+}\mathbf{1}_{\{\kappa_\epsilon^+ < \infty\}}\right]\right] \\
&\quad= \frac{\mathbb{E}_\epsilon\left[\mathbf{1}_{\{\nu_0^- < \infty\}}\mathrm{e}^{\Phi(\theta)Y_{\nu_0^-}}\right]}{\theta\left(\mathrm{e}^{\Phi(\theta)\epsilon} + \delta\Phi(\theta)\int_0^\epsilon \mathrm{e}^{\Phi(\theta)y}\mathbb{W}^{(\theta)}(\epsilon - y)\mathrm{d}y\right)} \\
&\quad= \frac{1 - (\theta - \delta\Phi(\theta))\int_0^\epsilon \mathrm{e}^{-\Phi(\theta)y}\mathbb{W}(y)\mathrm{d}y - \frac{\theta - \delta\Phi(\theta)}{\Phi(\theta)}\mathrm{e}^{-\Phi(\theta)\epsilon}\mathbb{W}(\epsilon)}{\theta\left(1 + \delta\Phi(\theta)\int_0^\epsilon \mathrm{e}^{-\Phi(\theta)y}\mathbb{W}^{(\theta)}(y)\mathrm{d}y\right)}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\theta r}\left(\frac{1-\mathbb{E}_\epsilon\left[\mathbb{P}_{Y_{\nu_0^-}}\left(\kappa_\epsilon^+ \le r\right)\mathbf{1}_{\{\nu_0^-<\infty\}}\right]}{\mathbb{W}(\epsilon)}\right)\mathrm{d}r \\
= \quad & \frac{1}{\theta\mathbb{W}(\epsilon)} - \frac{1-(\theta-\delta\Phi(\theta))\int_0^\epsilon \mathrm{e}^{-\Phi(\theta)y}\mathbb{W}(y)\mathrm{d}y - \frac{\theta-\delta\Phi(\theta)}{\Phi(\theta)}\mathrm{e}^{-\Phi(\theta)\epsilon}\mathbb{W}(\epsilon)}{\theta\mathbb{W}(\epsilon)\left(1+\delta\Phi(\theta)\int_0^\epsilon \mathrm{e}^{-\Phi(\theta)y}\mathbb{W}^{(\theta)}(y)\mathrm{d}y\right)} \\
= \quad & \frac{1}{\theta\mathbb{W}(\epsilon)} \\
& \times\left(\frac{\delta\Phi(\theta)\int_0^\epsilon \mathrm{e}^{-\Phi(\theta)y}\mathbb{W}^{(\theta)}(y)\mathrm{d}y + (\theta-\delta\Phi(\theta))\int_0^\epsilon \mathrm{e}^{-\Phi(\theta)y}\mathbb{W}(y)\mathrm{d}y + \frac{\theta-\delta\Phi(\theta)}{\Phi(\theta)}\mathrm{e}^{-\Phi(\theta)\epsilon}\mathbb{W}(\epsilon)}{1+\delta\Phi(\theta)\int_0^\epsilon \mathrm{e}^{-\Phi(\theta)y}\mathbb{W}^{(\theta)}(y)\mathrm{d}y}\right) \\
& \underset{\epsilon\to 0}{\longrightarrow} \frac{1}{\Phi(\theta)} - \frac{\delta}{\theta},
\end{aligned}$$

where we used the fact that, for all $\theta \geq 0$, we have

$$\lim_{\epsilon\to 0}\frac{\int_0^\epsilon \mathrm{e}^{-\Phi(\theta)y}\mathbb{W}^{(\theta)}(y)\mathrm{d}y}{\mathbb{W}(\epsilon)} = 0.$$

From (2.16), we have that $\theta \mapsto 1/\Phi(\theta) - \delta/\theta$ is the Laplace transform of

$$r \mapsto \int_0^\infty \frac{z}{r}\mathbb{P}\left(X_r \in \mathrm{d}z\right) - \delta.$$

By the Extended continuity theorem of Laplace transforms (see e.g. [27]), this concludes the proof for $x = 0$. We now prove the result for $x \in \mathbb{R}$. Now, X and Y can be of BV or of UBV. We can now write (2.26) as follows:

$$\begin{aligned}
\mathbb{P}_x(\kappa_r^U &= \infty) \\
&= \left(\mathbb{E}\left[X_1\right]-\delta\right)_+\mathbb{W}(x) + \frac{(\mathbb{E}[X_1]-\delta)_+}{\int_0^\infty \frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)-\delta}\mathbb{E}_x\left[\mathbb{P}_{Y_{\nu_0^-}}\left(\tau_0^+ \le r\right)\mathbf{1}_{\{\nu_0^-<\infty\}}\right].
\end{aligned}$$

Using (2.24), we get finally

$$\mathbb{P}_x\left(\kappa_r^U = \infty\right) = \left(\mathbb{E}[X_1]-\delta\right)_+\left(\frac{\int_0^\infty w(x;-z)z\mathbb{P}(X_r\in\mathrm{d}z)}{\int_0^\infty z\mathbb{P}(X_r\in\mathrm{d}z)-\delta r}\right),$$

which holds for all $x \in \mathbb{R}$.

■

2.5.3 Proof of Theorem 21

For $x < 0$, using the strong Markov property of U and the fact that it is skip-free upward, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right] = \mathrm{e}^{-qr}\mathbb{P}_x(\kappa_0^+ > r) + \mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^+}\mathbf{1}_{\{\kappa_0^+\leq r\}}\right]\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right].$$

Since $\{X_t,\ t<\tau_0^+\}$ and $\{U_t,\ t<\kappa_0^+\}$ have the same law under $\mathbb{P}_x$ when $x<0$, we obtain

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right] = \mathrm{e}^{-qr}\mathbb{P}_x(\tau_0^+ > r) + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+\leq r\}}\right]\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right]. \tag{2.29}$$

For $0 \leq x \leq a$, using the strong Markov property again, we get

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right] = \mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{E}_{U_{\kappa_0^-}}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right]\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right].$$

Using the fact that $\{Y_t,\ t<\nu_0^-\}$ and $\{U_t,\ t<\kappa_0^-\}$ have the same law under $\mathbb{P}_x$ when $x\geq 0$ and injecting (2.29) in the last expectation, we have, for all $x\in\mathbb{R}$

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right] &= \mathrm{e}^{-qr}\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right] - \mathrm{e}^{-qr}\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbb{P}_{Y_{\nu_0^-}}(\tau_0^+\leq r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]\\ &\quad + \mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right]\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbb{E}_{Y_{\nu_0^-}}\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+\leq r\}}\right]\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right].\end{aligned}$$

For $x=0$ and using the last equation

$$\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U<\kappa_a^+\}}\right] = \frac{\mathrm{e}^{-qr}\mathbb{E}\left[\mathrm{e}^{-q\nu_0^-}\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right] - \mathrm{e}^{-qr}\mathbb{E}\left[\mathrm{e}^{-q\nu_0^-}\Lambda(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]}{1-\mathrm{e}^{-qr}\mathbb{E}\left[\mathrm{e}^{-q\nu_0^-}\Lambda^{(q)}(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]},$$

where, from (2.22),

$$\begin{aligned}&\mathrm{e}^{-qr}\mathbb{E}\left[\mathrm{e}^{-q\nu_0^-}\Lambda^{(q)}(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]\\ &\quad = \int_0^\infty \mathrm{e}^{-qr}\left(W^{(q)}(z) - \frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a)}w^{(q)}(a;-z)\right)\frac{z}{r}\mathbb{P}(X_r\in \mathrm{d}z),\end{aligned}$$

and, from (2.23),

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-q\nu_0^-}\mathbb{P}_{Y_{\nu_0^-}}(\tau_0^+ \le r)\mathbf{1}_{\{\nu_0^- < \nu_a^+\}}\right] \\
&\quad = \int_0^\infty \left(W(z) - \frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a)}\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z).
\end{aligned}$$

With the help of (2.18), (2.6) and the fact that $\mathbb{W}(0) > 0$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U < \kappa_a^+\}}\right] &= \frac{-\mathrm{e}^{-qr}\frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a)}\mathbb{Z}^{(q)}(a) + \mathrm{e}^{-qr}\int_0^\infty \frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a)}\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}{\frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a)}\int_0^\infty \mathrm{e}^{-qr}w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)} \\
&= 1 - \frac{\mathbb{Z}^{(q)}(a) + \int_0^\infty \left(w^{(q)}(a;-z) - \mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}{\int_0^\infty w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}.
\end{aligned} \tag{2.30}$$

Then

$$\begin{aligned}
\mathrm{e}^{qr}\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U < \kappa_a^+\}}\right] &= \mathbb{Z}^{(q)}(x) - \mathbb{Z}^{(q)}(a)\frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)} \\
&\quad - \int_0^\infty \left(\mathcal{W}_{x,\delta}^{(q,-q)}(x+z) - \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z) \\
&\quad + \mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U < \kappa_a^+\}}\right]\int_0^\infty \left(w^{(q)}(x;-z) - \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}w^{(q)}(a;-z)\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z) \\
&= \mathbb{Z}^{(q)}(x) + \int_0^\infty \left(w^{(q)}(x;-z)\,\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U < \kappa_a^+\}}\right] - \mathcal{W}_{x,\delta}^{(q,-q)}(x+z)\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z).
\end{aligned}$$

When X has paths of unbounded variation, we can use the same approximation procedure as in the proof of Theorem 19. The details are left to the reader. Identity (ii) follows from (i) by taking limit. Indeed, we have

$$\begin{aligned}
\lim_{a\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-q(\kappa_r^U - r)}\mathbf{1}_{\{\kappa_r^U < \kappa_a^+\}}\right] &= \lim_{a\to\infty}\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{\{\kappa_r^U < \kappa_a^+\}}\right]\int_0^\infty w^{(q)}(x;-z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z) \\
&\quad + \mathbb{Z}^{(q)}(x) - \int_0^\infty \mathcal{W}_{x,\delta}^{(q,-q)}(x+z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z),
\end{aligned}$$

and, from (2.30),

$$\lim_{a\to\infty}\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^U}\mathbf{1}_{(\kappa_r^U < \kappa_a^+)}\right] = \lim_{a\to\infty}\frac{-\mathbb{Z}^{(q)}(a) + \int_0^\infty \mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}{\int_0^\infty w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}.$$

As shown before, we have

$$\lim_{a\to\infty}\frac{w^{(q)}(a;-z)}{\mathbb{W}^{(q)}(a)}=-\delta W^{(q)}(z)+\mathrm{e}^{\varphi(q)z}\left(1-\delta\varphi(q)\int_0^z \mathrm{e}^{-\varphi(q)y}W^{(q)}(y)\,\mathrm{d}y\right),$$

Then

$$\begin{aligned}&\lim_{a\to\infty}\int_0^\infty \frac{w^{(q)}(a;-z)}{\mathbb{W}^{(q)}(a)}\frac{z}{r}\mathbb{P}(X_r\in \mathrm{d}z)\\ &\quad=\int_0^\infty\left(1-\delta\varphi(q)\int_0^z \mathrm{e}^{-\varphi(q)v}W^{(q)}(v)\,\mathrm{d}v\right)\mathrm{e}^{\varphi(q)z}\frac{z}{r}\mathbb{P}(X_r\in \mathrm{d}z)-\delta\mathrm{e}^{qr}.\end{aligned}$$

Finally, from the definition of $\mathcal{W}_{a,\delta}^{(q,-q)}$, using Lebesgue's dominated convergence theorem and performing an integration by parts,

$$\begin{aligned}&\lim_{a\to\infty}\frac{-\mathbb{Z}^{(q)}(a)+\int_0^\infty\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)}{\mathbb{W}^{(q)}(a)}\\ =\;&-\frac{q}{\varphi(q)}+\lim_{a\to\infty}\int_0^\infty\left(\frac{\mathbb{W}^{(q)}(a+z)-\delta W(z)\mathbb{W}^{(q)}(a)}{\mathbb{W}^{(q)}(a)}\right)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)\\ &+\lim_{a\to\infty}\int_0^\infty\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)\int_0^z(qW(z-y)-\delta W'(z-y))\frac{\mathbb{W}^{(q)}(a+y)}{\mathbb{W}^{(q)}(a)}\mathrm{d}y\\ =\;&-\frac{q}{\varphi(q)}-\delta+\int_0^\infty\mathrm{e}^{\varphi(q)z}\left(1+(q-\delta\varphi(q))\int_0^z\mathrm{e}^{-\varphi(q)y}W(y)\mathrm{d}y\right)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z).\end{aligned}$$

To prove (iii), we use first the strong Markov property and the fact that U has only downward jumps to get, for $x<0$

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^+}\mathbf{1}_{\{\kappa_0^+<r\}}\right]\mathbb{E}\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right].$$

Since $\{X_t,\ t<\tau_0^+\}$ and $\{U_t,\ t<\kappa_0^+\}$ have the same law under $\mathbb{P}_x$ when $x<0$, we obtain

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+<r\}}\right]\mathbb{E}\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right].\qquad(2.31)$$

For $0\le x\le a$, using again the strong Markov property, we get

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{E}_{U_{\kappa_0^-}}\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right]\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right].$$

Using the fact that $\{Y_t,\ t<\nu_0^-\}$ and $\{U_t,\ t<\kappa_0^-\}$ have the same law under $\mathbb{P}_x$ when $x\geq 0$ and injecting (2.31) in the last expectation, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-q\nu_a^+}\mathbf{1}_{\{\nu_a^+<\nu_0^-\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbb{E}_{Y_{\nu_0^-}}\left[\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\kappa_r^U\}}\right]\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right].$$

Putting the pieces together we obtain, for $x\leq a$

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right] &= \mathbb{E}_x\left[\mathrm{e}^{-q\nu_a^+}\mathbf{1}_{\{\nu_a^+<\nu_0^-\}}\right]\\ &\quad+\mathrm{e}^{-qr}\mathbb{E}\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right]\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\Lambda^{(q)}(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right].\end{aligned}$$

If we assume that X is of BV, then, setting $x=0$ in the last equation and combining (2.5) and (2.22), we get

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-p\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right] &= \frac{\mathbb{E}\left[\mathrm{e}^{-q\nu_a^+}\mathbf{1}_{\{\nu_a^+<\nu_0^-\}}\right]}{1-\mathrm{e}^{-qr}\mathbb{E}\left[\mathrm{e}^{-q\nu_0^-}\Lambda^{(q)}(Y_{\nu_0^-},r)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]}\\ &= \frac{\frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a)}}{1-\int_0^\infty \mathrm{e}^{-qr}\left(w^{(q)}(0;-z)-\frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a)}w^{(q)}(a;-z)\right)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right)}\\ &= \frac{1}{\int_0^\infty \mathrm{e}^{-qr}w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right)}.\end{aligned}$$

Then,

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_r^U\}}\right]\\ &= \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}+\frac{\int_0^\infty \mathrm{e}^{-qr}\left(w^{(q)}(x;-z)-\frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}w^{(q)}(a;-z)\right)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right)}{\int_0^\infty \mathrm{e}^{-qr}w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right)}\\ &= \frac{\int_0^\infty w^{(q)}(x;-z)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right)}{\int_0^\infty w^{(q)}(a;-z)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right)}.\end{aligned}$$

■

2.6 The Gerber-Shiu distribution at Parisian ruin with exponential delays

In this section, we are interested in computing the Gerber-Shiu distribution at Parisian ruin with exponential delays for the refracted Lévy process. We denote the time of Parisian ruin with exponential delays for the refracted Lévy process U is defined as

$$\kappa_U^q = \inf\left\{t > 0 \mid t - g_t^U > \mathrm{e}_q^{g_t^U}\right\},$$

where $\mathrm{e}_q^{g_t^U}$ is exponentially distributed with rate $q > 0$.

We obtain the following main result.

Theorem 26. *For θ, a, $b \geq 0$, $x \in [-a, b)$ and $y \in [-a, 0)$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-\theta\kappa_U^q}, U_{\kappa_U^q} \in \mathrm{d}y, \kappa_U^q < \kappa_b^+ \wedge \kappa_{-a}^-\right]$$
$$= q\left[\mathcal{W}_{x,\delta}^{(\theta,q)}(a+x)\frac{\mathcal{W}_{b,\delta}^{(\theta,q)}(b-y)}{\mathcal{W}_{b,\delta}^{(\theta,q)}(a+b)} - \mathcal{W}_{x,\delta}^{(\theta,q)}(x-y)\right]\mathrm{d}y.$$

Letting a go to infinity, we obtain the following corollary.

Corollary 27. *For $\theta, b \geq 0$, $x \leq b$ and $y \in [-\infty, 0)$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-\theta\kappa_U^q}, U_{\kappa_U^q} \in \mathrm{d}y, \kappa_U^q < \kappa_b^+\right]$$
$$= q\left[\mathcal{W}_{b,\delta}^{(\theta,q)}(b-y)\frac{\mathcal{H}^{(\theta+q,-q,\delta)}(x)}{\mathcal{H}^{(\theta+q,-q,\delta)}(b)} - \mathcal{W}_{x,\delta}^{(\theta,q)}(x-y)\right]\mathrm{d}y,$$

where

$$\mathcal{H}^{(p,q,\delta)}(x) = e^{\Phi(p)x}\left(1 + \int_0^x (q - \delta\Phi(p))\, e^{-\Phi(p)z}\mathbb{W}^{(p+q)}(z)\,\mathrm{d}z\right),$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-\theta\kappa_U^q}, \kappa_U^q < \kappa_b^+\right] = \frac{q}{\theta+q}\left(\mathbb{Z}^{(\theta)}(x) - \frac{\mathcal{H}^{(\theta+q,-q,\delta)}(x)}{\mathcal{H}^{(\theta+q,-q,\delta)}(b)}\mathbb{Z}^{(\theta)}(b)\right). \quad (2.32)$$

To avoid repetition, we will omit the proof of the last theorem. However, the proof is based on the same techniques as in [8] and [44] combined with Equation (3.3) in Pérez and Yamazaki [66], that is : for $x \in \mathbb{R}$ and $p, q \geq 0$

$$\int_0^x \mathbb{W}^{(p)}(x-y)\left[\delta W^{(q)}(y) - (q-p)\int_0^y W^{(q)}(z)\,\mathrm{d}z\right]dy$$
$$= \int_0^x \mathbb{W}^{(p)}(z)\,dz - \int_0^x W^{(q)}(z)\,\mathrm{d}z.$$

Letting $\theta = 0$ and $b \to \infty$ respectively in (2.32), we recover the results in [68, Corollary 2], that is

$$\mathbb{P}_x\left(\kappa_U^q < \kappa_b^+\right) = \frac{\mathcal{H}^{(q,-q,\delta)}(x)}{\mathcal{H}^{(q,-q,\delta)}(b)},$$

and

$$\mathbb{P}_x\left(\kappa_U^q < \infty\right) = 1 - \frac{\Phi(q)\left(\mathbb{E}\left[X_1\right] - \delta\right)}{\left(q - \delta\Phi(q)\right)}\mathcal{H}^{(q,-q,\delta)}(x).$$

CHAPTER III

A UNIFIED APPROACH TO RUIN PROBABILITIES WITH DELAYS

3.1 Introduction

In this Chapter, we unify the definitions of Parisian ruin with deterministic delays and Parisian ruin with exponentially distributed delays by considering *mixed delays*. Indeed, for this unified version of Parisian ruin, the race is between the duration of an excursion in the red zone, a deterministic implementation delay $r > 0$ and a random delay described by an exponential random variable with rate $q > 0$. For our new definition of Parisian ruin, the time of ruin is defined as

$$\kappa_r^q = \kappa^q \wedge \kappa_r = \inf \left\{ t > 0 \colon t - g_t > \left(\mathrm{e}_q^{g_t} \wedge r \right) \right\}. \tag{3.1}$$

More precisely, ruin occurs the first time an excursion below zero lasts longer than one of the two delays. Our main contributions are generalizations of several recent results obtained by Loeffen *et al.* [59] and Lkabous *et al.* [55]. The identities involve *second-generation scale functions* and also the distribution of the spectrally negative Lévy process at a fixed time. As they have a similar structure as the ones in [55], [58] and [59], we can then analyze limiting cases in order to recover previous results related to other definitions of Parisian ruin. The rest of the Chapter is organized as follows. The main results are presented in Section 3.2, followed by a discussion on those results. In Section 3.3, we provide explicit computations of the probability of Parisian ruin with mixed delays for two specific Lévy risk processes. Finally, in Section 3.4, we derive new technical identities and then provide proofs for the main results.

In the main results, we will use the following auxiliary function: for $x \in \mathbb{R}$ and $p, p+s, \lambda \geq 0$, set

$$\begin{aligned}\mathcal{F}^{(p,\lambda)}(x;r,s) &= \frac{1}{\psi_{p+s}(\lambda)}\left(\psi_p(\lambda)\mathrm{e}^{\psi_{p+s}(\lambda)r} - s\right) Z_p(x,\lambda) \\ &\quad -\mathrm{e}^{\psi_{p+s}(\lambda)r}\psi_p(\lambda)\int_0^r \mathrm{e}^{-\psi(\lambda)u}\Lambda^{(p)}(x;u,s)\,\mathrm{d}u, \qquad (3.2)\end{aligned}$$

where ψ_{p+s} is defined in (1.4). For $\lambda = 0$, we write $\mathcal{F}^{(p,0)} = \mathcal{F}^{(p)}$, where

$$\mathcal{F}^{(p)}(x;r,s) = \frac{1}{(p+s)}\left(s + p\mathrm{e}^{-(p+s)r}\right) Z_p(x,0) + p\mathrm{e}^{-(p+s)r}\int_0^r \Lambda^{(p)}(x;u,s)\,\mathrm{d}u,$$

and for $s = 0$, we denote $\mathcal{F}^{(p,\lambda)}(x;r,0) = \mathcal{F}^{(p,\lambda)}(x,r)$.

3.2 Main results

We are now ready to state our main results. They are generalizations of those presented in the previous section in the sense that κ^q or κ_r is replaced by the more general time of ruin κ_r^q. First, here is the joint distribution of our new time of Parisian ruin and the corresponding deficit at ruin:

Theorem 28. *For $p, \lambda \geq 0$, $b, q, r > 0$ and $x \leq b$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_r^q + \lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q < \tau_b^+\}}\right] = \mathcal{F}^{(p,\lambda)}(x;r,q) - \frac{\Lambda^{(p)}(x;r,q)}{\Lambda^{(p)}(b;r,q)}\mathcal{F}^{(p,\lambda)}(b;r,q) \qquad (3.3)$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+ < \kappa_r^q\}}\right] = \frac{\Lambda^{(p)}(x;r,q)}{\Lambda^{(p)}(b;r,q)}. \qquad (3.4)$$

Setting $\lambda = 0$ in the previous Theorem, we obtain the following Laplace transforms for the Parisian time of ruin:

Corollary 29. *Let $p \geq 0$ and $b, q, r > 0$. For $x \leq b$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_r^q}\mathbf{1}_{\{\kappa_r^q < \tau_b^+\}}\right] = \mathcal{F}^{(p)}(x;r,q) - \frac{\Lambda^{(p)}(x;r,q)}{\Lambda^{(p)}(b;r,q)}\mathcal{F}^{(p)}(b;r,q). \qquad (3.5)$$

and, for $x \in \mathbb{R}$, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_r^q}\mathbf{1}_{\{\kappa_r^q<\infty\}}\right] = \mathcal{F}^{(p)}(x;r,q) - \Omega(p,r,q) \times \Lambda^{(p)}(x;r,q), \tag{3.6}$$

where

$$\Omega(p,r,q) = \frac{\frac{p}{(p+q)\Phi(p)}\left(q + p\mathrm{e}^{-(p+q)r}\right) + p\mathrm{e}^{-(p+q)r}\int_0^r\left(\int_0^\infty Z_{p+q}(z,\Phi(p))\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z)\right)\mathrm{d}s}{\int_0^\infty Z_{p+q}(z,\Phi(p))\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}.$$

Setting $p = 0$ in (3.6), we obtain the following expression for the probability of Parisian ruin with mixed delays:

Corollary 30. *For $x \in \mathbb{R}$ and $q, r > 0$, we have*

$$\mathbb{P}_x(\kappa_r^q < \infty) = 1 - (\mathbb{E}[X_1])_+ \frac{\Lambda(x;r,q)}{\int_0^\infty Z_q(u,\Phi(0))\frac{u}{r}\mathbb{P}(X_r \in \mathrm{d}u)}. \tag{3.7}$$

3.2.1 Discussion on the results

Our Parisian fluctuation identities are arguably compact and have a similar structure as classical fluctuation identities (without delays) as well as previously-obtained Parisian fluctuation identities (see e.g. [55, 58]).

Indeed, in Equation (3.3), the (q,r)-delayed (p,λ)-scale function $\mathcal{F}^{(p,q,\lambda)}(\cdot,r)$ plays a similar rôle as the one played by the scale function $Z_p(\cdot,\lambda)$ in the following classical fluctuation identity: for $x \leq b$, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^- + \lambda X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^- < \tau_b^+\}}\right] = Z_p(x,\lambda) - \frac{W^{(p)}(x)}{W^{(p)}(b)}Z_p(b,\lambda). \tag{3.8}$$

See [41] for the solution to the two-sided exit problem and see e.g. [5] for the latter identity.

For the rest of this section, we will demonstrate that our results are simultaneously generalizing known identities for Parisian ruin with exponential delays and

Parisian ruin with deterministic delays. As we have seen in the previous section, the results obtained so far in the literature, for either one definition of Parisian ruin or the other, did not seem to have strong connections allowing to recover the identity for one definition of ruin from the corresponding identity for the other definition of ruin.

3.2.2 Parisian ruin with exponential delays

First, we will show that the identity in (3.4) converges, as $r \to \infty$, to the solution of the delayed version of the two-sided exit problem when the implementation delay is exponentially distributed, namely the identity in (1.42).

Using (1.37), Lebesgue's convergence theorem and (1.19), we have

$$\begin{aligned}
\lim_{r\to\infty} \frac{\Lambda^{(p+q)}(x,r)}{\Lambda^{(p+q)}(b,r)} &= \lim_{r\to\infty} \mathbb{E}_x\left[\mathrm{e}^{-(p+q)\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r\}}\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-(p+q)\tau_b^+}\mathbf{1}_{\{\tau_b^+<\infty\}}\right] = \mathrm{e}^{\Phi(p+q)(x-b)}.
\end{aligned}$$

Consequently, using Lebesgue's convergence theorem and (1.15), we have

$$\begin{aligned}
\lim_{r\to\infty} \frac{\Lambda^{(p)}(x;r,q)}{\Lambda^{(p+q)}(b,r)} &= \lim_{r\to\infty} \frac{\Lambda^{(p+q)}(x,r)}{\Lambda^{(p+q)}(b,r)} - q\int_0^x W^{(p)}(x-u)\left(\lim_{r\to\infty}\frac{\Lambda^{(p+q)}(u,r)}{\Lambda^{(p+q)}(b,r)}\right)\mathrm{d}u \\
&= \mathrm{e}^{\Phi(p+q)(x-b)} - q\int_0^x W^{(p)}(x-u)\,\mathrm{e}^{\Phi(p+q)(u-b)}\mathrm{d}u \\
&= \mathrm{e}^{-\Phi(p+q)b} Z_p\left(x,\Phi(p+q)\right).
\end{aligned}$$

Finally, taking the limit as $r \to \infty$ of the identity in (3.4), we get

$$\begin{aligned}
\lim_{r\to\infty} \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right] &= \lim_{r\to\infty} \frac{\Lambda^{(p)}(x;r,q)}{\Lambda^{(p)}(b;r,q)} \\
&= \lim_{r\to\infty} \frac{\Lambda^{(p)}(x;r,q)/\Lambda^{(p+q)}(b,r)}{\Lambda^{(p)}(b;r,q)/\Lambda^{(p+q)}(b,r)} \\
&= \frac{\mathrm{e}^{-\Phi(p+q)b} Z_p\left(x,\Phi(p+q)\right)}{\mathrm{e}^{-\Phi(p+q)b} Z_p\left(b,\Phi(p+q)\right)} \\
&= \frac{Z_p\left(x,\Phi(p+q)\right)}{Z_p\left(b,\Phi(p+q)\right)},
\end{aligned}$$

which is, as announced, the corresponding identity when there is no deterministic component in the delays; see (1.42).

Second, we will show that the identity in (3.5) converges, as $r \to \infty$, to the solution of the delayed version of the two-sided exit problem when the implementation delay is exponentially distributed, namely the identity in (1.43).

But before, let us show that

$$\lim_{r\to\infty} \mathcal{F}^{(p)}(x;r,q) = \frac{q}{p+q} Z_p(x,0). \tag{3.9}$$

We want to compute the following limit:

$$\begin{aligned}
\lim_{r\to\infty} \mathcal{F}^{(p)}(x;r,q) &= \lim_{r\to\infty} \frac{1}{p+q}\left(q + p\mathrm{e}^{-(p+q)r}\right) Z_p(x,0) \\
&\quad + \lim_{r\to\infty} p\mathrm{e}^{-(p+q)r} \int_0^r \Lambda^{(p)}(x;s,q)\,\mathrm{d}s \\
&= \frac{q}{p+q} Z_p(x,0) + \lim_{r\to\infty} p\mathrm{e}^{-(p+q)r} \int_0^r \Lambda^{(p)}(x;s,q)\,\mathrm{d}s.
\end{aligned}$$

Using Kendall's identity and Tonelli's theorem, we have

$$\int_0^r \Lambda^{(p)}(x;s,q)\,\mathrm{d}s = \int_0^\infty \mathcal{W}_z^{(p+q,-q)}(x+z)\,\mathbb{P}\left(\tau_z^+ \le r\right)\mathrm{d}z.$$

Taking Laplace transforms in r on both sides, together with (1.19) and the fact that

$$\int_0^\infty \mathrm{e}^{-\theta z} \mathcal{W}_z^{(p,s)}(a+z)\,\mathrm{d}z = \frac{Z_p(a,\theta)}{\psi_{p+s}(\theta)}, \quad \theta > \Phi(p+s), \tag{3.10}$$

yields

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\theta r}\left(\int_0^r \Lambda^{(p)}(x;s,q)\,\mathrm{d}s\right)\mathrm{d}r &= \frac{1}{\theta}\int_0^\infty \mathrm{e}^{-\Phi(\theta)z}\mathcal{W}_z^{(p+q,-q)}(x+z)\,\mathrm{d}z \\
&= \frac{Z_{p+q}(x,\Phi(\theta))}{\theta(\theta-p)}.
\end{aligned}$$

Then, using the Final value theorem, we obtain

$$\lim_{r\to\infty} \int_0^r \Lambda^{(p)}(x;s,q)\,\mathrm{d}s = \lim_{\theta\to 0} \frac{Z_{p+q}(x,\Phi(\theta))}{\theta-p} = \frac{-Z_{p+q}(x,\Phi(0))}{p}.$$

To prove that the identity in (3.5) converges, as $r \to \infty$, to the identity in (1.43), it suffices to use the fact that

$$\lim_{r\to\infty} \frac{\Lambda^{(p)}(x;r,q)}{\Lambda^{(p)}(b;r,q)} = \frac{Z_p(x,\Phi(p+q))}{Z_p(b,\Phi(p+q))},$$

as shown above, together with (3.9).

3.2.3 Parisian ruin with deterministic delays

It is straightforward to verify that our results are generalizing known identities for Parisian ruin with deterministic delays.

Indeed, in identity (3.3) of Theorem 28, if we take the limit when $q \to 0$, then we get

$$\lim_{q\to 0} \mathbb{E}_x \left[\mathrm{e}^{-p\kappa_r^q + \lambda X_{\kappa_r^q}} \mathbf{1}_{\{\kappa_r^q < \tau_b^+\}} \right] = \mathcal{F}^{(p,\lambda)}(x,r) - \frac{\Lambda^{(p)}(x,r)}{\Lambda^{(p)}(b,r)} \mathcal{F}^{(p,\lambda)}(b,r),$$

as already obtained in [59], with a slightly different notation.

Remark 31. Using the same techniques as in Subsection 3.2.2, we can also show that identities in Theorem 28 and Corollary 30 converge to the identities (1.17), (3.8) and (5.2) related to classical ruin.

3.3 Examples

We now present two classical models for which we can compute easily the probability of mixed Parisian ruin, as given in Corollary 30. Note that, to use the formula in (3.7), one needs to have an expression for the 0-scale function W and the distribution of the underlying Lévy risk process X.

We will also verify that we recover the known expressions for the probability of Parisian ruin with exponentially distributed delays, i.e. without the deterministic component.

3.3.1 Brownian risk process

Let X be a Brownian risk process, i.e.

$$X_t - X_0 = ct + B_t,$$

where $B = \{B_t, t \geq 0\}$ is a standard Brownian motion.

In this case, for $x \geq 0$ and $q > 0$, the scale functions are given by

$$\begin{aligned}
W(x) &= \frac{1}{c}\left(1 - \mathrm{e}^{-2cx}\right), \\
W^{(q)}(x) &= \frac{1}{\Phi(q) + c}\left(\mathrm{e}^{\Phi(q)x} - \mathrm{e}^{-x(\Phi(q)+2c)}\right), \\
Z_q(x, 0) &= \frac{q}{\Phi(q) + c}\left(\frac{\mathrm{e}^{\Phi(q)x}}{\Phi(q)} + \frac{\mathrm{e}^{-(\Phi(q)+2c)x}}{\Phi(q) + 2c}\right),
\end{aligned}$$

where

$$\Phi(q) = \left(\sqrt{c^2 + 2q} - c\right).$$

Also, we have

$$\begin{aligned}
\mathcal{W}_z^{(q,-q)}(x + z) &= W(x + z) + q\int_0^z W(x + z - y)\, W^{(q)}(y)\, \mathrm{d}y \\
&= \mathrm{e}^{\Phi(q)z}\left(\frac{q}{c\Phi(q)\left(\Phi(q) + c\right)} - \frac{q\mathrm{e}^{-2cx}}{c\left(\Phi(q) + 2c\right)\left(\Phi(q) + c\right)}\right) \\
&\quad + \mathrm{e}^{-(\Phi(q)+2c)z}\left(\frac{q}{c\left(\Phi(q) + c\right)\left(\Phi(q) + 2c\right)} - \frac{q\mathrm{e}^{-2cx}}{c\Phi(q)\left(\Phi(q) + c\right)}\right) \\
&= \mathrm{e}^{\Phi(q)z}A_1(x) + \mathrm{e}^{-(\Phi(q)+2c)z}A_2(x),
\end{aligned}$$

where

$$\begin{aligned}
A_1(x) &= \frac{q}{c\sqrt{c^2 + 2q}\left(\sqrt{c^2 + 2q} - c\right)} - \frac{q\mathrm{e}^{-2cx}}{c\sqrt{c^2 + 2q}\left(\sqrt{c^2 + 2q} + c\right)}, \\
A_2(x) &= \frac{q}{c\sqrt{c^2 + 2q}\left(\sqrt{c^2 + 2q} + c\right)} - \frac{q\mathrm{e}^{-2cx}}{c\sqrt{c^2 + 2q}\left(\sqrt{c^2 + 2q} - c\right)}.
\end{aligned}$$

First, we need to compute the following quantity

$$\mathbb{E}\left[X_1\right]\int_0^\infty \mathcal{W}_z^{(q,-q)}(x + z)\,\frac{z}{r}\mathbb{P}\left(X_r \in \mathrm{d}z\right). \tag{3.11}$$

Making the change of variable $y = \left(z - r\sqrt{c^2 + 2q}\right)/\sqrt{r}$, we have

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi r}} \int_0^\infty \mathrm{e}^{z\Phi(q)} \frac{z}{r} \mathrm{e}^{-(z-cr)^2/(2r)} \mathrm{d}z \\
= & \frac{1}{\sqrt{2r\pi}} \mathrm{e}^{-rc^2/2} + \mathrm{e}^{rq} \left(\Phi(q) + c\right) \mathcal{N}\left(\sqrt{r}\left(\Phi(q) + c\right)\right) \\
= & \Psi_1\left(c, r, q\right), \qquad (3.12)
\end{aligned}$$

and, setting $y = -\left(z + r\sqrt{c^2 + 2q}\right)/\sqrt{r}$, we get

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi r}} \int_0^\infty \mathrm{e}^{-(\Phi(q)+2cq)z} \frac{z}{r} \mathrm{e}^{-(z-cr)^2/(2r)} \mathrm{d}z \\
= & \frac{1}{\sqrt{2r\pi}} \mathrm{e}^{-rc^2/(2)} - \mathrm{e}^{rq} \left(\Phi(q) - c\right) \mathcal{N}\left(-\sqrt{r}\left(\Phi(q) - c\right)\right) \\
= & \Psi_2\left(c, r, q\right), \qquad (3.13)
\end{aligned}$$

where $\mathcal{N}$ is the cumulative distribution of the standard normal distribution.

$$\begin{aligned}
\mathbb{E}\left[X_1\right] & \int_0^\infty \mathcal{W}_z^{(q,-q)}\left(x + z\right) \frac{z}{r} \mathbb{P}\left(X_r \in \mathrm{d}z\right) \\
& = \frac{1}{\sqrt{2\pi r}} \int_0^\infty \mathcal{W}_z^{(0,q)}\left(x + z\right) z \mathrm{e}^{-(z-cr)^2/(2r)} \mathrm{d}z \\
& = A_1(x)\Psi_1\left(c, r, q\right) + A_2(x)\Psi_2\left(c, r, q\right), \quad (3.14)
\end{aligned}$$

where $\mathcal{N}$ is the cumulative distribution of the standard normal distribution. Using (3.12) and (3.14) in (3.11), we obtain

$$\begin{aligned}
& \int_0^\infty Z^{(q)}(z) \frac{z}{r} \mathbb{P}\left(X_r \in \mathrm{d}z\right) \\
= & \frac{q}{\sqrt{c^2 + 2q}} \int_0^\infty \left(\frac{\mathrm{e}^{z\left(\sqrt{c^2+2q}-c\right)}}{\sqrt{c^2 + 2q} - c} + \frac{\mathrm{e}^{-z\left(\sqrt{c^2+2q}+c\right)}}{\sqrt{c^2 + 2q} + c} \right) \frac{z}{r} \mathbb{P}\left(X_r \in \mathrm{d}z\right) \\
= & \frac{q}{\left(\sqrt{c^2 + 2q} - c\right)\sqrt{c^2 + 2q}} \Psi_1\left(c, r, q\right) \\
& + \frac{q}{\left(\sqrt{c^2 + 2q} + c\right)\sqrt{c^2 + 2q}} \Psi_2\left(c, r, q\right).
\end{aligned}$$

Putting all the terms together, we get

$$\mathbb{P}_x\left(\kappa_r^q < \infty\right) = 1 - \frac{A_1(x)\Psi_1\left(c,r,q\right) + A_2(x)\Psi_2\left(c,r,q\right)}{\frac{q}{\left(\sqrt{c^2+2q}-c\right)\sqrt{c^2+2q}}\Psi_1\left(c,r,q\right) + \frac{q}{\left(\sqrt{c^2+2q}+c\right)\sqrt{c^2+2q}}\Psi_2\left(c,r,q\right)}.$$

Note that since

$$\lim_{r\to\infty} \frac{\Psi_1\left(c,r,q\right)}{\mathrm{e}^{rq}} = \sqrt{c^2+2q},$$

and

$$\lim_{r\to\infty} \frac{\Psi_2\left(c,r,q\right)}{\mathrm{e}^{rq}} = 0,$$

we recover

$$\lim_{r\to\infty} \mathbb{P}_x\left(\kappa_r^q < \infty\right) = \frac{\mathrm{e}^{-2xc}\left(\sqrt{c^2+2q}-c\right)}{\sqrt{c^2+2q}+c} = \mathbb{P}_x\left(\kappa^q < \infty\right),$$

which is the probability of Parisian ruin with exponentially distributed delays, as given in (1.41), for the Brownian risk model.

3.3.2 Cramér-Lundberg process with exponential claims

Let X be a Cramér-Lundberg risk processes with exponentially distributed claims, i.e.

$$X_t - X_0 = ct - \sum_{i=1}^{N_t} C_i,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity $\eta > 0$, and where $\{C_1, C_2, \dots\}$ are independent and exponentially distributed random variables with parameter α. The Poisson process and the random variables are mutually independent.

In this case, for $x \geq 0$ and $q > 0$, the scale functions are given by

$$W^{(q)}(x) = \frac{1}{c\left(\Phi(q)-\theta_q\right)}\left((\alpha+\Phi(q))\mathrm{e}^{\Phi(q)x} - (\alpha+\theta_q)\mathrm{e}^{\theta_q x}\right),$$

$$Z_q(x,0) = \frac{q}{\sqrt{\Delta_q}}\left(\frac{\alpha+\Phi(q)}{\Phi(q)}\mathrm{e}^{\Phi(q)x} - \frac{\alpha+\theta_q}{\theta_q}\mathrm{e}^{\theta_q x}\right),$$

where

$$\begin{aligned}
\Phi(q) &= \frac{1}{2c}\left(q+\lambda-c\alpha+\sqrt{\Delta_q}\right),\\
\theta_q &= \frac{1}{2c}\left(q+\lambda-c\alpha-\sqrt{\Delta_q}\right),\\
\Delta_q &= (q+\lambda-c\alpha)^2+4c\alpha q.
\end{aligned}$$

Then, for $x \geq z$, we get

$$\begin{aligned}
\mathcal{W}_z^{(p,q)}(x) &= q\frac{\alpha+\Phi(p+q)}{\sqrt{\Delta_{p+q}\Delta_p}}\mathrm{e}^{\Phi(p+q)(x-z)}\left[\frac{\alpha+\Phi(p)}{\Phi(p+q)-\Phi(p)}\mathrm{e}^{\Phi(p)z}-\frac{\alpha+\theta_p}{\Phi(p+q)-\theta_p}\mathrm{e}^{\theta_p z}\right]\\
&\quad -q\frac{\alpha+\theta_{p+q}}{\sqrt{\Delta_{p+q}\Delta_p}}\mathrm{e}^{\theta_{p+q}(x-z)}\left[\frac{\alpha+\Phi(p)}{\theta_{p+q}-\Phi(p)}\mathrm{e}^{\Phi(p)z}-\frac{\alpha+\theta_p}{\theta_{p+q}-\theta_p}\mathrm{e}^{\theta_p z}\right].
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}_z^{(q,-q)}(x+z) &= q\frac{\alpha}{\sqrt{\Delta_0\Delta_q}}\left[\frac{\alpha}{\Phi(q)}\mathrm{e}^{\Phi(q)z}-\frac{\alpha+\theta_q}{\theta_q}\mathrm{e}^{\theta_p z}\right]\\
&\quad +q\frac{\alpha+\theta_0}{\sqrt{\Delta_0\Delta_q}}\mathrm{e}^{\theta_0 x}\left[\frac{\alpha+\Phi(q)}{\theta_0-\Phi(q)}\mathrm{e}^{\Phi(q)z}-\frac{\alpha+\theta_q}{\theta_0-\theta_q}\mathrm{e}^{\theta_p z}\right].
\end{aligned}$$

As noted in [58], we have

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{N_r} C_i \in \mathrm{d}y\right) &= \sum_{k=0}^{\infty}\mathbb{P}\left(\sum_{i=0}^{k} C_i \in \mathrm{d}y\right)\mathbb{P}(N_r=k)\\
&= \mathrm{e}^{-\eta r}\left(\delta_0(\mathrm{d}y)+\mathrm{e}^{-\alpha y}\sum_{m=0}^{\infty}\frac{(\alpha\eta r)^{m+1}}{m!(m+1)!}y^m\mathrm{d}y\right),
\end{aligned}$$

where $\delta_0(\mathrm{d}y)$ is a Dirac mass at 0. We also have

$$\begin{aligned}
&\int_0^{\infty}\mathrm{e}^{f(q)}z\mathbb{P}\left(X_r\in\mathrm{d}z\right)\\
&\quad= \int_0^{\infty}\mathrm{e}^{f(q)z}z\mathrm{e}^{-\eta r}\left(\delta_0\left(cr-\mathrm{d}z\right)+\mathrm{e}^{-\alpha(cr-z)}\sum_{m=0}^{\infty}\frac{(\alpha\eta r)^{m+1}}{m!\,(m+1)!}\left(cr-z\right)^m\mathrm{d}z\right)\\
&\quad= \mathrm{e}^{(f(q)c-\eta)r}\left(cr+\sum_{m=0}^{\infty}\frac{(\alpha\eta r)^{m+1}}{m!\,(m+1)!}\int_0^{cr}\mathrm{e}^{-(\alpha+f(q))z}\left(cr-z\right)^m\mathrm{d}z\right)\\
&\quad= \mathrm{e}^{(f(q)c-\eta)r}cr+\mathrm{e}^{(f(q)c-\eta)r}\sum_{m=0}^{\infty}\frac{(\alpha\eta r)^{m+1}}{m!\,(m+1)!}\left[\frac{cr}{(\alpha+f(q))^{m+1}}\Gamma\left(m+1,cr\left(\alpha+f(q)\right)\right)\right.\\
&\qquad\left.-\frac{1}{(\alpha+f(q))^{m+2}}\Gamma\left(m+2,cr\left(\alpha+f(q)\right)\right)\right],
\end{aligned}$$

where $\Gamma(a,x) = \int_0^x \mathrm{e}^{-t} t^{a-1} \mathrm{d}t$ is the incomplete gamma function and $f(q)$ is equal to either $\Phi(q)$ or θ_q.

Putting all the pieces together, we obtain an expression for the probability of Parisian ruin with mixed delays.

3.4 Intermediate results and proofs

Before presenting the proofs of the main results, we need a few intermediate lemmas. Recall that, for $\theta, r, q > 0$ and $y \geq 0$, we have

$$\Lambda^{(q)}(0,r) = \mathrm{e}^{qr}, \tag{3.15}$$

$$\int_0^\infty \mathrm{e}^{-\theta r} \Lambda^{(q)}(-y,r)\mathrm{d}r = \frac{\mathrm{e}^{-\Phi(\theta)y}}{\theta - q}. \tag{3.16}$$

The next two lemmas are the reasons our main results can be expressed explicitly in terms of scale functions. To prove Lemma 33 below, we will need first to prove Lemma 32 which provides a solution to the *race* between the mixed clock and the underlying process trying to get back above zero. Despite the similarities with [59, Lemma 4.2 and Lemma 4.3], we will take another direct and simple approach.

Lemma 32. *For $x \leq 0$, $p, \lambda \geq 0$ and $q, r > 0$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^+} \mathbf{1}_{\left\{\tau_0^+ \leq \mathrm{e}_q \wedge r\right\}}\right] = \mathrm{e}^{-(p+q)r} \Lambda^{(p+q)}(x,r), \tag{3.17}$$

and

$$\begin{aligned} \mathbb{E}_x\left[\mathrm{e}^{-p(\mathrm{e}_q \wedge r) + \lambda X_{\mathrm{e}_q \wedge r}} \mathbf{1}_{\left\{\tau_0^+ > \mathrm{e}_q \wedge r\right\}}\right] &= \frac{\mathrm{e}^{\lambda x}}{\psi_{p+q}(\lambda)} \left(\psi_p(\lambda)\, \mathrm{e}^{\psi_{p+q}(\lambda) r} - q\right) \\ &\quad - \mathrm{e}^{\psi_{p+q}(\lambda) r} \psi_p(\lambda) \int_0^r \mathrm{e}^{-\psi(\lambda)s} \Lambda^{(p+q)}(x,s)\, \mathrm{d}s \\ &\quad - \mathrm{e}^{-(p+q)r} \Lambda^{(p+q)}(x,r), \end{aligned} \tag{3.18}$$

where, in the case $\lambda = \Phi(p+q)$*, the ratio* $\frac{\psi_p(\lambda)\mathrm{e}^{\psi_{p+q}(\lambda)r}-q}{\psi_{p+q}(\lambda)}$ *is understood in the limiting sense, i.e.*

$$\lim_{\lambda\to\Phi(p+q)} \frac{\psi_p(\lambda)\,\mathrm{e}^{\psi_{p+q}(\lambda)r}-q}{\psi_{p+q}(\lambda)} = 1+qr.$$

Proof. The result in (3.17) follows from Equation (2.20).

For $\theta > 0$, using the potential measure in (1.23), we have

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\theta r}\mathbb{E}_x\left[\mathrm{e}^{-p(\mathrm{e}_q\wedge r)+\lambda X_{\mathrm{e}_q\wedge r}}\mathbf{1}_{\{\tau_0^+>\mathrm{e}_q\wedge r\}}\right]\mathrm{d}r &= \frac{1}{\theta}\mathbb{E}_x\left[\mathrm{e}^{-p(\mathrm{e}_q\wedge \mathrm{e}_\theta)+\lambda X_{\mathrm{e}_q\wedge \mathrm{e}_\theta}}\mathbf{1}_{\{\tau_0^+>\mathrm{e}_q\wedge \mathrm{e}_\theta\}}\right] \\
&= \mathrm{e}^{\Phi(p+q+\theta)x}\frac{(q+\theta)}{\theta}\int_{-\infty}^0 \mathrm{e}^{\lambda y}W^{(p+q+\theta)}(-y)\,\mathrm{d}y - \frac{(q+\theta)}{\theta}\int_{-\infty}^0 \mathrm{e}^{\lambda y}W^{(p+q+\theta)}(x-y)\,\mathrm{d}y \\
&= \left(\frac{\psi_p(\lambda)}{\theta\left(\psi_{p+q+\theta}(\lambda)\right)} - \frac{1}{\theta}\right)\times\left(\mathrm{e}^{\Phi(\theta+p+q)x}-\mathrm{e}^{\lambda x}\right),
\end{aligned}$$

where, in the last equality, we used (1.3) for $\lambda > \Phi(p+q+\theta)$. Then, for $\theta > \psi_{p+q}(\lambda)$, we have

$$\frac{1}{\theta\psi_{p+q+\theta}(\lambda)} = \int_0^\infty \mathrm{e}^{-\theta r}\left(\int_0^r \mathrm{e}^{\psi_{p+q}(\lambda)s}\mathrm{d}s\right)\mathrm{d}r = \int_0^\infty \mathrm{e}^{-\theta r}\left(\frac{\mathrm{e}^{\psi_{p+q}(\lambda)}-1}{\psi_{p+q}(\lambda)}\right)\mathrm{d}r,$$

and by (3.16) we also obtain

$$\frac{\mathrm{e}^{\Phi(\theta+p+q)x}}{\theta\psi_{p+q+\theta}(\lambda)} = \int_0^\infty \mathrm{e}^{-\theta r}\left(\int_0^r \mathrm{e}^{\psi_{p+q}(\lambda)(r-s)}\mathrm{e}^{-(p+q)s}\Lambda^{p+q}(x,s)\mathrm{d}s\right)\mathrm{d}r.$$

By Laplace inversion and with further simplifications, the result in (3.18) follows.

■

Lemma 33. *For* $x\in\mathbb{R}$*,* $p,\lambda\geq 0$ *and* $b,q,r>0$*, we have*

$$\begin{aligned}
\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+\leq \mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]& \\
= \mathrm{e}^{-(p+q)r}\left(\Lambda^{(p)}(x;r,q) - \frac{W^{(p)}(x)}{W^{(p)}(b)}\Lambda^{(p)}(b;r,q)\right) \quad &(3.19)
\end{aligned}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p(\mathrm{e}_q\wedge r)+\lambda X_{\mathrm{e}_q\wedge r}}\mathbf{1}_{\{\tau_0^+>\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]$$
$$=\left(\mathcal{F}^{(p,\lambda)}(x;r,q)-\mathrm{e}^{-(p+q)r}\Lambda^{(p)}(x;r,q)\right)$$
$$-\frac{W^{(p)}(x)}{W^{(p)}(b)}\left(\mathcal{F}^{(p,\lambda)}(b;r,q)-\mathrm{e}^{-(p+q)r}\Lambda^{(p)}(b;r,q)\right). \quad (3.20)$$

Proof. The proof consists in using Lemma 32 to compute the inner expectations and then use the following relationship:

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\Lambda^{(p+q)}\left(X_{\tau_0^-},r\right)\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]=\Lambda^{(p)}(x;r,q)-\frac{W^{(p)}(x)}{W^{(p)}(b)}\Lambda^{(p)}(b;r,q).$$

This is proved with the result in Equation (1.21).

Identity (1.16) is also needed to complete the proof of (3.20). The details are left to the reader. ■

3.4.1 Proof of Theorem 28

The steps of the proof of Theorem 28 is based on the Lemma (33) together with standard probabilistic decompositions. For $x<0$, using the strong Markov property and the fact that X is skip-free upward, we get

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_r^q+\lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q<\tau_b^+\}}\right] &=\mathbb{E}_x\left[\mathrm{e}^{-p(\mathrm{e}_q\wedge r)+\lambda X_{\mathrm{e}_q\wedge r}}\mathbf{1}_{\{\tau_0^+>\mathrm{e}_q\wedge r\}}\right]\\ &+\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+\le \mathrm{e}_q\wedge r\}}\right]\mathbb{E}\left[\mathrm{e}^{-p\kappa_r^q+\lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q<\tau_b^+\}}\right].\end{aligned} \quad (3.21)$$

Now, for $x\geq 0$, using again the strong Markov property, we get

$$\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_r^q+\lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q<\tau_b^+\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\kappa_r^q+\lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q<\tau_b^+\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right].$$

Injecting (3.21) in this last expectation, we obtain

$$\mathbb{E}_x\left[\mathrm{e}^{-p\kappa_r^q+\lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q<\tau_b^+\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p(\mathrm{e}_q\wedge r)+\lambda X_{\mathrm{e}_q\wedge r}}\mathbf{1}_{\{\tau_0^+>\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]$$
$$+\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+\leq\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]\mathbb{E}\left[\mathrm{e}^{-p\kappa_r^q+\lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q<\tau_b^+\}}\right]. \tag{3.22}$$

Note that this decomposition holds for all $x\in\mathbb{R}$.

We will first prove the result in (3.3) for $x=0$. We split this part of the proof in two steps: for processes with paths of bounded variation (BV) and then for processes with paths of unbounded variation (UBV).

First, we assume that X has paths of BV. Setting $x=0$ in (3.22), yields

$$\mathbb{E}\left[\mathrm{e}^{-p\kappa_r^q+\lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q<\tau_b^+\}}\right]=\frac{\mathbb{E}\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p(\mathrm{e}_q\wedge r)+\lambda X_{\mathrm{e}_q\wedge r}}\mathbf{1}_{\{\tau_0^+>\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]}{1-\mathbb{E}\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+\leq\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]}.$$

Using (3.15) and (3.20), the numerator can be written as

$$\mathbb{E}\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p(\mathrm{e}_q\wedge r)+\lambda X_{\mathrm{e}_q\wedge r}}\mathbf{1}_{\{\tau_0^+>\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]$$
$$=-\frac{W^{(p)}(0)}{W^{(p)}(b)}\left(\mathcal{F}^{(p,\lambda)}(b;r,q)-\mathrm{e}^{-(p+q)r}\Lambda^{(p)}(b;r,q)\right), \tag{3.23}$$

while the denominator can be written as

$$1-\mathbb{E}\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+\leq\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]$$
$$=1-\mathrm{e}^{-(p+q)r}\left(\Lambda^{(p)}(0;r,q)-\frac{W^{(p)}(0)}{W^{(p)}(b)}\Lambda^{(p)}(b;r,q)\right)$$
$$=\frac{W^{(p)}(0)}{W^{(p)}(b)}\mathrm{e}^{-(p+q)r}\Lambda^{(p)}(b;r,q), \tag{3.24}$$

where in the last equality we used (3.15). Note that, since X is assumed to be of BV, we have $W(0)>0$. Consequently, we have obtained

$$\mathbb{E}\left[\mathrm{e}^{-p\kappa_r^q+\lambda X_{\kappa_r^q}}\mathbf{1}_{\{\kappa_r^q<\tau_b^+\}}\right]=1-\frac{\mathcal{F}^{(p,\lambda)}(b;r,q)}{\mathrm{e}^{-(p+q)r}\Lambda^{(p)}(b;r,q)}. \tag{3.25}$$

Now, we assume X has paths of UBV. Let us approximate the situation as follows (as in [58]). We denote by $\kappa_{r,\epsilon}^{q}$ the first time an excursion, starting when X gets below zero and ending before X gets back up to ϵ, is longer than $\mathrm{e}_q \wedge r$. Mathematically,

$$\kappa_{r,\epsilon}^{q} = \inf\left\{t > 0 : t - g_t^{\epsilon} > (\mathrm{e}_q \wedge r), X_{t-(\mathrm{e}_q\wedge r)} < 0\right\}.$$

where $g_t^{\epsilon} = \sup\{0 \leq s \leq t : \ X_s \geq \epsilon\}$. Using similar arguments as in the BV case, we can write

$$\mathbb{E}_{\epsilon}\left[\mathrm{e}^{-p\kappa_{r,\epsilon}^{q}+\lambda X_{\kappa_{r,\epsilon}^{q}}}\mathbf{1}_{\{\kappa_{r,\epsilon}^{q}<\tau_b^+\}}\right] = \frac{\mathbb{E}_{\epsilon}\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p(\mathrm{e}_q\wedge r)+\lambda X_{\mathrm{e}_q\wedge r}}\mathbf{1}_{\{\tau_\epsilon^+>\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]}{1-\mathbb{E}_{\epsilon}\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\tau_\epsilon^+}\mathbf{1}_{\{\tau_\epsilon^+\leq \mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]}$$

$$= \frac{\mathcal{F}_{\epsilon}^{(p,\lambda)}(\epsilon;r,q) - \mathrm{e}^{-(p+q)r}\Lambda_{\epsilon}^{(p)}(0;r,q) - \frac{W^{(p)}(\epsilon)}{W^{(p)}(b)}\left(\mathcal{F}^{(p,\lambda)}(b;r,q) - \mathrm{e}^{-(p+q)r}\Lambda_{\epsilon}^{(p)}(b-\epsilon;r,q)\right)}{1-\mathrm{e}^{-(p+q)r}\left(\Lambda_{\epsilon}^{(p)}(0;r,q) - \frac{W^{(p)}(\epsilon)}{W^{(p)}(b)}\Lambda_{\epsilon}^{(p)}(b-\epsilon;r,q)\right)},$$

where, from (1.10), we define temporarily

$$\Lambda_{\epsilon}^{(p)}(x,r,q) = \int_{\epsilon}^{\infty}\mathcal{W}_{z-\epsilon}^{(p+q,-q)}(x+z)\,\frac{z}{r}\mathbb{P}(X_r\in \mathrm{d}z),$$

and

$$\begin{aligned}\mathcal{F}_{\epsilon}^{(p,\lambda)}(x;r,s) &= \frac{1}{\psi_{p+s}(\lambda)}\left(\psi_p(\lambda)\mathrm{e}^{\psi_{p+s}(\lambda)r}-s\right)Z_p(x,\lambda)\\ &\quad -\mathrm{e}^{\psi_{p+s}(\lambda)r}\psi_p(\lambda)\,\mathrm{e}^{\lambda\epsilon}\int_0^r \mathrm{e}^{-\psi(\lambda)u}\Lambda_{\epsilon}^{(p)}(x;u,s)\,\mathrm{d}u.\end{aligned}$$

We will now compute the limit, as $\epsilon\to 0$, of the denominator and the numerator with an appropriate scaling. We can write

$$\frac{1-\mathrm{e}^{-(p+q)r}\left(\Lambda_{\epsilon}^{(p)}(\epsilon;r,q) - \frac{W^{(p)}(\epsilon)}{W^{(p)}(b)}\Lambda_{\epsilon}^{(p)}(b;r,q)\right)}{W^{(p)}(\epsilon)}$$

$$= \frac{1-\mathrm{e}^{-(p+q)r}\Lambda_{\epsilon}^{(p)}(\epsilon;r,q)}{W^{(p)}(\epsilon)} + \frac{\mathrm{e}^{-(p+q)r}\Lambda_{\epsilon}^{(p)}(b;r,q)}{W^{(p)}(b)},$$

where, using (1.13), we have

$$\frac{1-\mathrm{e}^{-(p+q)r}\Lambda_\epsilon^{(p)}(0;r,q)}{W^{(p)}(\epsilon)} = \frac{1-\mathrm{e}^{-(p+q)r}\int_\epsilon^\infty W^{(p+q)}(z)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)}{W^{(p)}(\epsilon)} + q\mathrm{e}^{-(p+q)r}\frac{\int_\epsilon^\infty\left[\int_{z-\epsilon}^z W^{(p)}(z-y)W^{(p+q)}(y)\mathrm{d}y\right]\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)}{W^{(p)}(\epsilon)}.$$

We will show that this last expression converges to zero. First, using (3.16) and then using the fact that $W^{(p+q)}$ is an increasing function, we can write

$$\begin{aligned}\frac{1-\mathrm{e}^{-(p+q)r}\int_\epsilon^\infty W^{(p+q)}(z)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)}{W^{(p)}(\epsilon)} &= \frac{\mathrm{e}^{-(p+q)r}\int_0^\epsilon W^{(p+q)}(z)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)}{W^{(p)}(\epsilon)}\\ &\le \frac{\mathrm{e}^{-(p+q)r}}{r}\frac{W^{(p+q)}(\epsilon)}{W^{(p)}(\epsilon)/\epsilon}\longrightarrow_{\epsilon\to0}0,\end{aligned}$$

since

$$\lim_{\epsilon\to0}\frac{W^{(p)}(\epsilon)}{\epsilon}=\begin{cases}\frac{2}{\sigma^2} & \text{if } \sigma>0,\\ \infty & \text{otherwise.}\end{cases}$$

Similarly, using Lebesgue's convergence theorem, we can write

$$\begin{aligned}&\frac{\int_\epsilon^\infty\left[\int_{z-\epsilon}^z W^{(p)}(z-y)W^{(p+q)}(y)\mathrm{d}y\right]\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)}{W^{(p)}(\epsilon)}\\ &\qquad\le\int_0^\infty\left[\int_{z-\epsilon}^z W^{(p+q)}(y)\mathrm{d}y\right]\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)\longrightarrow_{\epsilon\to0}0.\end{aligned}$$

Therefore, we have obtained

$$\lim_{\epsilon\to0}\frac{1-\mathrm{e}^{-(p+q)r}\left(\Lambda_\epsilon^{(p)}(0;r,q)-\frac{W^{(p)}(\epsilon)}{W^{(p)}(b)}\Lambda_\epsilon^{(p)}(b-\epsilon;r,q)\right)}{W^{(p)}(\epsilon)}=\frac{\mathrm{e}^{-(p+q)r}\Lambda^{(p)}(b;r,q)}{W^{(p)}(b)}.$$

Using similar arguments, we can also show that

$$\begin{aligned}&\lim_{\epsilon\to0}\frac{\mathcal{F}_\epsilon^{(p,\lambda)}(\epsilon;r,q)-\mathrm{e}^{-(p+q)r}\Lambda_\epsilon^{(p)}(0;r,q)-\frac{W^{(p)}(\epsilon)}{W^{(p)}(b)}\left(\mathcal{F}^{(p,\lambda)}(b;r,q)-\mathrm{e}^{-(p+q)r}\Lambda_\epsilon^{(p)}(b-\epsilon;r,q)\right)}{W^{(p)}(\epsilon)}\\ &\qquad=\frac{\mathcal{F}^{(p,\lambda)}(b;r,q)-\mathrm{e}^{-(p+q)r}\Lambda^{(p)}(b;r,q)}{W^{(p)}(b)}.\end{aligned}$$

This concludes the proof for $x=0$.

Finally, no matter if X is of BV or of UBV, using Equation (3.22), Equation (3.25) and identities in Lemma 33, we can finish the proof of Theorem 28 .

To prove (3.4), we use again the strong Markov property and spectral negativity of X. Then, for $x<0$

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+<\mathrm{e}_q\wedge r\}}\right]\mathbb{E}\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right],$$

and for $0\le x\le b$, we get

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right].$$

Putting the pieces together we obtain

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right] &= \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right]\\ &+\mathbb{E}\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right]\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+<\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right].\end{aligned}$$

If we assume that X is of BV, then, setting $x=0$ in the last equation and combining (3.8) and (3.19), we get

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right] &= \frac{\mathbb{E}\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right]}{1-\mathbb{E}\left[\mathrm{e}^{-p\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+<\mathrm{e}_q\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]}\\ &= \frac{\frac{W^{(p)}(0)}{W^{(p)}(b)}}{1-\mathrm{e}^{-(p+q)r}\left(\Lambda^{(p,q)}(x,r)-\frac{W^{(p)}(0)}{W^{(p)}(b)}\Lambda^{(p,q)}(b,r)\right)}\\ &= \frac{1}{\mathrm{e}^{-(p+q)r}\Lambda^{(p,q)}(b,r)}.\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^q\}}\right] &= \frac{W^{(p)}(x)}{W^{(p)}(b)}+\frac{\mathrm{e}^{-(p+q)r}\left(\Lambda^{(p,q)}(x,r)-\frac{W^{(p)}(x)}{W^{(p)}(b)}\Lambda^{(p,q)}(b,r)\right)}{\mathrm{e}^{-(p+q)r}\Lambda^{(p,q)}(b,r)}\\ &= \frac{\Lambda^{(p,q)}(x,r)}{\Lambda^{(p,q)}(b,r)}.\end{aligned}$$

If X is a general SNLP, we can use the same limiting argument as in the proof of identity (3.5). The details are left to the reader.

3.4.2 Proof of Corollary 29

To deal with the limit as $b \to \infty$, we use (1.18) and the fact that, for $\theta > \Phi(p)$

$$\lim_{b\to\infty} \frac{Z_p(p,\theta)}{W^{(p)}(b)} = \frac{\psi_p(\theta)}{\theta - \Phi(p)}, \tag{3.26}$$

to obtain

$$\begin{aligned}
\lim_{b\to\infty} \frac{\mathcal{W}_z^{(p+q,-q)}(b+z)}{W^{(p)}(b)} &= e^{\Phi(p)z} + q \lim_{b\to\infty} \int_0^z \frac{W^{(p)}(b+z-y)}{W^{(p)}(b)} W^{(p+q)}(y)\,dy \\
&= e^{\Phi(p)z} + q \int_0^z e^{\Phi(p)(z-y)} W^{(p+q)}(y)\,dy \\
&= e^{\Phi(p)z} \left(1 + q \int_0^z e^{-\Phi(p)y} W^{(p+q)}(y)\,dy\right) \\
&= Z_{p+q}(z, \Phi(p)),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{b\to\infty} \frac{\mathcal{F}^{(p)}(b;r,q)}{\Lambda^{(p)}(b;r,q)} = \lim_{b\to\infty} \frac{\mathcal{F}^{(p)}(b;r,q)/W^{(p)}(b)}{\Lambda^{(p)}(b;r,q)/W^{(p)}(b)} \\
= &\frac{\frac{p}{(p+q)\Phi(p)}\left(q + pe^{-(p+q)r}\right) + pe^{-(p+q)r} \int_0^r \left(\int_0^\infty Z_{p+q}(z,\Phi(p)) \frac{z}{s} \mathbb{P}(X_s \in dz)\right) ds}{\int_0^\infty Z_{p+q}(z,\Phi(p)) \frac{z}{r} \mathbb{P}(X_r \in dz)}.
\end{aligned}$$

CHAPTER IV

A NOTE ON PARISIAN RUIN UNDER A HYBRID OBSERVATION SCHEME

4.1 Introduction

In most Parisian ruin theory literature, the surplus process is monitored continuously and the level of ticking and resetting the clock is the same. However, it is rather impractical to observe the business continuously on a ongoing basis. Following this idea, Li et al. [49] introduced the hybrid observation scheme. More specifically, the surplus process is observed at discrete time-points (Poisson observation) whenever the business is financially healthy (above a) and it is continuously observed during periods of financial distress (below a). In [49], Parisian ruin is also studied. Once the surplus is observed below 0 at Poisson arrival times, the process is monitored continuously and Parisian ruin is declared if the duration of such period of distress is greater than a fixed delay r (see Figure 4.1 for a graphical interpretation). If the process recovers to the positive level a before the grace period r, the clock is stopped and the continuous monitoring will be switched back to discrete monitoring. Even though the recovery barrier a has no mathematical role in this model, in a risk management point of view, it makes sense to impose a prudent capital requirement after a solvency issue has occurred. In this chapter, we improve the main result in [49] which is the probability of Parisian ruin under the hybrid observation scheme. We also obtain the expressions for the two-sided exit problem, Laplace transform and the probability of Parisian ruin, all terms of scale functions. Our approach is based on the expression of the Gerber–Shiu distribution at Parisian ruin with exponential implementation delays in [8].

4.1.1 Parisian ruin under a hybrid observation scheme

First, in order to compare our results with those in [49], we suppose $\{T_i\}_{i\geq 0}$ are the arrival times of an independent Poisson process of rate λ.
The time of Parisian ruin under a hybrid observation scheme with a recovery barrier $a \geq 0$ and a fixed delay $r > 0$ is defined by

$$\tilde{\kappa}_{a,r}^{\lambda} = \inf\left\{t \in \left(T_n, \tau_a^+ \circ \theta_{T_n}\right) : X_{T_n} < 0 \text{ and } t - T_n \geq r,\, n \in \mathbb{N}\right\},$$

where θ is the Markov shift operator, i.e. $X_s \circ \theta_t = X_{s+t}$.

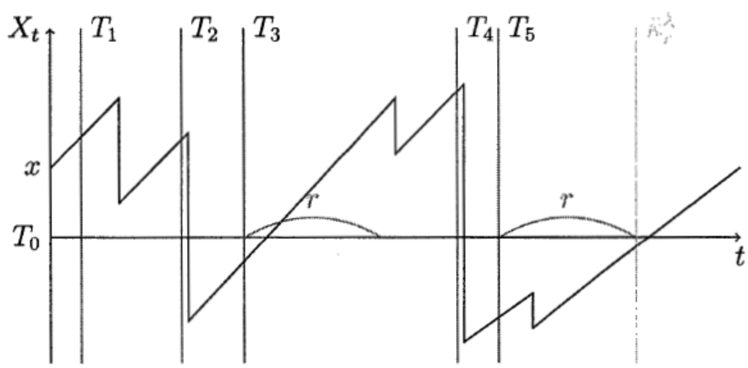

Figure 4.1 Illustration of Parisian ruin under the hybrid observation scheme for $a = 0$.

Li et al. [49] obtained the following expression for the probability of Parisian ruin $\mathbb{P}_x\left(\tilde{\kappa}_{a,r}^{\lambda} < \infty\right)$ using a probabilistic decomposition and using the technique of taking Laplace transforms with respect to the delay r.

Theorem 34. *For $r, \lambda > 0$, $a \geq 0$ and $x \in \mathbb{R}$, if $\psi'(0+) > 0$, we have*

$$\begin{aligned}\mathbb{P}_x\left(\tilde{\kappa}_{a,r}^{\lambda} < \infty\right) = 1 - \psi'(0+)\frac{\Phi(\lambda)}{\lambda} Z(x, \Phi(\lambda)) \\ - \psi'(0+)\Phi(\lambda)\frac{Z(a, \Phi(\lambda))\int_0^r \mathrm{e}^{\lambda(r-s)} g_{x,a,\lambda}(s)\,\mathrm{d}s}{1 - \lambda\int_0^r \mathrm{e}^{\lambda(r-s)} g_{a,a,\lambda}(s)\,\mathrm{d}s},\end{aligned} \tag{4.1}$$

where

$$g_{x,a,\lambda}(s) = \int_a^{\infty}\left(\frac{\Phi(\lambda)}{\lambda} Z(x, \Phi(\lambda)) - W(x+z-a)\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z).$$

We want to improve on this result by making it more close to Equation (1.35) using a probabilistic approach. Without loss of generality we will assume that the recovery barrier $a = 0$ and we will write $\tilde{\kappa}_{0,r}^{\lambda} = \tilde{\kappa}_r^{\lambda}$.

4.2 Main results

We now present our main results. First, we derive the two-sided exit problem when a hybrid Parisian delay is added as an improvement over Theorem 34. We present a probabilistic analysis and simple resulting expressions for the two-sided exit problem, Laplace transform and the probability of the Parisian ruin under a hybrid observation scheme, all expressed in terms of the delayed scale functions.

Theorem 35. *For $q \geq 0$, $b, r, \lambda > 0$ and $x \leq b$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-q(\tilde{\kappa}_r^{\lambda} - r)}\mathbf{1}_{\{\tilde{\kappa}_r^{\lambda} < \tau_b^+\}}\right] = \frac{\lambda}{\lambda + q}\left(\mathcal{S}^{(q)}(x, r) - \frac{\Theta^{(q)}(x; r, \lambda)}{\Theta^{(q)}(b; r, \lambda)}\mathcal{S}^{(q)}(b, r)\right) \tag{4.2}$$

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+ < \tilde{\kappa}_r^{\lambda}\}}\right] = \frac{\Theta^{(q)}(x; r, \lambda)}{\Theta^{(q)}(b; r, \lambda)}, \tag{4.3}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q(\tilde{\kappa}_r^{\lambda} - r)}\mathbf{1}_{\{\tilde{\kappa}_r^{\lambda} < \infty\}}\right] = \frac{\lambda}{\lambda + q}\left(\mathcal{S}^{(q)}(x, r) - \Theta^{(q)}(x; r, \lambda)\,\tilde{\mathcal{S}}^{(q,\lambda)}(r)\right), \tag{4.4}$$

where

$$\begin{aligned}\mathcal{S}^{(q)}(x,r) &= Z^{(q)}(x) + \Lambda^{(q)}(x,r) - \Lambda^{(q)}(x;r,-q),\\ \Theta^{(q)}(x;r,\lambda) &= \mathrm{e}^{(\lambda+q)r} Z_q(x,\Phi(\lambda+q)) + \Lambda^{(q)}(x;r) - \Lambda^{(q)}(x;r,\lambda),\end{aligned}$$

and

$$\tilde{\mathcal{S}}^{(q,\lambda)}(r) = \frac{q/\Phi(q) - \int_0^\infty \left(Z(z,\Phi(q)) - \mathrm{e}^{\Phi(q)z}\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}{\mathrm{e}^{(\lambda+q)r}\lambda/\left(\Phi(\lambda+q) - \Phi(q)\right) - \int_0^\infty \left(Z_{q+\lambda}(z,\Phi(q)) - \mathrm{e}^{\Phi(q)z}\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}.$$

Remark 36. It is interesting to note that the function $\Theta^{(q)}(x;r,\lambda)$ is expressed in terms of scale functions $\Lambda^{(q)}(x;r,\lambda)$, $\Lambda^{(q)}(x;r)$ and $Z_q(x,\Phi(\lambda+q))$ related to Parisian ruins (3.1), (1.24) and (1.46) respectively. It could be called the *hybrid* scale function.

Setting $q=0$ in (4.2), we obtain the following new expression for the probability of hybrid Parisian ruin.

Corollary 37. *For $x \in \mathbb{R}$ and $\lambda, r > 0$, we have*

$$\mathbb{P}_x\left(\tilde{\kappa}_r^\lambda < \infty\right) = 1 - \Theta(x;r,\lambda)\,\tilde{\mathcal{S}}^{(0,\lambda)}(r), \tag{4.5}$$

where $\Theta = \Theta^{(0)}$.

Remark 38. When $\psi'(0+) > 0$, we have

$$\tilde{\mathcal{S}}^{(0,\lambda)}(r) = \frac{\mathbb{E}[X_1]}{\mathrm{e}^{\lambda r}\lambda/\Phi(\lambda) - \int_0^\infty \left(Z^{(\lambda)}(z) - 1\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}.$$

Then,

$$\mathbb{P}_x\left(\tilde{\kappa}_r^\lambda < \infty\right) = 1 - \mathbb{E}[X_1]\frac{\Theta(x;r,\lambda)}{\mathrm{e}^{\lambda r}\lambda/\Phi(\lambda) - \int_0^\infty \left(Z^{(\lambda)}(z) - 1\right)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)}. \tag{4.6}$$

Our expression of the probability of Parisian ruin is different than the one in Theorem 34, which is because in the proof we use a different approach, and has a similar structure as the one in Equation (1.35).

4.3 Discussion on the results

The fluctuation identities in Theorem 35 have a similar structure as Parisian fluctuation identities (1.43) and (1.42). Indeed, in Equation (4.2), the functions $\mathcal{S}^{(q)}(\cdot, r)$ and $\Theta^{(q)}(\cdot; r, \lambda)$ play a similar rôle as the one played by the classical q-scale functions $Z^{(q)}(\cdot)$ and $Z_q(\cdot, \Phi(\lambda+q))$ respectively.
First, we will show that the function $\mathcal{S}^{(q)}(x, r)$ converges, as $r \to 0$, to the scale function $Z^{(q)}(x)$. Taking Laplace transforms in r, together with Kendall's identity, Tonelli's theorem and (3.10), we have

$$\int_0^\infty \mathrm{e}^{-\theta r} \left(\mathrm{e}^{-qr} \Lambda^{(q)}(x; r, -q)\right) \mathrm{d}r = \frac{Z_q(x, \Phi(\theta+q))}{\theta+q},$$

and also

$$\int_0^\infty \mathrm{e}^{-\theta r} \mathrm{e}^{-qr} \Lambda^{(q)}(x; r) \,\mathrm{d}r = \int_0^\infty \mathrm{e}^{-\Phi(\theta+q)z} W^{(q)}(x+z)\,\mathrm{d}z.$$

Then, using the initial value theorem, we obtain

$$\begin{aligned}
&\lim_{r\to 0} \mathrm{e}^{-qr} \left(\Lambda^{(q)}(x; r) - \Lambda^{(q)}(x; r, -q)\right) \\
&\quad = \lim_{\theta\to\infty} \theta \int_0^\infty \mathrm{e}^{-(\theta+q)r} \left(\Lambda^{(q)}(x; r) - \Lambda^{(q)}(x; r, -q)\right) \mathrm{d}r \\
&\quad\quad = \lim_{\theta\to\infty} \frac{q}{\theta+q} Z_q(x, \Phi(\theta+q)) = 0,
\end{aligned}$$

where the last equality follows from the fact that

$$\lim_{\theta\to\infty} \frac{q}{\theta+q} Z_q(x, \Phi(\theta+q)) = \lim_{\theta\to\infty} \left(\frac{q\theta}{(\theta+q)\,\Phi(\theta+q)}\right) \left(\frac{\Phi(\theta+q)}{\theta} Z_q(x, \Phi(\theta+q))\right),$$

and

$$\begin{aligned}
\lim_{\theta\to\infty} \frac{\Phi(\theta+q)}{\theta} Z_q(x, \Phi(\theta+q)) &= \lim_{\theta\to\infty} \Phi(\theta+q) \int_0^\infty \mathrm{e}^{-\Phi(\theta+q)y} W^{(q)}(x+y)\mathrm{d}y \\
&= W^{(q)}(x),
\end{aligned}$$

which follows the initial value theorem. Thus,

$$\lim_{r\to 0} \mathrm{e}^{-qr} \mathcal{S}^{(q)}(x, r) = Z^{(q)}(x).$$

Lastly, by the same techniques, we can also show that

$$\lim_{r\to 0}\Theta^{(q)}(x;r,\lambda)=Z_q\left(x,\Phi(\lambda+q)\right),$$

and then

$$\begin{aligned}\lim_{r\to 0}\mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right] &= \mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]\\ &= Z^{(q)}(x)-\frac{Z_q\left(x,\Phi(\lambda+q)\right)}{Z_q\left(b,\Phi(\lambda+q)\right)}Z^{(q)}(b).\end{aligned}$$

4.4 Proofs

The proofs of our main results are based on technical but important lemmas, as well as more standard probabilistic decompositions. We use the expression of the Gerber–Shiu distribution at Parisian ruin with exponential implementation delays from [8] and results from [55] to obtain our key lemma (Lemma 39 below). First, our main interest is to obtain a closed-form for the following expectation

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}W^{(p)}\left(X_{T_0^-}+z\right)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right],$$

where $p,q\geq 0$, $x\leq b$.

Lemma 39. *For $p,q,b\geq 0$, $\lambda,r,z>0$ and $x\leq b$, we have*

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}W^{(p)}\left(X_{T_0^-}+z\right)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]\\ &\quad=\frac{\lambda}{p-(q+\lambda)}\frac{Z_q\left(x,\Phi(\lambda+q)\right)}{Z_q\left(b,\Phi(\lambda+q)\right)}\left(\mathcal{W}_b^{(q,p-q)}\left(b+z\right)-\mathcal{W}_b^{(q,\lambda)}\left(b+z\right)\right)\\ &\qquad\qquad-\frac{\lambda}{p-(q+\lambda)}\left(\mathcal{W}_x^{(q,p-q)}\left(x+z\right)-\mathcal{W}_x^{(q,\lambda)}\left(x+z\right)\right),\end{aligned}\tag{4.7}$$

and, for $x\in\mathbb{R}$, we have

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}W^{(p)}\left(X_{T_0^-}+z\right)\mathbf{1}_{\{T_0^-<\infty\}}\right]\\ &\quad=\frac{\left(\Phi(\lambda+q)-\Phi(q)\right)}{p-(q+\lambda)}Z_p\left(x,\Phi(\lambda+p)\right)\left(Z_p\left(x+z,\Phi(q)\right)-Z_{\lambda+q}\left(x+z,\Phi(q)\right)\right)\\ &\qquad\qquad-\frac{\lambda}{p-(q+\lambda)}\left(\mathcal{W}_x^{(q,p-q)}\left(x+z\right)-\mathcal{W}_x^{(q,\lambda)}\left(x+z\right)\right).\end{aligned}\tag{4.8}$$

Proof. Using (1.44), we have

$$
\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}W^{(p)}\left(X_{T_0^-}+z\right)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]\\
&\quad=\int_{-\infty}^{0}W^{(p)}(y+z)\,\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-},X_{T_0^-}\in\mathrm{d}y,T_0^-<\tau_b^+\right]\\
&\quad=\lambda\frac{Z_q(x,\Phi(\lambda+))}{Z_q(b,\Phi(\lambda+q))}\int_{-\infty}^{0}W^{(p)}(y+z)\,\mathcal{W}_b^{(q,\lambda)}(b-y)\,\mathrm{d}y\\
&\qquad-\lambda\int_{-\infty}^{0}W^{(p)}(y+z)\,\mathcal{W}_x^{(q,\lambda)}(x-y)\,\mathrm{d}y\\
&\quad=\frac{Z_q(x,\Phi(\lambda+q))}{Z_q(b,\Phi(\lambda+q))}\lambda\int_0^z W^{(p)}(z-y)\,\mathcal{W}_b^{(q,\lambda)}(b+y)\,\mathrm{d}y\\
&\qquad-\lambda\int_0^z W^{(p)}(z-y)\,\mathcal{W}_x^{(q,\lambda)}(x+y)\,\mathrm{d}y\\
&\quad=\frac{\lambda}{p-(q+\lambda)}\frac{Z_q(x,\Phi(\lambda+q))}{Z_q(b,\Phi(\lambda+q))}\left(\mathcal{W}_b^{(q,p-q)}(b+z)-\mathcal{W}_b^{(q,\lambda)}(b+z)\right)\\
&\qquad-\frac{\lambda}{p-(q+\lambda)}\left(\mathcal{W}_x^{(q,p-q)}(x+z)-\mathcal{W}_x^{(q,\lambda)}(x+z)\right),
\end{aligned}
\tag{4.9}
$$

where in the last equality we applied the following useful identity taken from [51]

$$
(s-(p+q))\int_a^x W^{(s)}(x-y)\mathcal{W}_a^{(p,q)}(y)\,\mathrm{d}y=\mathcal{W}_a^{(p,s-p)}(x)-\mathcal{W}_a^{(p,q)}(x)\,. \tag{4.10}
$$

The second identity follows using (1.45) and the fact that

$$
(p-q)\int_0^a W^{(p)}(a-x)\,Z_q(x,\theta)\,\mathrm{d}x=Z_p(a,\theta)-Z_q(a,\theta)\,, \tag{4.11}
$$

or by letting $b\to\infty$. ■

The identities in Lemma (39) generalizes the classical identities (1.21) and (1.22) respectively (in spite of the fact that it is hard to prove the convergence when $\lambda\to\infty$).

The following identity is new and crucial for the proof of our main results.

Lemma 40. *For $p, q, b \geq 0$, $\lambda, r > 0$ and $x \leq b$, we have*

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\Lambda^{(p)}(X_{T_0^-}, r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\ = \frac{\lambda}{p-(q+\lambda)}\frac{Z_q(x,\Phi(\lambda+q))}{Z_q(b,\Phi(\lambda+q))}\left(\Lambda^{(q)}(b;r,p-q)-\Lambda^{(q)}(b;r,\lambda)\right) \\ -\frac{\lambda}{p-(q+\lambda)}\left(\Lambda^{(q)}(x;r,p-q)-\Lambda^{(q)}(x;r,\lambda)\right). \quad (4.12)\end{aligned}$$

Proof. Using (4.7) and Tonelli's theorem, we obtain

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\Lambda^{(p)}(X_{T_0^-}, r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\ = \mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\int_0^\infty W^{(p)}\left(X_{T_0^-}+z\right)\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right)\mathbf{1}_{\{T_0^-<\infty\}}\right] \\ = \int_0^\infty \mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}W^{(p)}\left(X_{T_0^-}+z\right)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]\frac{z}{r}\mathbb{P}\left(X_r\in \mathrm{d}z\right) \\ = \frac{\lambda}{p-(q+\lambda)}\frac{Z_q(x,\Phi(\lambda+q))}{Z_q(b,\Phi(\lambda+q))}\left(\Lambda^{(q)}(b;r,p-q)-\Lambda^{(q)}(b;r,\lambda)\right) \\ -\frac{\lambda}{p-(q+\lambda)}\left(\Lambda^{(q)}(x;r,p-q)-\Lambda^{(q)}(x;r,\lambda)\right),\end{aligned}$$

and the result follows. ■

4.4.1 Proof of Theorem 35

We use a standard probabilistic decomposition of the sample paths of X. For $x < 0$, from the strong Markov property and the fact that X is skip-free upward, we have

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right] = \mathrm{e}^{-qr}\mathbb{P}_x\left(\tau_0^+ > r\right) \\ +\mathbb{E}\left[\mathrm{e}^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right]\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+<r\}}\right]. \quad (4.13)\end{aligned}$$

Consequently, for $0 \leq x \leq b$, using again the strong Markov property, we obtain

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right] = \mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\mathbb{E}_{X_{T_0^-}}\left[\mathrm{e}^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right]\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right].$$

Injecting (4.13) in the last expectation, we have, for all $x \in \mathbb{R}$

$$\begin{aligned}
\mathbb{E}_x\left[e^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right] &= e^{-qr}\mathbb{E}_x\left[e^{-qT_0^-}\mathbb{P}_{X_{T_0^-}}\left(\tau_0^+>r\right)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\
&\quad +\mathbb{E}\left[e^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right]\mathbb{E}_x\left[e^{-qT_0^-}\mathbb{E}_{X_{T_0^-}}\left[e^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+<r\}}\right]\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\
&= e^{-qr}\mathbb{E}_x\left[e^{-qT_0^-}\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\
&\quad -e^{-qr}\mathbb{E}_x\left[e^{-qT_0^-}\Lambda(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\
&\quad +e^{-qr}\mathbb{E}\left[e^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right]\mathbb{E}_x\left[e^{-qT_0^-}\Lambda^{(q)}(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \qquad (4.14)
\end{aligned}$$

Setting $x=0$ in (4.14), we egt

$$\mathbb{E}\left[e^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right] = \frac{e^{-qr}\mathbb{E}\left[e^{-qT_0^-}\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]-e^{-qr}\mathbb{E}\left[e^{-qT_0^-}\Lambda(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]}{1-e^{-qr}\mathbb{E}\left[e^{-qT_0^-}\Lambda^{(q)}(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]}. \tag{4.15}$$

Taking $p=0$ and $p=q$ in (4.12) respectively with $x=0$, we have

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-qT_0^-}\Lambda(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\
&=\frac{\lambda}{\lambda+q}\left(\Lambda^{(q)}(x;r,-q)-\Lambda^{(q)}(x;r,\lambda)-\frac{Z_q(x,\Phi(\lambda+q))}{Z_q(b,\Phi(\lambda+q))}\left(\Lambda^{(q)}(b;r,-q)-\Lambda^{(q)}(b;r,\lambda)\right)\right).
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-qT_0^-}\Lambda^{(q)}(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\
&=\Lambda^{(q)}(x;r)-\Lambda^{(q)}(x;r,\lambda)-\frac{Z_q(x,\Phi(\lambda+q))}{Z_q(b,\Phi(\lambda+q))}\left(\Lambda^{(q)}(b;r)-\Lambda^{(q)}(b;r,\lambda)\right).
\end{aligned} \tag{4.17}$$

Thus, using (3.15), we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-qT_0^-}\Lambda(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] \\
&\qquad=\frac{\lambda}{\lambda+q}\left(\left(1-e^{(\lambda+q)r}\right)-\frac{\Lambda^{(q)}(b;r,-q)-\Lambda^{(q)}(b;r,\lambda)}{Z_q(b,\Phi(\lambda+q))}\right),
\end{aligned} \tag{4.18}$$

and

$$\mathbb{E}\left[e^{-qT_0^-}\Lambda^{(q)}(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] = e^{qr}-e^{(\lambda+q)r}-\frac{\Lambda^{(q)}(b;r)-\Lambda^{(q)}(b;r,\lambda)}{Z_q(b,\Phi(\lambda+q))}. \tag{4.19}$$

Plugging (4.18) and (4.19) in (4.15), we get

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-q\tilde{\kappa}_r^\lambda}\mathbf{1}_{\{\tilde{\kappa}_r^\lambda<\tau_b^+\}}\right] &= \frac{\lambda}{\lambda+q}\left(1-\frac{Z_q(b)+\Lambda^{(q)}(b;r)-\Lambda^{(q)}(b;r,-q)}{Z_q(b,\Phi(\lambda+q))\,\mathrm{e}^{(\lambda+q)r}+\Lambda^{(q)}(b;r)-\Lambda^{(q)}(b;r,\lambda)}\right)\\ &= \frac{\lambda}{\lambda+q}\left(1-\frac{S^{(q,\lambda)}(b,r)}{\Theta^{(q)}(b;r,\lambda)}\right). \qquad (4.20)\end{aligned}$$

We finally obtain the result by plugging the last expectation in (4.14) together with (4.16) and (4.17).

To prove (4.3), we use again the strong Markov property and spectral negativity of X. For $x<0$, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tilde{\kappa}_r^\lambda\}}\right]=\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tilde{\kappa}_r^\lambda\}}\right]\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+<r\}}\right]. \qquad (4.21)$$

For $0\le x\le b$ and using (1.42) and (4.21), we get

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tilde{\kappa}_r^\lambda\}}\right] &= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^-\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\mathbb{E}_{X_{T_0^-}}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tilde{\kappa}_r^\lambda\}}\right]\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]\\ &= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^-\}}\right]\\ &\quad+\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tilde{\kappa}_r^\lambda\}}\right]\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\mathbb{E}_{X_{T_0^-}}\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+<r\}}\right]\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]\\ &= \frac{Z_q(x,\Phi(\lambda+q))}{Z_q(b,\Phi(\lambda+q))}\\ &\quad+\mathrm{e}^{-qr}\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tilde{\kappa}_r^\lambda\}}\right]\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\Lambda^{(q)}(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right].\end{aligned}$$

Setting $x=0$, yields

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tilde{\kappa}_r^\lambda\}}\right]=\frac{\frac{1}{Z_q(b,\Phi(\lambda+q))}}{\mathrm{e}^{\lambda r}+\mathrm{e}^{-qr}\frac{\Lambda^{(q)}(b;r)-\Lambda^{(q)}(b;r,\lambda)}{Z_q(b,\Phi(\lambda+q))}}=\frac{\mathrm{e}^{qr}}{\Theta^{(q)}(b;r,\lambda)}.$$

Then, putting all the pieces together, we have

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tilde{\kappa}_r^\lambda\}}\right] &= \frac{Z_q(x,\Phi(\lambda+q))}{Z_q(b,\Phi(\lambda+q))}+\frac{\mathrm{e}^{-qr}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\Lambda^{(q)}(X_{T_0^-},r)\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right]}{\Theta^{(q)}(b;r,\lambda)}\\ &= \frac{\Theta^{(q)}(x;r,\lambda)}{\Theta^{(q)}(b;r,\lambda)}.\end{aligned}$$

To deal with the limit as $b\to\infty$ in (4.2), we apply the same machinery using identity (4.8). We can also compute the limit directly using (3.26), (1.18) and the fact that

$$
\begin{aligned}
\lim_{b\to\infty}\frac{\mathcal{W}_z^{(q,\lambda)}(b+z)}{W^{(q)}(b)} &= \mathrm{e}^{\Phi(q)z}+\lambda\lim_{b\to\infty}\int_0^z \frac{W^{(q)}(b+z-y)}{W^{(q)}(b)}W^{(q+\lambda)}(y)\,\mathrm{d}y \\
&= \mathrm{e}^{\Phi(q)z}+\lambda\int_0^z \mathrm{e}^{\Phi(q)(z-y)}W^{(q+\lambda)}(y)\,\mathrm{d}y \\
&= \mathrm{e}^{\Phi(q)z}\left(1+\lambda\int_0^z \mathrm{e}^{-\Phi(q)y}W^{(q+\lambda)}(y)\,\mathrm{d}y\right)=Z_{q+\lambda}(z,\Phi(q)).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\lim_{b\to\infty}\frac{\mathcal{S}^{(q)}(b,r)}{\Theta^{(q)}(b;r,\lambda)} \\
=\ &\lim_{b\to\infty}\frac{\mathcal{S}^{(q)}(b,r)/W^{(q)}(b)}{\Theta^{(q)}(b;r,\lambda)/W^{(q)}(b)} \\
=\ &\frac{q/\Phi(q)-\int_0^\infty\left(Z(z,\Phi(q))-\mathrm{e}^{\Phi(q)z}\right)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)}{\mathrm{e}^{(\lambda+q)r}\lambda/(\Phi(\lambda+q)-\Phi(q))-\int_0^\infty\left(Z_{q+\lambda}(z,\Phi(q))-\mathrm{e}^{\Phi(q)z}\right)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)}.
\end{aligned}
$$

CHAPTER V

ON A VAR-TYPE RISK MEASURE BASED ON CUMULATIVE PARISIAN RUIN

5.1 Introduction

Over the last few years, several *dynamic* risk measures, i.e. risk measures based on ruin-theoretic quantities, have been studied. For example, in the classical compound Poisson risk model, Trufin et al. [73] considered a VaR-type risk measure defined as the smallest initial capital needed to ensure a certain probability of solvency throughout the lifetime of the surplus process. This risk measure has been extended by Mitric and Trufin [64] who defined a risk measure taking into account both the probability of ruin and the expected deficit at ruin. Also, Loisel and Trufin [62] used the expected area below the solvency threshold as a risk indicator to introduce a new risk measure with some interesting properties.

Very recently, implementation delays in the recognition of ruin and occupation times of the surplus process have been used as alternative risk management tools to assess the quality of an insurance portfolio. In this direction, Guérin and Renaud [33] have introduced the concept of cumulative Parisian ruin, which is based on the *time spent in the red* by the underlying surplus process. The time of cumulative Parisian ruin is the first time the surplus process stays cumulatively below a critical level longer than a pre-determined grace period. Inspired by the risk measure of Trufin et al. [73], they have defined a VaR-type risk measure based on cumulative Parisian ruin. It is also defined as the smallest amount of capital for which the associated cumulative Parisian ruin probability is less than or equal to a tolerable level.

The rest of the Chapter is organized as follows. In Section 5.2, we recall some background on the Cramér-Lundberg model and we define the concept of cumulative Parisian ruin. In Section 5.3, we introduce our risk measure and we give some of its properties.

5.2 Insurance risk model

In the Cramér-Lundberg model, the surplus process of an insurance company is modelled by

$$X_t = x + ct - S_t, \tag{5.1}$$

where $x \geq 0$ and $c > 0$, and where $S_t = \sum_{i=1}^{N_t} C_i$ is a compound Poisson process with $N = \{N_t, t \geq 0\}$ a Poisson process of intensity $\lambda > 0$ and with $\{C_1, C_2, \dots\}$ positive random variables following a common cumulative distribution function F_C. Recall that in this setup the claim sizes $\{C_1, C_2, \dots\}$ are mutually independent and are also independent of the number-of-claim process N. The process $S = \{S_t, t \geq 0\}$ is known as the aggregate claim amount process. We will call x the initial capital and c the premium rate.

We will use the following equivalent notations $\mathbb{P}_x(\cdot) \equiv \mathbb{P}(\cdot | X_0 = x)$ to emphasize that the process X starts at level x. The notation $\mathbb{E}_x$ corresponds to $\mathbb{P}_x$. When $X_0 = 0$, we drop the index. In this model, the premium rate c is chosen usually to satisfy the net profit condition $\mathbb{E}[X_1] = c - \lambda\mathbb{E}[C_1] > 0$, which means that we can define the safety loading factor $\eta > 0$ by $\eta := (c - \lambda\mathbb{E}[C_1]) / \lambda\mathbb{E}[C_1]$.

The time of classical ruin associated to X is defined as

$$\tau_0^- = \inf\{t > 0 | X_t < 0\},$$

and we denote the corresponding finite-time probability of ruin, for $x \geq 0$ and $t > 0$, by

$$\psi(t,x) = \mathbb{P}_x\left(\tau_0^- \leq t\right), \tag{5.2}$$

and the infinite-time probability of ruin by

$$\psi(x) = \mathbb{P}_x\left(\tau_0^- < \infty\right). \tag{5.3}$$

Of course, we have $\psi(x) = \lim\limits_{t\to\infty} \psi(t,x)$.

In [73], assuming that the safety loading η is fixed, the following ruin-consistent VaR risk measure is defined and analyzed: for $\epsilon > 0$,

$$\zeta_\epsilon[C] = \inf\{x \geq 0 \colon \psi(x) \leq \epsilon\}.$$

It is well known that we can compute $\psi(x)$ using the Pollaczeck–Khinchine formula (also known in the actuarial literature as the Beekman's convolution formula, see Beekman [9]) which states that the probability of classical ruin is equal to the tail distribution function of a compound geometric random variable. First, let us define the aggregate loss at time t by $L_t = S_t - ct$ and the maximal aggregate loss of the process by $L = \max\limits_{t\geq 0}\{L_t\}$ which can be expressed as a sum

$$L = \sum_{i=1}^{M} D_i, \tag{5.4}$$

where M is the number of record highs, which has a geometric distribution with success probability $\eta/(\eta+1)$, and where $\{D_1, D_2, \dots\}$ are the ladder heights with common distribution $F_D(u) = \int_0^u (1-F_C(y))\mathrm{d}y/\mathbb{E}[C_1]$. The Pollaczeck–Khinchine formula for the probability of ruin is then given by

$$\psi(x) = \mathbb{P}(L > x) = 1 - \frac{\eta}{\eta+1}\sum_{k=1}^{\infty}\left(\frac{1}{\eta+1}\right)^k F_D^{*(k)}(u), \tag{5.5}$$

where $F_D^{*(k)}$ denotes the k-th convolution of the distribution F_D. Thus, the risk measure is such that

$$\zeta_\epsilon[C] = \inf\{x \geq 0 \colon \mathbb{P}(L > x) \leq \epsilon\} = F_L^{-1}(1-\epsilon). \tag{5.6}$$

In some sense, the focus of this risk measure is shifted from the surplus process X to the distribution of the maximal aggregate loss L. This important relationship is at the core of the analysis done in [73]. However, this relationship with the maximal aggregate loss L does not exist for the finite-time ruin probability. This is also the case for the risk measure in [64].

5.2.1 Cumulative Parisian ruin

Very recently, Guérin and Renaud [33] introduced a new definition of actuarial ruin based on the occupation-time process (below 0) associated with the surplus process X. The occupation-time process $\mathcal{O}^L = \left\{\mathcal{O}_t^L, t \geq 0\right\}$ is defined as

$$\mathcal{O}_t^L = \int_0^t \mathbf{1}_{\{X_u<0\}} \mathrm{d}u = \int_0^t \mathbf{1}_{\{L_u>X_0\}} \mathrm{d}u.$$

Then, the time of cumulative Parisian ruin, with delay $r > 0$, is given by

$$\sigma_r = \inf\left\{t > 0 \colon \mathcal{O}_t^L > r\right\}.$$

In the definition of cumulative Parisian ruin, we aggregate the duration of all periods of financial distress until we accumulate r units of time spent in that red zone. Consequently, ruin is not declared as soon as X goes below zero. In fact, for $x \geq 0$, $t > 0$ and $r > 0$, we have

$$\mathbb{P}_x(\sigma_r \leq t) \leq \mathbb{P}_x\left(\tau_0^- \leq t\right). \tag{5.7}$$

Cumulative Parisian ruin is clearly a generalization of classical ruin and, when $r \to 0$, we recover the classical definition (see [33] for the details and see Figure 5.1 for a graphical comparison).

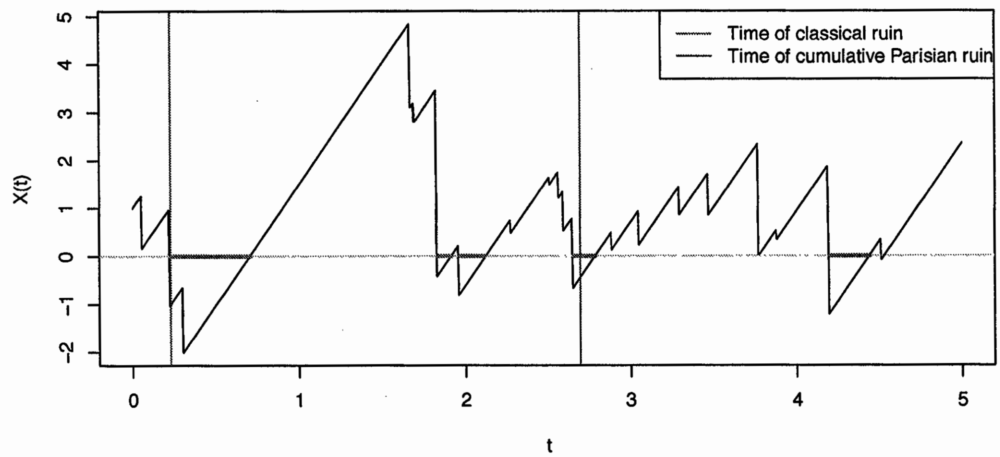

Figure 5.1 A sample path of a Cramér-Lundberg process X_t. The time of ruin τ_0^- is in red and cumulative Parisian ruin time κ_r is shown in blue.

We denote the finite-time probability of cumulative Parisian ruin by

$$\psi_r(t,x) = \mathbb{P}_x(\sigma_r \leq t) = \mathbb{P}_x\left(\mathcal{O}_t^L > r\right). \tag{5.8}$$

and the infinite-time version by

$$\psi_r(x) = \mathbb{P}_x(\sigma_r < \infty).$$

Of course, we have $\psi_r(x) = \lim_{t\to\infty}\psi_r(t,x)$. With this new notation in hand, we can re-write the inequality in (5.7) as follows: for $x \geq 0$, $t > 0$ and $r > 0$, we have

$$\psi_r(t,x) \leq \psi(t,x). \tag{5.9}$$

We also have

$$\psi_r(t,x) = \lim_{r\to 0}\psi(t,x) \quad \text{and} \quad \psi_r(x) = \lim_{r\to 0}\psi(x).$$

Before going any further, let us give some background material on stochastic dominance.

5.2.2 Stochastic dominance

Consider two random variables X and Y, and let $\bar{F}_X$ and $\bar{F}_Y$ be their survival functions. We say that X is smaller than Y in the stochastic dominance order, which is denoted by $X \preceq_{st} Y$, if

$$\bar{F}_X(u) \leq \bar{F}_Y(u), \text{ for all } u. \tag{5.10}$$

Theorem 41 (Shaked and Shanthikumar [70]). *(i) Let $\{X_1, X_2, \ldots, X_m\}$ and $\{Y_1, Y_2, \ldots, Y_m\}$ be two sets of independent random variables such that $X_i \preceq_{st} Y_i$, for each $i = 1, \ldots, m$. Then, for any increasing function $g\colon \mathbb{R}^m \to \mathbb{R}$, we have*

$$g(X_1, X_2, \ldots, X_m) \preceq_{st} g(Y_1, Y_2, \ldots, Y_m). \tag{5.11}$$

(ii) Consider two sequences of random variables $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$ and two random variables X and Y such that

$$X_n \xrightarrow{d} X \quad \text{and} \quad Y_n \xrightarrow{d} Y,$$

where $\xrightarrow{d}$ denotes convergence in distribution. If $X_n \preceq_{st} Y_n$ for each n, then $X \preceq_{st} Y$.

(iii) Let the positive integer-valued random variable N be independent of the family of random variables $\{C_1, C_2, \ldots\}$ and define $S = \sum_{i=1}^{N} C_i$. Define similarly $\tilde{S} = \sum_{i=1}^{\tilde{N}} \tilde{C}_i$.

If $N \preceq_{st} \tilde{N}$ and if $C_i \preceq_{st} \tilde{C}_i$ for each i, then

$$S \preceq_{st} \tilde{S}. \tag{5.12}$$

If $X = \{X_t, t \geq 0\}$ and $Y = \{Y_t, t \geq 0\}$ are stochastic processes, then we write

$X \preceq_{st} Y$ if, for each $t \geq 0$, we have

$$X_t \preceq_{st} Y_t.$$

The reader is referred to Shaked and Shanthikumar [70], Kaas et al. [35] and Dhaene et al. [34] for more details on stochastic ordering and applications in actuarial science.

5.3 A VaR-type risk measure derived from cumulative Parisian ruin

Using the definition of cumulative Parisian ruin, Guérin and Renaud [33] have defined the following VaR-type risk measure: for a given tolerance level $\epsilon > 0$, set

$$\rho_\epsilon^{(r,t)}[L] = \inf\left\{x \geq 0 \colon \mathbb{P}_x\left(\mathcal{O}_t^L > r\right) \leq \epsilon\right\}.$$

It gives the amount of initial capital needed in order to bound the probability of cumulative Parisian ruin with delay r by ϵ. Consequently, this risk measure is based on the distribution of $\mathcal{O}_t^L$. This is the analog of the random variable L for the risk measure in (5.6). A major improvement is that we can now vary the time horizon and the implementation delay with the variables t and r, respectively. The trade-off is that we need the distribution of a strongly path-dependent random variable, namely $\mathcal{O}_t^L$.

For the rest of this Chapter, we aim at studying the properties of this VaR-type cumulative Parisian risk measure and use it in the context of an optimal allocation problem. In [33], this risk measure is proposed as a motivational reason to study the concept of cumulative Parisian ruin; the risk measure itself is not analyzed nor used for any particular application. Also, we will compare the infinite-time version to the infinite-time risk measure defined in [73]. Then, we will also study the finite-time version as this is possible as soon as the distribution of $\mathcal{O}_t^L$ is available.

5.3.1 Properties of the risk measure $\rho_\epsilon^{(r,t)}$

Recall that our main object of study is the following VaR-type risk measure: for $r > 0$, $\epsilon > 0$ and $t > 0$,

$$\rho_\epsilon^{(r,t)}[L] = \inf\{x \geq 0 \colon \psi_r(t,x) \leq \epsilon\} = \inf\left\{x \geq 0 \colon \mathbb{P}_x\left(\mathcal{O}_t^L > r\right) \leq \epsilon\right\}. \quad (5.13)$$

When $t = \infty$, we write $\rho_\epsilon^{(r)}$.

We are also interested in the risk measure based on the finite-time probability of classical ruin:

$$\zeta_\epsilon^{(t)}[L] = \inf\{x \geq 0 \colon \psi(t,x) \leq \epsilon\}. \quad (5.14)$$

Using inequality (5.9) and the discussions in the previous section, we deduce the following first proposition :

Proposition 42. *For a given time horizon $0 < t \leq \infty$ and an acceptance level $\epsilon > 0$, the risk measure $\rho_\epsilon^{(r,t)}$ is less conservative than the risk measure $\zeta_\epsilon^{(t)}$, i.e.*

$$\rho_\epsilon^{(r,t)}[L] \leq \zeta_\epsilon^{(t)}[L], \quad (5.15)$$

and, when $r \to 0$, it converges to $\zeta_\epsilon^{(t)}$, i.e.

$$\rho_\epsilon^{(r,t)}[L] \uparrow \zeta_\epsilon^{(t)}[L], \text{ as } r \to 0. \quad (5.16)$$

In what follows, let L and $\tilde{L}$ be two aggregate loss amount processes associated with two aggregate loss amount S and $\tilde{S}$ of two Cramér-Lundberg processes X and $\tilde{X}$ as defined in (5.1).

Theorem 43. *For $r > 0$, $\varepsilon > 0$ and $t > 0$, we have*

(i) ***Invariance by translation:*** *For all $a > 0$,*

$$\rho_\epsilon^{(r,t)}[L + a] = \rho_\epsilon^{(r,t)}[L] + a. \quad (5.17)$$

(ii) ***Positive homogeneity:*** *For all $b > 0$,*

$$\rho_\epsilon^{(r,t)}\left[bL\right] = b\rho_\epsilon^{(r,t)}\left[L\right]. \tag{5.18}$$

(iii) ***Monotonicity:*** *If $L \preccurlyeq_{st} \tilde{L}$, then*

$$\rho_\epsilon^{(r,t)}\left[L\right] \leq \rho_\epsilon^{(r,t)}[\tilde{L}]. \tag{5.19}$$

Proof. First, note that

$$\mathbb{P}_x\left(\mathcal{O}_t^{L+a} > r\right) = \mathbb{P}_x\left(\int_0^t \mathbf{1}_{\{L_u > x-a\}} \mathrm{d}u > r\right) = \mathbb{P}_{x-a}\left(\mathcal{O}_t^L > r\right).$$

Consequently,

$$\begin{aligned}\rho_\epsilon^{(r,t)}[L+a] &= \inf\left\{x \geq 0 \colon \mathbb{P}_x\left(\mathcal{O}_t^{L+a} > r\right) \geq \epsilon\right\} \\ &= \inf\left\{x \geq 0 \colon \mathbb{P}_{x-a}\left(\mathcal{O}_t^{L} > r\right) \geq \epsilon\right\} \\ &= \rho_\epsilon^{(r,t)}[L] + a.\end{aligned}$$

This proves (5.17).

Similarly, if we note that

$$\mathbb{P}_x\left(\mathcal{O}_t^{bL} > r\right) = \mathbb{P}_x\left(\int_0^t \mathbf{1}_{\{L_u > x/b\}} \mathrm{d}u > r\right) = \mathbb{P}_{x/b}\left(\mathcal{O}_t^L > r\right),$$

then (5.18) follows.

In order to prove this property, we fix $t > 0$ and we will show that, if $L_u \preccurlyeq_{st} \tilde{L}_u$ for all $u \leq t$, then

$$\mathcal{O}_t^L \preccurlyeq_{st} \mathcal{O}_t^{\tilde{L}}.$$

First, let us define a sequence of discretized versions of the occupation-time process $\mathcal{O}_t^L$. For each $n \geq 1$, choose $0 = t_0 < t_1 < \ldots < t_n = t$ such that $\max_{0 \leq i \leq n}(t_i - t_{i-1}) \to 0$, as $n \to \infty$, and define

$$\mathcal{O}_t^{(n)} = \sum_{i=1}^n (t_i - t_{i-1})\, \mathbf{1}_{\{L_{t_i} > x\}}.$$

We define $\widetilde{\mathcal{O}}_t^{(n)}$ in the obvious way, i.e. when S is replaced by $\tilde{S}$. We can re-write $\mathcal{O}_t^{(n)}$ as follows:

$$\mathcal{O}_t^{(n)} = \phi_n \left(L_{t_1} - L_{t_0}, L_{t_2} - L_{t_1}, \ldots, L_{t_n} - L_{t_{n-1}} \right),$$

where $\phi_n (u_1, \ldots, u_n) = \sum_{i=1}^{n} (t_i - t_{i-1}) \mathbf{1}_{\left\{ \sum_{j=1}^{i} u_j > x + ct_i \right\}}$.

Since $L_u \preccurlyeq_{st} \tilde{L}_u$ for all $u \leq t$, then we have $L_{t_i - t_{i-1}} \preceq_{st} \tilde{L}_{t_i - t_{i-1}}$ for each i. Then, since

$$L_{t_i - t_{i-1}} \stackrel{d}{=} L_{t_i} - L_{t_{i-1}} \quad \text{and} \quad \tilde{L}_{t_i - t_{i-1}} \stackrel{d}{=} \tilde{L}_{t_i} - \tilde{L}_{t_{i-1}},$$

we have that $L_{t_i} - L_{t_{i-1}} \preceq_{st} \tilde{L}_{t_i} - \tilde{L}_{t_{i-1}}$ for each i. From (5.11), we obtain

$$\phi_n \left(L_{t_1} - L_{t_0}, L_{t_2} - L_{t_1}, \ldots, L_{t_n} - L_{t_{n-1}} \right) \preceq_{st} \phi_n \left(\tilde{L}_{t_1} - \tilde{L}_{t_0}, \tilde{L}_{t_2} - \tilde{L}_{t_1}, \ldots, \tilde{L}_{t_n} - \tilde{L}_{t_{n-1}} \right),$$

or equivalently

$$\mathcal{O}_t^{(n)} \preceq_{st} \widetilde{\mathcal{O}}_t^{(n)}.$$

Since $\mathcal{O}_t^{(n)} \xrightarrow{d} \mathcal{O}_t^{L}$ and $\widetilde{\mathcal{O}}_t^{(n)} \xrightarrow{d} \mathcal{O}_t^{\tilde{L}}$, by the second part of Theorem 41, we get

$$\mathcal{O}_t^{L} \preceq_{st} \mathcal{O}_t^{\tilde{L}}.$$

This means that

$$\mathbb{P}_x \left(\mathcal{O}_t^{L} > r \right) \leq \mathbb{P}_x \left(\mathcal{O}_t^{\tilde{L}} > r \right), \text{ for all } r.$$

The monotonicity property in (5.19) follows.

■

The monotonicity property in (5.19) says that the risk measure $\rho_\epsilon^{(r,t)} [L]$ is increasing (in its second variable) with respect to the stochastic dominance order. Note that if $\mathbb{P} \left(L_t \leq \tilde{L}_t \right) = 1$ for all $t \geq 0$, then we can also prove that

$$\rho_\epsilon^{(r,t)} [L] \leq \rho_\epsilon^{(r,t)} \left[\tilde{L} \right].$$

If we put together the monotonicity property in (5.19) and (5.12), then we can deduce the following intuitive relationship: a smaller frequency and a smaller severity yield less occupation time in the red zone and thus a smaller probability of cumulative Parisian ruin. For example, by the third part of Theorem 41, if C and $\tilde{C}$ are exponentially distributed random variables with parameters α and $\tilde{\alpha}$ respectively, and if $\alpha \geq \tilde{\alpha}$ and $\lambda \leq \tilde{\lambda}$, then, for a given common premium rate c, the initial capital needed at a given tolerance level ϵ is larger for X than $\tilde{X}$.

It is worthwhile to mention that, as an immediate consequence of Proposition 42, Theorem 43 is also satisfied for the infinite-time horizon risk measure ζ_ϵ. Thus, we have recovered some of the results in Properties 3.1 and 3.2 in [73]. Also, an important consequence of Proposition 42 is the stochastic order for the finite-time ruin probability $\psi(t, x)$.

At first sight, our risk measure appears to be only related to the risk measures $\zeta_\epsilon^{(t)}$ and ζ_ϵ. However, if we consider a finite-time version of the infinite-horizon risk measure defined by Loisel and Trufin [62], then we can define

$$\omega_a^{(t)}[L] := \inf\left\{x \geq 0 \colon \mathbb{E}_x\left[\mathcal{A}_t^L\right] \leq a\right\},$$

where $a > 0$ is a tolerance level for the expected *area in the red* defined as

$$\mathcal{A}_t^L = \int_0^t (L_u - x)_+ \,\mathrm{d}u,$$

where $(x)_+ = \max(x, 0)$. Furthermore, we can use Theorem 1 of Loisel [61] and then write

$$\mathbb{E}_x\left[\mathcal{A}_t^L\right] = \int_x^\infty \mathbb{E}_v\left[\mathcal{O}_t^L\right] \mathrm{d}v = \int_x^\infty \int_0^\infty \mathbb{P}_v\left(\mathcal{O}_t^L \geq u\right) \mathrm{d}u\mathrm{d}v. \tag{5.20}$$

Consequently, if we suppose that $L \preccurlyeq_{st} \tilde{L}$, then $\mathcal{O}_t^L \preccurlyeq_{st} \mathcal{O}_t^{\tilde{L}}$ and then, from (5.10) and (5.20), we have

$$\mathbb{E}_x\left[\mathcal{A}_t^L\right] \leq \mathbb{E}_x\left[\mathcal{A}_t^{\tilde{L}}\right].$$

Thus,

$$\omega_a^{(t)}\left[L\right] \leq \omega_a^{(t)}\left[\tilde{L}\right], \tag{5.21}$$

which corresponds Property 3.1 in [62].

Remark 44. Note also that with the distribution in Theorem 45 below, it is possible to compute the finite-time version of this risk measure based on the area in the red in the case of a Cramér-Lundberg process with exponential claims.

5.3.2 Sensitivity analysis in the case of exponential claims

In this section, we want to see how $\rho_\epsilon^{(r,t)}$ reacts to changes in the value of its parameters. In other words, we want to perform a sensitivity analysis.

In general, we could use Monte Carlo simulations to computes values for $\rho_\epsilon^{(r,t)}$. However, if we consider a Cramér-Lundberg process with exponentially distributed claims $\{C_1, C_2, \dots\}$ with rate parameter $\alpha > 0$, then there exists an explicit expression for the distribution of the occupation time for a finite-time horizon. Unfortunately, such formulas are not available for most claim distributions.

Theorem 45 (Guérin and Renaud [33]). *For $t > 0$, we have*

$$\mathbb{P}_x\left(\mathcal{O}_t^L \in \mathrm{d}s\right) = a_t^x \delta_0\left(\mathrm{d}s\right) + \left(a_{t-s}^x + k_{t-s}^x\right)\left(\lambda - c\alpha\left(1 - a_s^0\right)\right)\mathbf{1}_{(0,t)}\left(s\right)\mathrm{d}s,$$

with

$$a_t^x = 1 - \lambda \mathrm{e}^{-\alpha x}\int_0^t \mathrm{e}^{-(\lambda + c\alpha)s}\left[I_0\left(2\sqrt{\lambda c\alpha s\left(s + x/c\right)}\right) - \frac{s}{s + x/c} I_2\left(2\sqrt{\lambda c \alpha s\left(s + x/c\right)}\right)\right]\mathrm{d}s$$

and

$$k_t^x = \mathrm{e}^{-\alpha x} - 1 + \lambda x \alpha \mathrm{e}^{-\alpha x}\int_0^t \mathrm{e}^{-(\lambda + c\alpha)s}\left[I_0\left(2\sqrt{\lambda c\alpha s\left(s + x/c\right)}\right) - I_2\left(2\sqrt{\lambda c\alpha s\left(s + x/c\right)}\right)\right]\mathrm{d}s,$$

where I_ν represents the modified Bessel function of the first kind of order ν.

In Theorem 45, a_t^x is the survival ruin probability over $[0,t]$, that is

$$\begin{aligned} a_t^x &= 1-\psi(t,x) \\ &= 1-\lambda \mathrm{e}^{-\alpha x}\int_0^t \mathrm{e}^{-(\lambda+c\alpha)s}\left[I_0\left(2\sqrt{\lambda c\alpha s\left(s+x/c\right)}\right)-\frac{s}{s+x/c}I_2\left(2\sqrt{\lambda c\alpha s\left(s+x/c\right)}\right)\right]\mathrm{d}s. \end{aligned}$$

For an infinite-time horizon, we have the well-known expression:

$$a^x = \lim_{t\to\infty} a_t^x = 1-\psi(x) = \frac{\lambda}{c\alpha}\mathrm{e}^{x(\lambda/c-\alpha)} = \frac{1}{1+\eta}\mathrm{e}^{-x\alpha\eta/(1+\eta)}.$$

From Corollary 2 in [68], we can deduce the following expression for the distribution of $\mathcal{O}_\infty^L$, when the claims are exponentially distributed.

Corollary 46. *For any $x\in\mathbb{R}$, we have*

$$\begin{aligned} &\mathbb{P}_x\left(\mathcal{O}_\infty^L \in \mathrm{d}s\right) = a^x\delta_0\left(\mathrm{d}s\right) \\ &+\frac{\lambda}{c}\left(1-\frac{\lambda}{c\alpha}\right)\mathrm{e}^{-cs\alpha}\mathrm{e}^{-x(\alpha-\lambda/c)}\left(c+\sum_{i=0}^{\infty}\frac{\left(\lambda s\right)^{i+1}}{i!\left(1+i\right)!}\left(c\Gamma\left(i+1,s\lambda\right)-\frac{c}{s\lambda}\Gamma\left(i+2,s\lambda\right)\right)\right), \end{aligned}$$

where $\Gamma(a,x)=\int_0^x \mathrm{e}^{-t}t^{a-1}\mathrm{d}t$ is the incomplete gamma function.

The explicit formula in Theorem 45 allows for a sensitivity analysis of the value of the probability of cumulative Parisian ruin when claims are exponentially distributed with respect to the delay parameter r and the time horizon t. In Figure 5.2, we observe that for a fixed delay parameter r, the probability of cumulative Parisian ruin increases when the time horizon t increases. This is because we accumulate more occupation time. On the other hand, it decreases when the delay r increases. For a fixed value of the time horizon t, increasing the initial capital x decreases the probability of cumulative Parisian ruin, as expected.

For the corresponding risk measures, Figure 5.3 illustrates the relationships in (5.15) and in (5.16) between $\rho_\epsilon^{(r,t)}$ and $\zeta_\epsilon^{(t)}$. As $r\to 0$, i.e. as the grace period gets smaller, the initial capital needed with $\rho_\epsilon^{(r,t)}$ increases toward that needed with $\zeta_\epsilon^{(t)}$, both at a tolerance level of $\epsilon=0.3$. When the time horizon t increases, both risk measures increase the initial capital needed for that tolerance level.

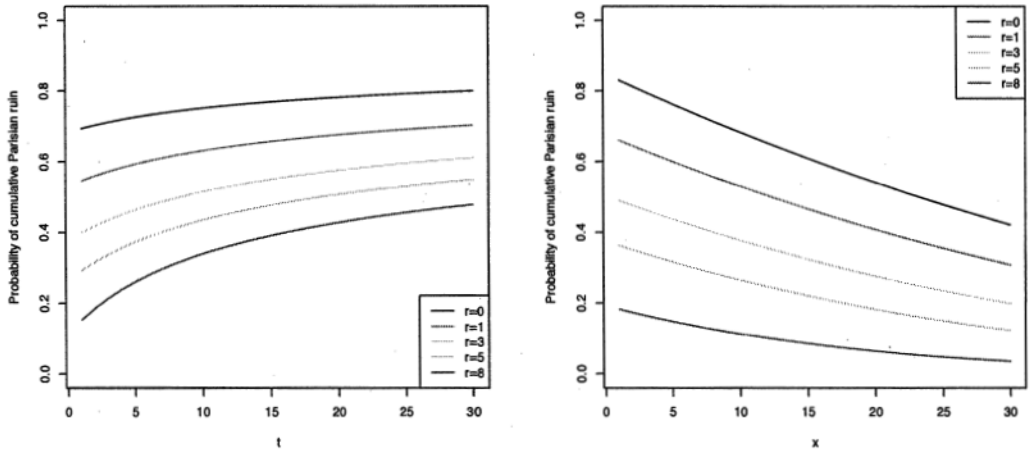

Figure 5.2 The probability of cumulative Parisian ruin for the Cramér-Lundberg process with $\alpha = 1/8$, $\lambda = 2$, $c = 17$, $r = 1$, $x = 10$ and $t = 10$.

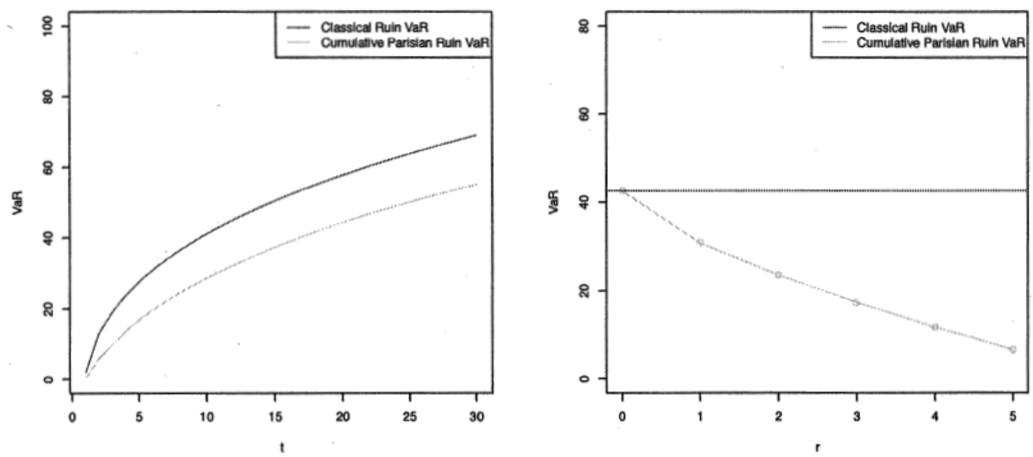

Figure 5.3 Risk measures $\rho_\epsilon^{(r,t)}$ and $\zeta_\epsilon^{(t)}$ for $\alpha = 1/8$, $\lambda = 2$, $c = 17$, $t = 10$, $r = 2$, $\epsilon = 0.3$

CHAPTER VI

CONCLUDING REMARKS AND FUTURE RESEARCH

The main objective of this thesis was to study some Parisian ruin problems for Lévy insurance risk processes. In Chapter 2, we first extended the work of Loeffen et al. [60] to the refracted Lévy process and we computed other fluctuation identities such as the Laplace transform of the time of Parisian ruin and the two-sided exit problem. In Chapter 3, we considered a type of ruin called *mixed Parisian ruin* that unified the two types of Parisian ruin. The expressions are expressed in terms of the delayed scale functions. In Chapter 4, we studied the concept of Parisian ruin under a hybrid observation scheme recently introduced by Li et al. [49]. We improved the expression of the probability of ruin originally obtained and we computed other fluctuation identities. In Chapter 5, we studied a VaR-type risk measure based on cumulative Parisian ruin. We derived some properties of this risk measure and we compared it to the risk measures of Trufin et al. [73] and of Loisel and Trufin [62].

In the same vein as that illustrated in Kyprianou and Loeffen [43], a potential research direction is to study the case where the refraction happens under Poissonian observation approach of Albrecher et al. [5]. Indeed, let T_i be the arrival times of an independent Poisson process of rate $\lambda > 0$. As discussed above, we define a surplus process U whose dynamics change whenever it is observed at T_i

below or above a critical level b. This means, when $U_{T_i} < b$, the process U behaves as a Lévy insurance risk process X^1 until the moment that $U_{T_i} > b$ then it behaves as another Lévy insurance risk process X^2. Also, the refracted process at Poisson arrival times converges, when the Poisson observation rate goes to infinity, to the classical refracted Lévy process defined in [43].

Also, analysing Parisian ruin quantities for the drawdown process is an interesting research direction. Indeed, we could adapt the hybrid observation scheme defined in [49] for the Parisian model introduced by Surya [72]. Thus, unlike the hybrid scheme defined in [49], when the drawdown process is below a certain level, it is observed at discrete Poisson arrival times. Once the process goes above this level, it is observed continuously and Parisian ruin occurs when the process stays above this level for consecutive period of time greater than a pre-specified delay. This is also equivalent to the fact that the risk process has gone below its last running maximum during the grace period.

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