UNIVERSITÉ DU QUÉBEC À MONTRÉAL

PROPRIÉTÉS ARITHMÉTIQUES DU E-POLYNÔME DU SCHÉMA DE HILBERT DE n POINTS DANS LE TORE BIDIMENSIONNEL

MÉMOIRE

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PAR

JOSÉ MANUEL RODRÍGUEZ CABALLERO

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AVERTISSEMENT

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RÉSUMÉ

Le E-polynôme du schéma de Hilbert à n points dans le tore bidimensionnel, noté $C_n(q)$, est donné par la fonction génératrice

$$\prod_{m=1}^{\infty} \frac{(1-t^m)^2}{(1-qt^m)(1-q^{-1}t^m)} = 1 + \sum_{n=1}^{\infty} \frac{C_n(q)}{q} t^n.$$

Le polynôme $C_n(q)$ est divisible par $(q-1)^2$ en $\mathbb{Z}[q]$. On note $P_n(q)$ le polynôme satisfaisant $C_n(q) = (q-1)^2 P_n(q)$. Le but de ce mémoire est de démontrer de nouvelles propriétés arithmétiques satisfaites par $C_n(q)$ et $P_n(q)$.

Mots clefs : E-polynôme, diviseur, triangle de Pythagore, partage en parties consécutives, mot de Dyck.

INTRODUCTION

La notion de *schéma de Hilbert* s'est présentée à Grothendieck (Grothendieck, 1960) comme un moyen de remplacer en théorie des schémas les variétés de Chow. Cette notion très féconde joue un rôle fondamental en combinatoire algébrique, notamment le schéma de Hilbert à n points dans le plan affine apparaît dans les démonstrations de conjectures de Garsia et Haiman (Haiman, 2001), en particulier les deux conjectures nommées $n!$ et $(n+1)^{n-1}$.

Strømme et Ellingsrud (Ellingsrud et Strømme, 1987) ont étudié les homologies des schémas de Hilbert à n points dans les plans affine et projectif à l'aide de la méthode de Birula-Bialynicki (Bialynicki-Birula, 1973). En introduisant les *cellules de Gröbner*, Conca et Valla ont trouvé en (Conca et Valla, 2007) une paramétrisation explicite de la décomposition de Strømme-Ellingsrud qui est particulièrement convenable pour le calcul. Kassel et Reutenauer (Kassel et Reutenauer, 2018b) ont raffiné la décomposition de Strømme-Ellingsrud, afin d'isoler les cellules de Gröbner qui sont *inversibles*, ce qui permet de trouver la décomposition cellulaire du schéma de Hilbert $\left(\mathbb{F}_q^\times \times \mathbb{F}_q^\times\right)^{[n]}$ de n points dans le tore $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$. En dénombrant certaines matrices inversibles associées à la décomposition cellulaire de $\left(\mathbb{F}_q^\times \times \mathbb{F}_q^\times\right)^{[n]}$, Kassel et Reutenauer (Kassel et Reutenauer, 2018a) ont trouvé une formule explicite pour le polynôme $C_n(q)$ qui compte le nombre de points de $\left(\mathbb{F}_q^\times \times \mathbb{F}_q^\times\right)^{[n]}$, ainsi que la fonction génératrice

$$\prod_{m=1}^{\infty} \frac{(1-t^m)^2}{(1-qt^m)\,(1-q^{-1}t^m)} = 1 + \sum_{n=1}^{\infty} \frac{C_n(q)}{q^n} t^n.$$

Göttsche et Soergel (Göttsche et Wolfgang, 1993, Théorème 2) ont trouvé un

isomorphisme canonique entre des structures de Hodge mixtes qui permet de calculer la cohomologie singulière d'un schéma de Hilbert $X^{[n]}$ à n points dans une variété donnée X en fonction de la cohomologie singulière de X. En utilisant cet isomorphisme, Hausel, Letellier et Rodriguez-Villegas (Tamás *et al.*,) ont trouvé la fonction génératrice des polynômes de Hodge du schéma de Hilbert $(\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]}$ à n points dans le tore $\mathbb{C}^\times \times \mathbb{C}^\times$. Il suit d'un résultat de N. M. Katz (Katz, 2008) que le E-polynôme de $(\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]}$ coïncide avec $C_n(q)$.

Le point de départ de la présente recherche est la constatation du fait que les coefficients de $C_n(q)$ sont liés aux expressions bien parenthésées (mots de Dyck). Un tel motif dans les coefficients d'un E-polynôme n'a pas été rapporté dans la littérature de géométrie algébrique. En plus, nous allons montrer des liens entre les expressions bien parenthésées associées à $C_n(q)$ et certains fonctions arithmétiques utilisées dans de divers contextes par Erdös, Hirschhorn, Tao et d'autres mathématiciens.

Le présent mémoire est composé de 4 chapitres. Au chapitre I, on montre que le polynôme $P_n(q) := \frac{C_n(q)}{(q-1)^2}$ projeté sur l'anneau fini $\mathbb{F}_3[X]/X^2\mathbb{F}_3[X]$, où $X = q-1$, est une fonction multiplicative. Au chapitre II, on montre un lien entre $P_n(q)$ et les partages de n en parties consécutives. Au chapitre III, on montre un lien entre $P_n(q)$ et les hypoténuses des triangles de Pythagore primitifs. Au chapitre IV, on montre un lien entre $P_n(q)$ et une fonction introduite par P. Erdös et J-L. Nicolas, qui est liée à la distribution des diviseurs d'un entier et aux semi-périmètres des triangles de Pythagore.

Ce mémoire repose sur plusieurs manuscripts qui ont été rédigés en Anglais pendant mes études de maîtrise et qui sont inclus dans le document. Le chapitre I consiste à l'insertion de l'article publié (Caballero, 2018).

CHAPITRE I

ON KASSEL-REUTENAUER q-ANALOG OF THE SUM OF DIVISORS AND THE RING $\mathbb{F}_3[X]/X^2\mathbb{F}_3[X]$

Abstract

A q-analog $P_n(q)$ of the sum of divisors of n was introduced by C. Kassel and C. Reutenauer in a combinatorial setting and by T. Hausel, E. Letellier, F. Rodriguez-Villegas in a Hodge-theoretic setting. We study the reduction modulo 3 of the polynomial $P_n(q)$ with respect to the ideal $(q^2+q+1)\mathbb{F}_3[q]$.

1.1 Introduction

Consider the infinite product

$$\theta(w) := (1-w)\prod_{n\geq 1}\frac{(1-q^n w)\,(1-q^n w^{-1})}{(1-q^n)^2}.$$

The identity

$$\frac{\theta(uv)}{\theta(u)\theta(v)} = \sum_{m,n\geq 0} q^{mn}u^m v^n - \sum_{m,n\geq 1} q^{mn}u^{-m}v^{-n}, \tag{1.1}$$

is attributed to L. Kronecker [1] (10). The particular case of (1.1),

$$\frac{1}{\theta(w)} - \frac{1}{1-w} = \sum_{\substack{n,m \geq 1 \\ n \not\equiv m \pmod 2}} (-1)^n q^{nm/2} w^{(m-n-1)/2}, \tag{1.2}$$

is attributed to C. Jordan (5, p. 453).

Let $T_n(w) \in \mathbb{Z}[w, w^{-1}]$ be the coefficient of q^n in the Taylor expansion of (1.2) at $q = 0$. Let $C_n(q) \in \mathbb{Z}[q]$ be defined by $C_n(q) := (q-1)q^n T_n(q)$. C. Kassel and C. Reutenauer (6; 7) proved that, if q is a prime power, then there are precisely $C_n(q)$ ideals I of the group algebra $\mathbb{F}_q[\mathbb{Z}^2]$ of the free abelian group of rank 2 such that the quotient $\mathbb{F}_q[\mathbb{Z}^2]/I$ is an n-dimensional vector space over $\mathbb{F}_q$.

T. Hausel, E. Letellier and F. Rodriguez-Villegas (3) proved that $C_n(q)$ is the E-polynomial of the Hilbert scheme $X^{[n]}$ of n points on the algebraic torus $X := \mathbb{C}^\times \times \mathbb{C}^\times$. It is natural to consider the obvious action of the group $\mathbb{C}^\times \times \mathbb{C}^\times$ on the variety X and to extend this action to the punctual Hilbert scheme $X^{[n]}$. Let $\widetilde{X}^{[n]} := X^{[n]} /\!/ (\mathbb{C}^\times \times \mathbb{C}^\times)$ be the corresponding GIT-quotient (11). Denoting $P_n(q) \in \mathbb{Z}[q]$ the E-polynomial of $\widetilde{X}^{[n]}$, it follows, using elementary Hodge Theory (4), that $(q-1)^2 P_n(q) = C_n(q)$.

In virtue of (1.2),

$$P_n(q) = \frac{q^{n-1}}{q-1} \sum_{\substack{d|n \\ d \text{ odd}}} \left(q^{\gamma(d)} - q^{1-\gamma(d)}\right),$$

where $\gamma(d) := \frac{1}{2}\left(\frac{2n}{d} - d + 1\right)$. Using L'Hôpital's rule, it follows that

$$P_n(1) = \lim_{q \to 1} P_n(q) = \sigma(n),$$

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1. Kronecker's original identity is rather different, but it can be transformed into this one.

where $\sigma(n)$ is the sum of divisors of n. In (1) we called $P_n(q)$ the *Kassel-Reutenauer polynomials* because C. Kassel and C. Reutenauer studied some of their number-theoretical properties (6; 7; 8). A more informative name should be *Kassel-Reutenauer q-analog of the sum of divisors.*

It is obvious that $P_n(1)$ is divisible by 3 if $P_n(q)$ belongs to the principal ideal $[3]_q\mathbb{Z}[q]$, where $[3]_q := q^2 + q + 1$ is the classical q-analog of 3. Nevertheless, the converse statement is not always true. In order to fix this correspondence, we will consider the reduction modulo 3 of the polynomials $P_n(q)$, denoted ${}_3P_n(q)$. The aim of this paper is to prove the following result.

Theorem 1. *For any integer $n \geq 1$, the following statements are equivalent :*

(i) $P_n(1)$ belongs to the principal ideal $3\mathbb{Z}$;

(ii) ${}_3P_n(q)$ belongs to the principal ideal $[3]_q\mathbb{F}_3[q]$.

It is worth mentioning that Theorem 1 depends upon the special properties of the number 3 in the sense that it cannot be generalized in the obvious way substituting 3 and q^2+q+1 by p and $q^{p-1}+q^{p-2}+...+q+1$, respectively, for an arbitrary prime number p. Indeed, $P_{81}(1) = 121$ is a multiple of 11, but the reduction modulo 11 of the polynomial $P_{81}(q)$ does not belong to the principal ideal $(q^{10} + q^9 + q^8 + ... + q + 1)\mathbb{F}_{11}[q]$.

Given a commutative ring with identity R, we define an *R-arithmetical function* to be a sequence taking values in R. Traditional arithmetical functions correspond to $\mathbb{C}$-arithmetical function. We will extend in the obvious way the definition of *multiplicative function* to all R-arithmetical function, i.e. $f(n)$ is multiplicative if and only if $f(uv) = f(u)f(v)$ provided that u and v are relatively primes.

Let $\mathcal{R}$ be the ring defined by the quotient $\mathcal{R} := \mathbb{F}_3[q]/[3]_q\mathbb{F}_3[q]$. Notice that the ring $\mathcal{R}$ is isomorphic to $\mathbb{F}_3[X]/X^2\mathbb{F}_3[X]$. We will use the notation $\langle a \rangle := a + [3]_q\mathbb{F}_3[q]$.

For each $n \in \mathbb{Z}_{\geq 1}$, let ${}_{\mathcal{R}}P_n(q) \in \mathcal{R}$ be defined by ${}_{\mathcal{R}}P_n(q) := \langle {}_3P_n(q) \rangle$. We will derive Theorem 1 from the following explicit formula.

Theorem 2. *The $\mathcal{R}$-arithmetical function $n \mapsto {}_{\mathcal{R}}P_n(q)$ is multiplicative. Furthermore,*

$$ {}_{\mathcal{R}}P_n(q) = \left\langle \sigma(n) - (q-1)\,\sigma\left(\frac{n}{3}\right) \right\rangle, \tag{1.3} $$

where $\sigma(x) := 0$ for all $x \in \mathbb{Q}\backslash\mathbb{Z}$.

In order to prove Theorem 2, we will use an η-product related to the arithmetical function $n \mapsto P_n\left(\frac{-1+\sqrt{-3}}{2}\right)$ and we will consider a version of this sequence in characteristic 3.

1.2 The η-product $\frac{(\eta(z))^3}{\eta(3z)}$

Let

$$ \eta(z) := q^{1/24} \prod_{n \geq 1} (1 - q^n), $$

be the Dedekind η-function, where $q := e^{2\pi\sqrt{-1}z}$. An *η-product* (9) is a function which can be expressed as

$$ \prod_{m|N} \eta(mz)^{a_m}, $$

for some finite sequence $(a_m)_{m|N}$ of integers. Throughout this paper, we will focus on the η-product (9, pp. 158)

$$ \frac{(\eta(z))^3}{\eta(3z)} = \sum_{(x,y)\in\mathbb{Z}^2} \omega^{x-y} q^{x^2+xy+y^2}, \tag{1.4} $$

where $\omega := \frac{-1+\sqrt{-3}}{2}$, which is related to the Diophantine equation

$$ x^2 + xy + y^2 = n. $$

Let $-3\lambda(n)$ be the coefficient of q^n in (1.4). The arithmetical function $\lambda(n)$ is multiplicative and for each prime number p and each integer $k \geq 1$ we have the

formula (7, Equation (1.9))

$$\lambda\left(p^k\right) := \begin{cases} -2, & \text{if } p = 3\,; \\ k+1, & \text{if } p \equiv 1 \pmod 3\,; \\ \frac{1+(-1)^k}{2}, & \text{if } p \equiv -1 \pmod 3. \end{cases} \tag{1.5}$$

Using the identity (2, Proposition 2.29.)

$$\frac{(\eta(z))^3}{\eta(3z)} = 1 - 3\sum_{n=1}^{\infty} \frac{P_n(\omega)}{\omega^{n-1}} q^n,$$

C. Kassel and C. Reutenauer obtained the explicit evaluation (7, Theorem 1.6. (b))

$$P_n(\omega) = \lambda(n)\omega^{n-1}. \tag{1.6}$$

The identity (1.6) is equivalent to the following result.

Proposition 1. *For each integer $n \geq 1$,*

$$P_n(q) \in \lambda(n)q^{n-1} + [3]_q\mathbb{Z}[q]. \tag{1.7}$$

Proof. Consider the auxiliary polynomial $Q_n(q) := P_n(q) - \lambda(n)q^{n-1}$. Using the fact that $\mathbb{Q}[q]$ is an Euclidean domain, there are two polynomials $U_n(q), V_n(q) \in \mathbb{Q}[q]$, such that $Q_n(q) = U_n(q) + [3]_q V_n(q)$ and $U_n(q) = aq + b$ for some $a, b \in \mathbb{Z}$. Furthermore, the polynomials $U_n(q)$ and $V_n(q)$ are uniquely determined by these two properties. Notice that the Euclidean division may be performed in $\mathbb{Z}[q]$, since $[3]_q$ is monic. Hence, $U_n(q), V_n(q) \in \mathbb{Z}[q]$. Therefore, (1.7) holds. □

1.3 Proof of the main results

Lemma 1. *Consider the multiplicative function $\delta_{-1,3} : \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{Z}$ given by*

$$\delta_{-1,3}\left(p^k\right) = \begin{cases} 0, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 1 \pmod 2\,; \\ 1, & \text{otherwise}\,; \end{cases}$$

for any prime p and any $k \geq 1$. The $\mathbb{Z}[q]/[3]_q\mathbb{Z}$-arithmetical function $n \mapsto \delta_{-1,3}(n)q^{n-1}+[3]_q\mathbb{Z}[q]$ is multiplicative.

Proof. Notice that $q^3 - 1 \in [3]_q\mathbb{Z}[q]$ in virtue of the identity $q^3 - 1 = (q - 1)(q^2 + q + 1)$. Consider a pair of relatively prime positive integers n and m.

Suppose that $n \not\equiv -1 \pmod 3$ and $m \not\equiv -1 \pmod 3$. It follows that either at $n \equiv 1 \pmod 3$ or $m \equiv 1 \pmod 3$, i.e. $(n-1)(m-1) \equiv 0 \pmod 3$. So, $n+m-1 \equiv nm \pmod 3$. Therefore,

$$\begin{aligned} & \left(\delta_{-1,3}(n)q^{n-1} + [3]_q\mathbb{Z}[q]\right)\left(\delta_{-1,3}(m)q^{m-1} + [3]_q\mathbb{Z}[q]\right) \\ = \; & \delta_{-1,3}(n\,m)q^{(n+m-1)-1} + [3]_q\mathbb{Z}[q] \\ = \; & \delta_{-1,3}(n\,m)q^{nm-1} + [3]_q\mathbb{Z}[q]. \end{aligned}$$

Now, negate the hypothesis that $n \not\equiv -1 \pmod 3$ and $m \not\equiv -1 \pmod 3$. Without loss of generality, suppose that $m \equiv -1 \pmod 3$. It follows that m is divisible by some prime number $p \equiv -1 \pmod 3$ and the exponent of p in the prime factorization of m, denoted k, is odd. Notice that the exponent of p in the prime factorization of nm is also k, because n is relatively prime with m.

On the one hand

$$\left(\delta_{-1,3}(n)q^{n-1} + [3]_q\mathbb{Z}[q]\right)\left(\delta_{-1,3}(m)q^{m-1} + [3]_q\mathbb{Z}[q]\right) = 0 + [3]_q\mathbb{Z}[q],$$

because $\delta_{-1,3}(m) = 0$. On the other hand,

$$\delta_{-1,3}(nm)q^{nm-1} + [3]_q\mathbb{Z}[q] = 0 + [3]_q\mathbb{Z}[q],$$

because $\delta_{-1,3}(nm) = 0$.

Therefore, $n \mapsto \delta_{-1,3}(n)q^{n-1} + [3]_q\mathbb{Z}[q]$ is multiplicative. □

Lemma 2. *The $\mathcal{R}$-arithmetic function $n \mapsto \langle \sigma(n) - (q-1)\,\sigma\left(\frac{n}{3}\right)\rangle$ is multiplicative. Furthermore, the equality*

$$\left\langle \sigma(p^k) - (q-1)\,\sigma\left(\frac{p^k}{3}\right)\right\rangle = \begin{cases} \langle -q-1\rangle, & \text{if } p = 3\,; \\ \langle -1\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 1 \pmod 3\,; \\ \langle 0\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 2 \pmod 3\,; \\ \langle 1\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3\,; \\ \langle 0\rangle, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 1 \pmod 2\,; \\ \langle 1\rangle, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 0 \pmod 2. \end{cases}$$

holds for any prime number p and any integer $k \geq 1$.

Proof. Let m and n be relatively prime positive integers. Without lost of generality, we will suppose that m is not divisible by 3. The $\mathcal{R}$-arithmetic function $n \mapsto \langle \sigma(n) - (q-1)\,\sigma\left(\frac{n}{3}\right)\rangle$ is multiplicative in virtue of the identity

$$\begin{aligned} &\left\langle \sigma(m) - (q-1)\,\sigma\left(\frac{m}{3}\right)\right\rangle \left\langle \sigma(n) - (q-1)\,\sigma\left(\frac{n}{3}\right)\right\rangle = \langle \sigma(m)\rangle \left\langle \sigma(n) - (q-1)\,\sigma\left(\frac{n}{3}\right)\right\rangle \\ = \;& \left\langle \sigma(m)\sigma(n) - (q-1)\,\sigma(m)\sigma\left(\frac{n}{3}\right)\right\rangle = \left\langle \sigma(mn) - (q-1)\,\sigma\left(\frac{mn}{3}\right)\right\rangle. \end{aligned}$$

Let p be a prime number. Consider an integer $k \geq 1$. For $p \neq 3$ we have

$$\left\langle \sigma(p^k) - (q-1)\,\sigma\left(\frac{p^k}{3}\right)\right\rangle = \langle \sigma\left(p^k\right)\rangle = \langle 1\rangle + \langle p\rangle + \langle p\rangle^2 + \ldots + \langle p\rangle^k$$

$$= \begin{cases} \langle -1\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 1 \pmod 3\,; \\ \langle 0\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 2 \pmod 3\,; \\ \langle 1\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3\,; \\ \langle 0\rangle, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 1 \pmod 2\,; \\ \langle 1\rangle, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 0 \pmod 2. \end{cases}$$

For $p = 3$ we have

$$\left\langle \sigma(3^k) - (q-1)\,\sigma\left(\frac{3^k}{3}\right)\right\rangle = \left\langle \frac{3^{k+1}-1}{2} - (q-1)\,\frac{3^k-1}{2}\right\rangle$$
$$= \left\langle 3^k + \frac{3^k-1}{2} - (q-1)\,\frac{3^k-1}{2}\right\rangle = \left\langle 3^k + \frac{3^k-1}{2}\,(-q+2)\right\rangle = \langle -q-1\rangle\,.$$

□

Now, we proceed to prove Theorem 2. **Proof.**(Theorem 2) Let $\delta_{-1,3}(n)$ be the arithmetical function defined in Lemma 1. Let $\lambda(n)$ be the arithmetical function defined in (1.5). Notice that $\delta_{-1,3}(n) = 1$ provided that $\lambda(n) \neq 0$. It follows that $\lambda(n)\delta_{-1,3}(n) = \lambda(n)$. Applying Proposition 1,

$$_{\mathcal{R}}P_{p^k}(q) = \lambda(n)q^{n-1} + [3]_q\mathbb{Z}[q] = \lambda(n)\left(\delta_{-1,3}(n)q^{n-1} + [3]_q\mathbb{Z}[q]\right). \tag{1.8}$$

In virtue of Lemma 1, $n \mapsto \delta_{-1,3}(n)q^{n-1} + [3]_q\mathbb{Z}[q]$ is multiplicative. Recall that $\lambda(n)$ is multiplicative. So, $n \mapsto \lambda(n)\left(\delta_{-1,3}(n)q^{n-1} + [3]_q\mathbb{Z}[q]\right)$ is multiplicative. We conclude that $n \mapsto {}_{\mathcal{R}}P_n(q)$ is multiplicative.

It follows from (1.5) that

$$\lambda\left(p^k\right) \bmod 3 = \begin{cases} 1, & \text{if } p = 3\,; \\ 2, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 1 \pmod 3\,; \\ 0, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 2 \pmod 3\,; \\ 1, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3\,; \\ 0, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 1 \pmod 2\,; \\ 1, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 0 \pmod 2. \end{cases}$$

Reducing q^{n-1} modulo $[3]_q\mathbb{Z}[q]$, we have that

$$[\delta_{-1,3}(p^k)q^{p^k-1}] = \begin{cases} [-q-1], & \text{if } p = 3; \\ [1], & \text{if } p \equiv 1 \pmod 3; \\ [0], & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 1 \pmod 2; \\ [1], & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 0 \pmod 2; \end{cases}$$

where $[a] := a + [3]_q\mathbb{Z}[q]$. Combining the expressions for $\lambda\left(p^k\right) \bmod 3$ and $[\delta_{-1,3}(p^k)q^{p^k-1}]$, the identity (1.8) implies that the equality

$$_{\mathcal{R}}P_{p^k}(q) = \begin{cases} \langle -q-1\rangle, & \text{if } p = 3; \\ \langle -1\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 1 \pmod 3; \\ \langle 0\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 2 \pmod 3; \\ \langle 1\rangle, & \text{if } p \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3; \\ \langle 0\rangle, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 1 \pmod 2; \\ \langle 1\rangle, & \text{if } p \equiv -1 \pmod 3 \text{ and } k \equiv 0 \pmod 2. \end{cases} \tag{1.9}$$

holds for any prime number p and any integer $k \geq 1$. In virtue of Lemma 2, the equality (1.3) holds. □

Remark. In virtue of (1.9), the $\mathcal{R}$-arithmetical function $n \mapsto {}_{\mathcal{R}}P_{p^k}(q)$ takes at most 5 values. These values are realized for each $n \in \{1, 2, 3, 7, 21\}$.

Now, we proceed to prove our main result.

Proof.(Theorem 1) In virtue of Theorem 2, ${}_{\mathcal{R}}P_n(q) = 0$ if and only if $\sigma(n) \equiv 0 \pmod 3$ and $\sigma\left(\frac{n}{3}\right) \equiv 0 \pmod 3$. Notice that $\sigma\left(\frac{n}{3}\right) \equiv 0 \pmod 3$ is a consequence of the statement $\sigma(n) \equiv 0 \pmod 3$. Hence, ${}_{\mathcal{R}}P_n(q) = 0$ if and only if $\sigma(n) \equiv 0 \pmod 3$. □

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CHAPITRE II

A HODGE-THEORETICAL CHARACTERIZATION OF THE INTEGERS WHICH CANNOT BE PARTITIONED INTO AN EVEN NUMBER OF CONSECUTIVE PARTS

Abstract

Consider the algebraic torus (as a variety) $X := \mathbb{C}^\times \times \mathbb{C}^\times$ and its Hilbert scheme of n points, denoted $X^{[n]}$. Let $E\left(\widetilde{X}^{[n]}; q\right)$ be the E-polynomial of the GIT [1] quotient $\widetilde{X}^{[n]} := X^{[n]}//G$, where $G := \mathbb{C}^\times \times \mathbb{C}^\times$ is the algebraic torus (as a Lie group) and the action of G on $X^{[n]}$ is induced by the obvious action of G on X. We show a relationship between the coefficients of $E\left(\widetilde{X}^{[n]}; q\right)$ and the existence of a partition

$$n = m + (m+1) + (m+2) + ... + (m+k-1),$$

with $m, k \in \mathbb{Z}_{\geq 1}$, such that k even.

2.1 Introduction

Let $X := \mathbb{C}^\times \times \mathbb{C}^\times$ be the algebraic torus as a variety. Denote $X^{[n]}$ the Hilbert scheme of n points on X. The Lie group $G := \mathbb{C}^\times \times \mathbb{C}^\times$ acts on X in the obvious way. This action can be extended to $X^{[n]}$. So, we can define the GIT quotient $\widetilde{X}^{[n]} := X^{[n]}//G$.

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1. GIT is the abbreviation for Geometric Invariant Theory.

The E-polynomial of $\widetilde{X}^{[n]}$, denoted $E\left(\widetilde{X}^{[n]};q\right)$, was studied by T. Hausel, E. Letellier and F. Rodriguez-Villegas (2) and independently [2] by C. Kassel and C. Reutenauer (4; 5; 6).

The degree of $E\left(\widetilde{X}^{[n]};q\right)$ is $2n-2$, it is self-reciprocal and all its coefficients are non-negative integers (5), i.e.

$$E\left(\widetilde{X}^{[n]};q\right) = a_{n,0}q^{n-1} + \sum_{i=1}^{n-1} a_{n,i}\left(q^{n-1+i} + q^{n-1-i}\right), \tag{2.1}$$

for some nonnegative integers $a_{n,0}, a_{n,1}, a_{n,2}, ..., a_{n,n-1}$. The aim of this paper is to prove the following result.

Theorem 3. *For each integer $n \geq 1$, the following statements are equivalent :*

(i) for all integers $m \geq 1$ and $k \geq 1$, the equality [3]

$$n = m + (m+1) + (m+2) + ... + (m+k-1)$$

implies that k is odd;

(ii) $a_{n,0} \geq a_{n,1} \geq a_{n,2} \geq ... \geq a_{n,n-1}$.

It is worth mentioning that a finite sequence $s_1, s_2, ..., s_n$ if *unimodal* if and only if there some $1 \leq t \leq n$ such that

$$s_1 \leq s_2 \leq s_3 \leq ... \leq s_t \geq s_{t+1} \geq s_{t+2} \geq s_{t+3} \geq ... \geq s_n.$$

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2. C. Kassel and C. Reutenauer defined $E\left(\widetilde{X}^{[n]};q\right)$, in a rather combinatorial way, as the unique polynomial $P_n(q) = E\left(\widetilde{X}^{[n]};q\right)$ satisfying $(q-1)^2 P_n(q) = C_n(q)$, where $C_n(q)$ is the number of n codimensional ideals of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$.

3. The expression of a number as a sum of consecutive numbers it is named *polite representation.*

A polynomial having non-negative coefficients is said to be *unimodal* if its sequence of coefficients is unimodal. So, if $E\left(\widetilde{X}^{[n]};q\right)$ satisfies condition (ii) in Theorem 3, then it is unimodal (the converse is not necessarily true).

2.2 Proof of the main result

We will use the generating function

$$\prod_{m=1}^{\infty}\frac{(1-t^m)^2}{(1-qt^m)(1-q^{-1}t^m)}=1+\left(q+q^{-1}-2\right)\sum_{n=1}^{\infty}\frac{E\left(\widetilde{X}^{[n]};q\right)}{q^{n-1}}t^n, \tag{2.2}$$

due to T. Hausel, E. Letellier and F. Rodriguez-Villegas (2) and independently to C. Kassel and C. Reutenauer (4).

Lemma 3. *The number of solutions* $(m,k)\in\left(\mathbb{Z}_{\geq 1}\right)^2$ *of the equation*

$$n=m+(m+1)+(m+2)+...+(m+k-1),$$

with k *even, coincides with the number of odd divisors* d *of* n *satisfying the inequality* $d>\sqrt{2n}$.

Proof. This result is due to M. D. Hirschhorn and P. M. Hirschhorn (3). □

Lemma 4. *Let* $n\geq 1$ *be an integer. For any divisor* d *of* $2n$, *if* $d>\sqrt{2n}$ *then*

$$n+\frac{1}{2}\left(d-\frac{2n}{d}-1\right)\geq n>n-1\geq n+\frac{1}{2}\left(\frac{2n}{d}-d-1\right)\geq 0.$$

Proof. Consider an integer $n\geq 1$. Let d be a divisor of n. Suppose that $d>\sqrt{2n}$. The inequality $d>\sqrt{2n}$ implies that $d-\frac{2n}{d}>0$. Using the fact that $d-\frac{2n}{d}$ is an integer, it follows that $d-\frac{2n}{d}\geq 1$. So, $n+\frac{1}{2}\left(d-\frac{2n}{d}-1\right)\geq n$.

The inequality $d-\frac{2n}{d}>0$ implies that $\frac{2n}{d}-d<0$. Using the fact that $\frac{2n}{d}-d$ is an integer, it follows that $\frac{2n}{d}-d\leq -1$. So, $n+\frac{1}{2}\left(\frac{2n}{d}-d-1\right)\leq n-1$.

For $x \geq 1$ and $y \geq 1$, we have the trivial inequality

$$(x-1)(y-1) \geq 2(1-y).$$

From the above inequality, it follows that $xy = x - y + 1$. Substituting $x = d$ and $y = \frac{2n}{d}$, we obtain

$$2n \geq d - \frac{2n}{d} + 1,$$

which is equivalent to

$$n + \frac{1}{2}\left(\frac{2n}{d} - d - 1\right) \geq 0.$$

□

Lemma 5. *Let $n \geq 1$ be an integer. For any divisor d of $2n$, if $d < \sqrt{2n}$, then*

$$n + \frac{1}{2}\left(\frac{2n}{d} - d - 1\right) \geq n > n - 1 \geq n + \frac{1}{2}\left(d - \frac{2n}{d} - 1\right).$$

Proof. It is enough to apply Lemma 4 with $d = \frac{2n}{d}$. □

We proceed to prove our main result.

Proof.(Theorem 3) The equality

$$E\left(\widetilde{X}^{[n]}; q\right) = \sum_{\substack{d|n \\ d \equiv 1 \pmod 2}} \frac{q^{n+(2n/d-d-1)/2} - q^{n+(d-2n/d-1)/2}}{q-1}$$

follows from the combination of (2.2) with the classical identity (2, p. 113)

$$\frac{1}{\theta(w)} - \frac{1}{1-w} = \sum_{\substack{n,m \geq 1 \\ n \not\equiv m \pmod 2}} (-1)^n q^{nm/2} w^{(m-n-1)/2},$$

attributed to L. Kronecker and C. Jordan, where $\theta(w)$ is the formal product

$$\theta(w) := (1-w) \prod_{n \geq 1} \frac{(1-q^n w)(1-q^n w^{-1})}{(1-q^n)^2}.$$

We can express $E\left(\widetilde{X}^{[n]};q\right)$ as the difference $E\left(\widetilde{X}^{[n]};q\right) = R_n(q) - S_n(q)$ of two polynomials given by[4]

$$S_n(q) = \sum_{\substack{d|n \\ d\equiv 1 \pmod 2 \\ d>\sqrt{2n}}} \frac{q^{n+(d-2n/d-1)/2} - q^{n+(2n/d-d-1)/2}}{q-1},$$

$$R_n(q) = \sum_{\substack{d|n \\ d\equiv 1 \pmod 2 \\ d<\sqrt{2n}}} \frac{q^{n+(2n/d-d-1)/2} - q^{n+(d-2n/d-1)/2}}{q-1}.$$

Applying Lemmas 4 and 5, the coefficients of $S_n(q)$ and $R_n(q)$ are non-negative integers. Using the expansion $\frac{q^n-1}{q-1} = 1 + q + q^2 + ... + q^{n-1}$, it follows from the explicit formulae for $S_n(q)$ and $R_n(q)$ that the coefficients from (2.1) can be expressed as $a_{n,i} = a^+_{n,i} - a^-_{n,i}$, where

$$a^+_{n,i} = \#\left\{d|n: \quad d \text{ odd}, d<\sqrt{2n}, i \le \frac{1}{2}\left(\frac{2n}{d} - d - 1\right)\right\},$$

$$a^-_{n,i} = \#\left\{d|n: \quad d \text{ odd}, d>\sqrt{2n}, i \le \frac{1}{2}\left(d - \frac{2n}{d} - 1\right)\right\}.$$

Notice that the functions $\mathbb{Z}_{\ge 0} \longrightarrow \mathbb{Z}_{\ge 0}$, given by $i \mapsto a^+_{n,i}$ and $i \mapsto a^-_{n,i}$, are both weakly decreasing[5]. Hence, condition (ii) holds provided that $d < \sqrt{2n}$ for each odd divisor d of n, because in this case, $a^-_{n,i} = 0$ for all i.

Suppose that $d_0 > \sqrt{2n}$ for a fixed odd divisor d_0 of n. On the one hand, $a^-_{n,i_0} > a^-_{n,i_0+1}$, where $i_0 := \frac{1}{2}\left(d_0 - \frac{2n}{d_0} - 1\right)$. On the other hand, $a^+_{n,i_0} = a^+_{n,i_0+1}$, because the equality $\frac{1}{2}\left(d_0 - \frac{2n}{d_0} - 1\right) = \frac{1}{2}\left(\frac{2n}{d} - d - 1\right)$ is impossible[6] for any odd divisor

4. Notice that, if $d = \sqrt{2n}$, for some integer d, then d is even.

5. A sequence $s_1, s_2, ..., s_n$ is ***weakly decreasing*** if $s_1 \ge s_2 \ge ... \ge s_n$.

6. This equality would imply that the product $d\,d_0 = 2n$ is even, while both d_0 and d are

d of n. So, $a_{n,i_0} < a_{n,i_0+1}$. Hence, condition (ii) does not hold provided that there is at least one odd divisor d of n satisfying $d > \sqrt{2n}$.

In virtue of Lemma 3, we conclude that conditions (i) and (ii) are equivalent. □

2.3 Final remarks

Consider the symmetric Dyck word (1) $\langle\langle n \rangle\rangle := w_1 w_2 ... w_k \in \{+, -\}^*$, whose letters are given by

$$w_i := \begin{cases} +, & \text{if } u_i \in D_n \setminus (2D_n); \\ -, & \text{if } u_i \in (2D_n) \setminus D_n; \end{cases}$$

where D_n is the set of divisors of n, $2D_n := \{2d : \quad d \in D_n\}$ and $u_1, u_2, ..., u_k$ are the elements of the symmetric difference $D_n \triangle 2D_n$ written in increasing order. This word encodes the non-zero coefficients of $(q-1)E\left(\widetilde{X}^{[n]}; q\right)$. Theorem 3 admits the language-theoretical reformulation : condition (i) is equivalent to $\langle\langle n \rangle\rangle = \underbrace{++...+}_{s \text{ times}}\underbrace{--...-}_{s \text{ times}}$, for some $s \in \mathbb{Z}_{\geq 1}$. For details, see Appendix I (Annexe I) in this Memoir.

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odd.

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CHAPITRE III

ON A q-ANALOGUE OF THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS A SUM OF TWO SQUARES

Abstract

Kassel and Reutenauer (5) introduced a q-analogue of the number of representations of an integer as a sum of two squares. We establish some connections between the prime factorization of n and the coefficients of this q-analogue.

3.1 Introduction

Let $\mathcal{A}_q := \mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ be the algebra of Laurent polynomials over the finite field $\mathbb{F}_q$ having precisely q elements. For any ideal I of $\mathcal{A}_q$, we consider the quotient $V_I := \mathcal{A}_q/I$ as a vector space over $\mathbb{F}_q$. Let $C_n(q)$ be the number of ideals I of $\mathcal{A}_q$ such that $\dim V_I = n$. It is known (2; 4) that $C_n(q)$ is a polynomial in q of degree $2n$. From a Hodge-theoretical point of view (2), $C_n(q)$ can be described as the E-polynomial of the Hilbert scheme $X^{[n]}$ of n points on the algebraic torus $X := \mathbb{C}^\times \times \mathbb{C}^\times$.

C. Kassel and C. Reutenauer (3; 5) proved that $\Gamma_n(q) := C_n(-q)$ is a q-analogue [1]

1. A q analogue of $f : \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{Z}_{\geq 0}$ is a sequence of polynomials $(P_n(q))_{n\geq 1}$ such that $P_n(1) = f(n)$.

of the number of solutions $(x,y) \in \mathbb{Z}^2$ of the equation $n = x^2 + y^2$.

A *primitive Pythagorean triple* is a triple $(x,y,z) \in (\mathbb{Z}_{\geq 0})^3$ satisfying $x^2+y^2=z^2$ and $\gcd(x,y,z) = 1$. Notice that, according to this definition, $(1,0,1)$ and $(0,1,1)$ are primitive Pythagorean triples. The aim of this paper is to prove the following results.

Theorem 4. *For any integer $n \geq 1$, all the coefficients of $\Gamma_n(q)$ are non-negative if and only if $n = 2^k z$, for some integer $k \geq 0$ and some primitive Pythagorean triple (x,y,z).*

3.2 Proof of the main result

Denote $\gamma_{n,i}$ the coefficients of

$$\Gamma_n(q) = \gamma_{n,0} q^n + \sum_{i=1}^{n} \gamma_{n,i} \left(q^{n+i} + q^{n-i} \right).$$

Lemma 6. *Let $n \geq 1$ be an integer. For each $i \in [0..n]$, $\gamma_{n,i} = \gamma^+_{n,i} - \gamma^-_{n,i}$, where*

$$\gamma^+_{n,i} := \#\left\{ k|2n : \quad \frac{k\left(k+2i+(-1)^{n+k+i}\right)}{2} = n \right\},$$

$$\gamma^-_{n,i} := \#\left\{ k|2n : \quad \frac{k\left(k+2i-(-1)^{n+k+i}\right)}{2} = n \right\}.$$

Proof. The coefficients of

$$C_n(q) = c_{n,0} q^n + \sum_{i=1}^{n} c_{n,i} \left(q^{n+i} + q^{n-i} \right)$$

are given by (3, Proposition 3.3.),

$$\begin{aligned} \sum_{n\geq 1} c_{n,0} t^n &= 2 \sum_{k\geq 1} (-1)^k t^{k(k+1)/2} \\ &= -1 + \sum_{k\geq 1} (-1)^k \left(t^{k(k+1)/2} - t^{k(k-1)/2} \right), \\ \sum_{n\geq 1} c_{n,i} t^n &= \sum_{k\geq 1} (-1)^k \left(t^{k(k+2i+1)/2} - t^{k(k+2i-1)/2} \right), \end{aligned}$$

for all $i \geq 1$. So, for each $i \geq 1$,

$$\begin{aligned}\sum_{n\geq 1} c_{n,0}t^n &= -1 + \sum_{k\geq 1}\left(t^{k(k+(-1)^k)/2} - t^{k(k-(-1)^k)/2}\right),\\ \sum_{n\geq 1} c_{n,i}t^n &= \sum_{k\geq 1}\left(t^{k(k+2i+(-1)^k)/2} - t^{k(k+2i-(-1)^k)/2}\right).\end{aligned}$$

It follows that $c_{n,i} = c^+_{n,i} - c^-_{n,i}$, where

$$\begin{aligned}c^+_{n,i} &:= \#\left\{k|2n : \quad \frac{k\left(k+2i+(-1)^k\right)}{2} = n\right\},\\ c^-_{n,i} &:= \#\left\{k|2n : \quad \frac{k\left(k+2i-(-1)^k\right)}{2} = n\right\}.\end{aligned}$$

By definition of $\Gamma_n(q)$, $\gamma_{n,i} = (-1)^{n+i}c_{n,i}$. Hence, $\gamma_{n,i} = \gamma^+_{n,i} - \gamma^-_{n,i}$. □

Lemma 7. *Let $n \geq 1$ be an integer. Consider a divisor k of $2n$ such that $k \not\equiv \frac{2n}{k}$ (mod 2) and $k < \frac{2n}{k}$. Define*

$$u := \begin{cases} k, & \text{if } k \text{ is odd;}\\ \frac{2n}{k}, & \text{if } k \text{ is even.}\end{cases} \tag{3.1}$$

For each $\lambda \in \{-1, 1\}$, the following statements are equivalent :

(i) there is some $i \in [0..n]$ satisfying

$$\frac{k\left(k+2i+\lambda(-1)^{n+k+i}\right)}{2} = n; \tag{3.2}$$

(ii) $u \equiv \lambda$ (mod 4).

Proof. Consider n, k and λ fixed. Notice that, for all $t \in \mathbb{Z}$ the congruence $2t + \lambda(-1)^t \equiv \lambda \pmod 4$ holds (it is enough to evaluate this expression at all the possible values of t and λ, assuming that λ is odd).

The equality (3.2) is equivalent to

$$u + v + 2n = 2t + \lambda(-1)^t, \tag{3.3}$$

where $v := \frac{2n}{u}$ and $t := n + k + i$. Notice that $2n + v$ is a multiple of 4. Indeed, if n is even, then both $2n$ and v are multiples of 4, so $2n + v$ is a multiple of 4. Also, if n is odd, then $\frac{v}{2}$ is an odd integer, so $2n + v = 2\left(n + \frac{v}{2}\right)$ is a multiple of 4.

The equality (3.3) implies the congruence

$$u \equiv \lambda \pmod 4. \tag{3.4}$$

Suppose that (i) holds. There is some $i \in [0..n]$ satisfying (3.2). The congruence (3.4) follows. Hence, (ii) holds.

Now, suppose that (ii) holds. There is some integer $t \geq 1$ satisfying $2t + \lambda(-1)^t = u + v + 2n$, because any positive integer $\equiv \lambda \pmod 4$ can be expressed as $2t + \lambda(-1)^t$ for some $t \geq 1$ and we have $u + v + 2n \equiv \lambda \pmod 4$.

Define $i := t - n - k$. In virtue of the hypothesis $k < \frac{2n}{k}$, we have that $i \geq 0$. The equality (3.2) follows. Hence, (i) holds. □

Let $d_{a,m}(n)$ be the number of $d|n$ such that $d \equiv a \pmod m$.

Proposition 2. *For all $n \geq 1$,*

(i) the sum of all positive coefficients of $\Gamma_n(q)$ is $4d_{1,4}(n)$;

(ii) the sum of all negative coefficients of $\Gamma_n(q)$ is $-4d_{3,4}(n)$.

Proof. Let $\lambda \in \{-1, 1\}$. In virtue of Lemma 7, $2d_{\lambda,4}(n)$ is the number of pair

(k, i), where k is a divisor of $2n$ and $i \in [0..n]$, satisfying (3.2). Applying Lemma 6, we conclude that

$$4\lambda d_{\lambda,4}(n) = \begin{cases} \gamma^+_{n,0} + 2\sum_{i=1}^{n} \gamma^+_{n,i}, & \text{if } \lambda = 1\,; \\ -\gamma^-_{n,0} - 2\sum_{i=1}^{n} \gamma^-_{n,i}, & \text{if } \lambda = -1\,; \end{cases}$$

where $\gamma^+_{n,0} + 2\sum_{i=1}^{n} \gamma^+_{n,i}$ is the sum of the positive coefficients of $\Gamma_n(q)$ and $-\gamma^-_{n,0} - 2\sum_{i=1}^{n} \gamma^-_{n,i}$ is the sum of the negative coefficients of $\Gamma_n(q)$. □

Now, we proceed to prove our main result.

Proof.(Theorem 4) Let $n \geq 1$ be an integer. Notice that n has no prime factors p satisfying $p \equiv 3 \pmod 4$ if and only if $d_{3,4}(n) = 0$. In virtue of Proposition 2 (ii), the condition $d_{3,4}(n) = 0$ is equivalent to the fact that all the coefficients of $\Gamma_n(q)$ are non-negative. E. J. Eckert (1) proved that the set of positive integers z for which $(x, y, z) \in (\mathbb{Z}_{\geq 0})^3$ is a primitive Pythagorean triple, for some $(x, y) \in (\mathbb{Z}_{\geq 0})^2$, are precisely the positive integers without prime factors $p \equiv 3 \pmod 4$. Therefore, all the coefficients of $\Gamma_n(q)$ are non-negative if and only if $n = 2^k z$, for some integer $k \geq 0$ and some primitive Pythagorean triple (x, y, z). □

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CHAPITRE IV

ON A FUNCTION INTRODUCED BY ERDÖS AND NICOLAS

Abstract

Erdös and Nicolas (Erdös et Nicolas, 1976) introduced an arithmetical function $F(n)$ related to divisors of n in short intervals $\left]\frac{t}{2}, t\right]$. The aim of this note is to prove that $F(n)$ is the largest coefficient of polynomial $P_n(q)$ introduced by Kassel and Reutenauer (6). We deduce that $P_n(q)$ has a coefficient larger than 1 if and only if $2n$ is the perimeter of a Pythagorean triangle. We improve a result due to Vatne (Vatne, 2017) concerning the coefficients of $P_n(q)$.

4.1 Introduction

Erdös and Nicolas introduced in (1) the function

$$F(n) = \max\{\mathrm{q}_t(n) : \quad t \in \mathbb{R}_+^*\}, \tag{4.1}$$

where $\mathrm{q}_t(n) = \#\left\{d : \quad d|n \quad \text{and} \quad \frac{1}{2}t < d \leq t\right\}$, and they proved that

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{n\leq x} F(n) = +\infty. \tag{4.2}$$

Kassel and Reutenauer introduced in (2) a q-analog of the sum of divisors, denoted

$P_n(q)$, by means of the generating function

$$\prod_{m\geq 1} \frac{(1-t^m)^2}{(1-qt^m)(1-q^{-1}t^m)} = 1 + (q+q^{-1}-2)\sum_{n=1}^{\infty} \frac{P_n(q)}{q^{n-1}}\, t^n \qquad (4.3)$$

and they proved that, for $q = \exp\left(\frac{2\pi}{k}\sqrt{-1}\right)$, with $k \in \{2,3,4,6\}$, this infinite product can be expressed by means of the Dedekind η-function (see (4)). A consequence of this coincidence is that the corresponding arithmetic functions $n \mapsto P_n(q)$, for each of the above-mentioned values of q, are related to the number of ways to express a given integer by means of a quadratic form (see (2) and (3)).

The aim of this paper is to prove the following theorem.

Theorem 5. *For each integer $n \geq 1$, the largest coefficient of $P_n(q)$ is $F(n)$.*

Using this result, we will derive that $P_n(q)$ has a coefficient larger than 1 if and only if $2n$ is the perimeter of a Pythagorean triangle. Also, we will prove that each nonnegative integer m is the coefficient of $P_n(q)$ for infinitely many positive integers n.

4.2 Proof of the main result

In order to simplify the notation in the proofs, we will consider two functions[1] $f : \mathbb{R} \longrightarrow \mathbb{R}_+^*$ and $g : \mathbb{R}_+^* \longrightarrow \mathbb{R}$, defined by

$$f(x) = \frac{1}{2}\left(x + \sqrt{8n+x^2}\right), \qquad (4.4)$$

$$g(y) = y - \frac{2n}{y}. \qquad (4.5)$$

1. The function $g(y)$ was implicitly used in Proposition 2.2. in (4).

Lemma 8. *The functions $f(x)$ and $g(y)$ are well-defined, strictly increasing and mutually inverse. Furthermore, $g(y)$ satisfies the identity*

$$g(y) = -g\left(\frac{2n}{y}\right). \tag{4.6}$$

Proof. It follows in a straightforward way from the explicit expressions (4.4) and (4.5) that $f(x)$ and $g(y)$ are well-defined and strictly increasing. In particular, the inequality $|x| < \sqrt{2n + x^2}$ guarantees that $f(x) \in \mathbb{R}_+^*$ for all $x \in \mathbb{R}$.

On the one hand, for all $x \in \mathbb{R}$, we have

$$g(f(x)) = \frac{\left(f(x) - x - \sqrt{2n + x^2}\right)\left(f(x) - x + \sqrt{2n + x^2}\right)}{2f(x)} + x = x.$$

On the other hand, for all $y \in \mathbb{R}_+^*$, we have

$$f(g(y)) = \frac{y}{2} - \frac{n}{y} + \sqrt{\left(\frac{y}{2} + \frac{n}{y}\right)^2} = \frac{y}{2} - \frac{n}{y} + \frac{y}{2} + \frac{n}{y} = y,$$

where we used the inequality $\frac{y}{2} + \frac{n}{y} > 0$, provided that $y > 0$, for the elimination of the square root.

Hence, $f(x)$ and $g(y)$ are mutually inverses. Furthermore, using the identity

$$-g\left(2\frac{n}{y}\right) = -\left(\frac{2\frac{n}{y}}{2} - \frac{n}{2\frac{n}{y}}\right) = -\left(\frac{n}{y} - \frac{y}{2}\right) = \frac{y}{2} - \frac{n}{y} = g(y),$$

we conclude that (4.6) holds for all $y \in \mathbb{R}_+^*$. □

Lemma 9. *For each integer $n \geq 1$,*

$$\frac{P_n(q)}{q^{n-1}} = \sum_{i \in \mathbb{Z}} a_{n,i} q^i, \tag{4.7}$$

where

$$a_{n,i} = \#\left\{d : \quad d|n \quad \text{and} \quad \frac{1}{2}g(d) \leq i < \frac{1}{2}g(2d)\right\}. \tag{4.8}$$

Proof. By Theorem 1.2 in (3),

$$P_n(q) = a_{n,0}\, q^{n-1} + \sum_{i=1}^{n-1} a_{n,i} \left(q^{n-1+i} + q^{n-1-i} \right), \tag{4.9}$$

where

$$a_{n,i} = \# \left\{ d : \quad d|n \quad \text{and} \quad \frac{f(2\,i)}{2} < d \leq f\,(2\,i) \right\}. \tag{4.10}$$

The condition $\frac{f(2\,i)}{2} < d \leq f(2\,i)$ is equivalent to $d \leq f(2\,i) < 2\,d$. So, since $g(y)$ is strictly increasing by Lemma 8, the expression (4.8) follows for all $0 \leq i \leq n-1$.

We will extend $a_{n,i}$ to any $i \in \mathbb{Z}$ using the expression (4.8) as the definition of $a_{n,i}$ for $i < 0$.

Applying the identity (4.6) to (4.8),

$$a_{n,i} = \# \left\{ d : \quad d|n \quad \text{and} \quad \frac{1}{2}\, g\,(d) < -i \leq \frac{1}{2}\, g\,(2\,d) \right\}. \tag{4.11}$$

Substituting i by $-i$ in (4.8),

$$a_{n,-i} = \# \left\{ d : \quad d|n \quad \text{and} \quad \frac{1}{2}\, g\,(d) \leq -i < \frac{1}{2}\, g\,(2\,d) \right\}. \tag{4.12}$$

Now, we will prove that

$$\begin{aligned} & \# \left\{ d : \quad d|n \quad \text{and} \quad \frac{1}{2}\, g\,(d) < -i \leq \frac{1}{2}\, g\,(2\,d) \right\} \\ = \; & \# \left\{ d : \quad d|n \quad \text{and} \quad \frac{1}{2}\, g\,(d) \leq -i < \frac{1}{2}\, g\,(2\,d) \right\}. \end{aligned} \tag{4.13}$$

Suppose that

$$\frac{1}{2}\, g\,(d) = -i, \tag{4.14}$$

for some $d|n$. Transforming (4.14) into $d = 2\left(\frac{n}{d} - i\right)$, it follows that d is even. So, $-i = \frac{1}{2}\, g\,(2\,d')$, where $d' = \frac{d}{2}$ is a divisor of n.

Conversely, suppose that

$$-\,i = \frac{1}{2}\, g\,(2\,d)\,, \tag{4.15}$$

for some $d|n$. Transforming (4.15) into $\frac{n}{d} = 2\,(d + i)$, it follows that $\frac{n}{d}$ is even. Furthermore, $2\,d$ divides n, because $2\,d\,\frac{\frac{n}{d}}{2} = n$ and $\frac{\frac{n}{d}}{2} \in \mathbb{Z}$. So, $\frac{1}{2}\, g\,(d') = -i$, where $d' = 2\,d$ is a divisor of n. Hence, (4.14) holds.

Combining (4.14), (4.11) and (4.12), we obtain that

$$a_{n,i} = a_{n,-i} \tag{4.16}$$

holds for all $0 \leq i \leq n - 1$.

Furthermore, the bound $-(2\,n - 1) \leq g(y) \leq 2\,n - 1$ for all $1 \leq y \leq 2\,n$ and the equality (4.8) imply that

$$a_{n,i} = 0 \tag{4.17}$$

for all $i \in \mathbb{Z}$ such that $|i| \geq n$.

Using that (4.16) holds for all $0 \leq i \leq n - 1$ and that (4.17) holds for all $i \in \mathbb{Z}$, with $|i| \geq n$, we conclude that the expression (4.9) can be transformed into (4.7), where $a_{n,i}$ is given by (4.8) for all $i \in \mathbb{Z}$. □

Lemma 10. *Let y_1 and y_2 be two divisors of $2\,n$. If $y_1 < y_2$ then*

$$g(y_1) + 2 \leq g(y_2). \tag{4.18}$$

Proof. Using the expression (4.5) we obtain that, for any real number $y > 0$,

$$g(y+1) - g(y) > 1, \tag{4.19}$$

because $g(y+1) - g(y) = 1 + \frac{2n}{y(y+1)}$.

Let y_1 and y_2 be two positive real numbers satisfying $y_2 - y_1 \geq 1$. By Lemma 8, the function $g(y)$ is strictly increasing. So, (4.19) implies that

$$g(y_2) - g(y_1) > 1. \tag{4.20}$$

Furthermore, suppose that y_1 and y_2 are divisors of $2n$. It follows that $g(y_2) - g(y_1)$ is an integer, because of (4.5). In this case, the inequality (4.20) becomes

$$g(y_2) - g(y_1) \geq 2. \tag{4.21}$$

Therefore, (4.18) holds. □

Now, we can prove our main result.

Proof.(Theorem 5) By Lemma 9, the coefficient $a_{n,i}$ is defined for all $i \in \mathbb{Z}$ by the expression (4.8).

First, we will prove that the largest coefficient of $P_n(q)$ is at most $F(n)$. Take some $j \in \mathbb{Z}$ satisfying $a_{n,j} = \max\{a_{n,i} : \quad i \in \mathbb{Z}\}$. By (4.8), there are $h = a_{n,j}$ divisors of n, denoted $d_1, d_2, ..., d_h$ satisfying

$$g(d_1) < g(d_2) < ... < g(d_h) \leq 2j < g(2d_1) < g(2d_2) < ... < g(2d_h). \tag{4.22}$$

In particular,

$$g(d_1) < g(d_2) < ... < g(d_h) < g(2d_1) < g(2d_2) < ... < g(2d_h). \tag{4.23}$$

Applying $f(x)$ to the inequalities (4.23) we obtain

$$d_1 < d_2 < \ldots < d_h < 2\,d_1 < 2\,d_2 < \ldots < 2\,d_h, \tag{4.24}$$

because $f(x)$ and $g(y)$ are mutually inverses in virtue of Lemma 8. So, we guarantee that

$$\frac{1}{2}\,t < d_1 < d_2 < \ldots < d_h \leq t, \tag{4.25}$$

where $t = 2\,d_1 - \varepsilon$ for all $\varepsilon > 0$ small enough. Hence, $a_{n,j} \leq F(n)$, because of (4.1).

Now, we will prove that there is at least one coefficient of $P_n(q)$ which reaches the value $F(n)$. Setting $h = F(n)$ and applying (4.1), it follows that there are h divisors of n satisfying (4.25) for some $t \in \mathbb{R}_+^*$. The inequalities (4.24) follow. Applying $g(y)$ to (4.24) we obtain (4.23).

Setting

$$j := \left\lceil \frac{g\,(d_h)}{2} \right\rceil, \tag{4.26}$$

we have the inequalities

$$\begin{aligned} g(d_h) &\leq 2j, && (4.27)\\ 2j &\leq g(d_h) + 1, && (4.28)\\ g(d_h) + 1 &< g(d_h) + 2, && (4.29)\\ g(d_h) + 2 &\leq g(2\,d_1). && (4.30) \end{aligned}$$

The inequality (4.27) follows from (4.26). The inequality $2\,j < g\,(d_h) + 2$ follows from (4.26) and the stronger inequality (4.28) is obtained using the fact $g\,(d_h) \in \mathbb{Z}$,

derived from (4.5). The inequality (4.29) is trivial. Finally, the inequality (4.30) follows by Lemma 10, because d_h and $2\,d_1$ are divisors of $2\,n$ satisfying $d_h < 2\,d_1$.

Combining (4.27), (4.28), (4.29) and (4.30) we obtain that

$$g(d_h) \leq 2j < g(2\,d_1). \tag{4.31}$$

The inequalities (4.22) holds, because of (4.31) and (4.23). Hence, $a_{n,j} = F(n)$. Therefore, the largest coefficient of $P_n(q)$ is $F(n)$. □

Corollary. *The largest coefficient of $P_n(q)$ is the largest value of h for which (4.24) holds.*

Example 1. The polynomial $P_{12}(q)$ was computed in (2),

$$P_{12}(q) = q^{22} + q^{21} + q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + 2q^{13}$$
$$+2q^{12} + 2q^{11} + 2q^{10} + 2q^9 + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1.$$

Let us compute j such that $a_{12,j} = a_{12,-j}$ are equal to the largest coefficient of $P_{12}(q)$.

d	1	2	3	4	6	12
$g(d)$	-23	-10	-5	-2	2	10
$g(2\,d)$	-10	-2	2	5	10	23

The equality $F(12) = 2$ implies the existence of 2 divisors of 12, for example $d_1 = 2$ and $d_2 = 3$, satisfying (4.24). In our case,

$$2 < 3 < 2 \cdot 2 < 2 \cdot 3.$$

Applying $g(y)$ to the above inequalities, we obtain a particular case of (4.23),

$$-10 < -5 < -2 < 2.$$

So, taking $2j = g(3) + 1 = -5 + 1 = -4$, we obtain $a_{12,-2} = a_{12,2} = 2$, which are the coefficients of q^9 and q^{13}.

4.3 Some consequences of the main result

Kassel and Reutenauer observed in (2) that $P_n(q)$ has a coefficient larger than 1 provided that n is a perfect number or an abundant number. The corresponding necessary and sufficient condition is given in the following result.

Corollary. *The polynomial $P_n(q)$ has a coefficient larger than 1 if and only if $2n$ is the perimeter of a Pythagorean triangle.*

Proof. From the explicit formula for Pythagorean triples (see (5)), it follows in a straightforward way that $2n$ is the perimeter of a Pythagorean triangle if and only if n has a pair of divisors d and d' satisfying the inequality $d < d' < 2d$. So, the result follows from Corollary 4.2. □

Vatne proved in (6) that the set of coefficients of $P_n(q)$ is unbounded. The following result is a stronger version of this property.

Corollary. *Let $a_{n,i}$ be the coefficients of $P_n(q)$ as shown in (4.7) and (4.8). For any integer $m \geq 0$, the equality $a_{n,i} = m$ holds for infinitely many $(n, i) \in \mathbb{Z}^2$, with $n \geq 1$.*

In the proof of Corollary 4.3 we will use the following auxiliary result.

Lemma 11. *Let $h \geq 1$ be an integer. If h is a coefficient of the polynomial $P_n(q)$, then $h - 1$ is also a coefficient of the same polynomial.*

Proof. Consider two fixed integers $n \geq 1$ and $h \geq 1$. Let j be the largest integer such that $a_{n,j} \geq h$, where $a_{n,j}$ is given by (4.8). The inequalities (4.22) hold for h divisors of n, denoted $d_1, d_2, ..., d_h$. Setting

$$i := \left\lceil \frac{g(2\,d_1)}{2} \right\rceil, \tag{4.32}$$

we have the inequalities

$$\begin{aligned} g(d_h) &< g(2\,d_1), && (4.33)\\ g(2\,d_1) &\leq 2\,i, && (4.34)\\ 2\,i &\leq g(2\,d_1) + 1, && (4.35)\\ g(2\,d_1) + 1 &< g(2\,d_1) + 2, && (4.36)\\ g(2\,d_1) + 2 &\leq g(2\,d_2). && (4.37) \end{aligned}$$

The inequality (4.33) follows by (4.22). The inequality (4.34) follows from (4.32). The inequality $2\,i < g(2\,d_1) + 2$ follows from (4.32) and the stronger inequality (4.35) is obtained using the fact $g(2\,d_1) \in \mathbb{Z}$, derived from (4.5). The inequality (4.36) is trivial. Finally, the inequality (4.37) follows by Lemma 10, because $2\,d_1 < 2\,d_2$ and both are divisors of $2\,n$.

Combining (4.33), (4.34), (4.35), (4.36) and (4.37) we obtain that

$$g(d_h) \leq 2i < g(2\,d_2). \tag{4.38}$$

Combining (4.38) with (4.24), it follows that

$$d_2 < d_3 < ... < d_h \leq 2\,i < 2\,d_2 < 2\,d_3 < ... < 2\,d_h. \tag{4.39}$$

Notice that (4.22) and (4.34) imply

$$j < i. \tag{4.40}$$

In virtue of the expression (4.8), the inequalities (4.39) imply that

$$a_{n,i} \geq h - 1. \tag{4.41}$$

The inequalities (4.40) and (4.41) imply that $a_{n,i} = h - 1$, because j is the largest integer satisfying $a_{n,j} \geq h$. □

Proof.(Corollary 4.3)

Using (4.2), it follows that the range of $F(n)$ is unbounded. By Theorem 5, the set of coefficients of $P_n(q)$, for all $n \geq 1$, is unbounded.

Take an integer $m \geq 0$. Consider a polynomial $P_n(q)$ whose largest coefficient is $h > m$. Applying Lemma 11 several times, we will obtain that m is a coefficient of $P_n(q)$. As there are infinitely many values of n such that $P_n(q)$ has a coefficient larger than m, the equality $a_{n,i} = m$ holds for infinitely many $(n, i) \in \mathbb{Z}^2$, with $n \geq 1$. □

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CONCLUSIONS

Plusieurs prédicats concernant les entiers positifs, qui sont apparus de dans le cadre classique de la théorie de nombres, peuvent être transformés de façon naturelle dans des propriétés des polynômes $C_n(q)$ et $P_n(q)$. Par exemple,

1. $\sigma(n)$ est divisible par 3 si et seulement si $P_n(q)$, réduit mod 3, appartient à l'idéal principal $(q^2+q+1)\mathbb{F}_3$;
2. Il n'y pas $m \geq 1$ et $k \geq 1$ tels que $n = m+(m+1)+(m+2)+...+(m+k-1)$ avec k pair si et seulement si $a_{n,0} \geq a_{n,1} \geq a_{n,2} \geq ... \geq a_{n,n-1}$;
3. $n = 2^k z$ pour $h \geq 0$ et quelque triplet pythagoricien primitif (x, y, z) si et seulement si tous les coefficients de $C_n(-q)$ sont non-négatifs ;
4. $n = \frac{1}{2}(x+y+z)$, pour triplet pythagoricien (x, y, z) si et seulement si $P_n(q)$ a un coefficient plus grand que 1.

Il y a une sorte d'« isomorphisme » entre une partie non-triviale de la théorie des nombres et la suite de polynômes $C_n(q)$. Par exemple, le théorème selon lequel le produit de deux nombres qui sont des hypoténuses de triangles de Pythagore primitifs est aussi un triangle de Pythagore primitif correspond au fait trivial suivant : le produit de deux polynômes à coefficients non-négatifs est un polynôme à coefficients non-négatifs. L'étude approfondi d'un tel « isomorphisme » mérite d'être l'objet de recherches ultérieures.

Appendices

ANNEXE I

FACTORIZATION OF DYCK WORDS AND THE DISTRIBUTION OF THE DIVISORS OF AN INTEGER

Abstract

In (2), we associated a Dyck word $\langle\langle n \rangle\rangle_\lambda$ to any pair (n, λ) consisting of an integer $n \geq 1$ and a real number $\lambda > 1$. The goal of the present paper is to show a relationship between the factorization of $\langle\langle n \rangle\rangle_\lambda$ as the concatenation of irreducible Dyck words and the distribution of the divisors of n. In particular, we will provide a characterization of λ-densely divisible numbers (these numbers were introduced in (1)).

A.1 Introduction

Zhang (7) established the first finite bound on gaps between prime numbers. In order to refine Zhang's result, the *polymath8* project led by Tao (1) introduced the so-called densely divisible numbers, which are a weak version of the classical smooth numbers. An integer $n \geq 1$ is *λ-densely divisible*, where $\lambda > 1$ is a real number, if for all $R \in [1, n]$, there is at least one divisor of n on the interval $[\lambda^{-1} R, R]$.

Let L be a finite set of real numbers. Consider the set

$$\mathcal{T}(L;t) := \bigcup_{\ell \in L} [\ell, \ell + t]\,, \tag{A.1}$$

endowed with the topology inherited from $\mathbb{R}$, where $t > 0$ is an arbitrary real number. It is natural to associate any integer $n \geq 1$ with the topological space

$$\mathcal{T}_\lambda(n) := \mathcal{T}(L;t)\,,$$

where $L := \{\ln d : \quad d|n\}$ and $t := \ln \lambda$. It follows that an integer $n \geq 1$ is λ-densely divisible if and only if $\mathcal{T}_\lambda(n)$ is connected (see Proposition 5).

In this paper, we will show a relationship between the number of connected components of $\mathcal{T}(L;t)$ and the factorization of the Dyck word $\langle\langle S \rangle\rangle_\lambda$ introduced in (2), provided that $L = \{\ln s : \quad s \in S\}$ and $t = \ln \lambda$. From this general result, we will derive a characterization of λ-densely divisible numbers in terms of the Dyck word $\langle\langle n \rangle\rangle_\lambda$, also introduced in (2). We recall the definitions of $\langle\langle S \rangle\rangle_\lambda$ and $\langle\langle n \rangle\rangle_\lambda$ given in (2).

Definition 1. Consider a real number $\lambda > 1$ and a 2-letter alphabet $\Sigma = \{a, b\}$.

(i) Given a finite set of positive real numbers S, the λ-*class* of S is the word

$$\langle\langle S \rangle\rangle_\lambda := w_0\, w_1\, w_2 \ldots w_{k-1} \in \Sigma^*, \tag{A.2}$$

such that each letter is given by

$$w_i := \begin{cases} a & \text{if } \mu_i \in S, \\ b & \text{if } \mu_i \in \lambda S, \end{cases} \tag{A.3}$$

for all $0 \leq i \leq k-1$, where μ_0, μ_1, ..., μ_{k-1} are the elements of the symmetric difference $S \triangle \lambda S$ written in increasing order, i.e.

$$\begin{aligned} \lambda S &:= \{\lambda s : \quad s \in S\}, \\ S \triangle \lambda S &= \{\mu_0 < \mu_1 < \ldots < \mu_{k-1}\}\,. \end{aligned} \tag{A.4}$$

(ii) If S is the set of divisors of n, then we will write $\langle\langle n\rangle\rangle_\lambda := \langle\langle S\rangle\rangle_\lambda$. The word $\langle\langle n\rangle\rangle_\lambda$ will be called the λ-*class* of n.

The proof that $\langle\langle n\rangle\rangle_\lambda$ and $\langle\langle S\rangle\rangle_\lambda$ are Dyck words was given in (2). Also, the height of the Dyck path associated to $\langle\langle n\rangle\rangle_\lambda$ coincides with the generalized Hooley's Δ_λ-function

$$\Delta_\lambda(n) := \max_{R>0} \#\left\{d|n: \quad d \in \left]\lambda^{-1} R, R\right]\right\},$$

where R runs over the positive real numbers (see (2)).

The main result in the present paper is the following theorem.

Theorem 6. *Let $\lambda > 1$ be a real number.*

(i) For any integer $n \geq 1$, the number of connected components of $\mathcal{T}_\lambda(n)$ is precisely $\Omega\left(\langle\langle n\rangle\rangle_\lambda\right)$.

(ii) An integer $n \geq 1$ is λ-densely divisible if and only if $\langle\langle n\rangle\rangle_\lambda$ is an irreducible Dyck word.

The function $\Omega(w)$, formally defined using diagram (A.5), is just the number of irreducible Dyck words needed to obtain the Dyck word w as a concatenation of them[1]. We will derive Theorem 6 taking S to be the set of divisors of n in the following more general result.

Proposition 3. *Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers S. Define $L := \{\ln s: \quad s \in S\}$ and $t := \ln \lambda$. The number of connected components of $\mathcal{T}(L;t)$ is $\Omega\left(\langle\langle S\rangle\rangle_\lambda\right)$.*

---

1. We use the notation $\Omega(w)$ in analogy to the arithmetical function $\Omega(n)$ which is equal to the number of prime factors of n counting their multiplicities.

A.2 Preliminaries

Consider a 2-letter alphabet $\Sigma = \{a, b\}$. The *bicyclic semigroup*[2] $\mathcal{B}$ is the monoid given by the presentation

$$\mathcal{B} := \langle a, b | \quad a\,b = \varepsilon \rangle\,,$$

where ε is the empty word.

Let $\pi : \Sigma^* \longrightarrow \mathcal{B}$ be the canonical projection. The *Dyck language* $\mathcal{D}$ is the kernel of π, i.e.

$$\mathcal{D} := \pi^{-1}\left(\pi\left(\varepsilon\right)\right).$$

Interpreting the letters a and b as the displacements $1+\sqrt{-1}$ and $1-\sqrt{-1}$ in the complex plane $\mathbb{C}$, we can represent each word $w \in \mathcal{H}$ by means of a Dyck path, i.e. a lattice path from 0 to $|w|$, using only the above-mentioned steps and always keeping the imaginary part on the upper half-plane $\{z \in \mathbb{C} : \quad \mathrm{Im}\ z \geq 0\}$. For an example of Dyck path, see Fig A.1. It is easy to check that $\mathcal{D}$ can be described as the language corresponding to all possible Dyck paths.

The language of *reducible Dyck words* is the submonoid

$$\widetilde{\mathcal{D}} := \{\varepsilon\} \cup \{u\,v : \quad u, v \in \mathcal{D}\backslash\{\varepsilon\}\}$$

of $\mathcal{D}$. The elements of the complement of $\widetilde{\mathcal{D}}$ in $\mathcal{D}$, denoted

$$\mathcal{P} := \mathcal{D}\backslash\widetilde{\mathcal{D}}$$

are called *irreducible Dyck words.*

---

2. In this paper, the bicyclic semigroup is not just a semigroup, but also a monoid. We preserved the word "semigroup" in the name for historical reasons.

It is well-known that $\mathcal{D}$ is freely generated by $\mathcal{P}$, i.e. every word in $\mathcal{D}$ may be formed in a unique way by concatenating a sequence of words from $\mathcal{P}$. So, there is a unique morphism of monoids $\Omega : \mathcal{D} \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the monoid of non-negative integers endowed with the ordinary addition, such that the diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{P}^* \\ & \overset{\Omega}{\searrow} & \downarrow \\ & & \mathbb{N} \end{array} \tag{A.5}$$

commutes, where $\mathcal{D} \longrightarrow \mathcal{P}^*$ is the identification of $\mathcal{D}$ with the free monoid $\mathcal{P}^*$ and $\mathcal{P}^* \longrightarrow \mathbb{N}$ is just the length of a word in $\mathcal{P}^*$ considering each element of the set $\mathcal{P}$ as a single letter (of length 1). In other words, $\Omega(w)$, with $w \in \mathcal{D}$, is the number of irreducible Dyck words that we need to obtain w as a concatenation of them.

We will use the following result proved in (2).

Proposition 4. *Let S be a finite set of positive real numbers. For any real number $\lambda > 1$ we have that $\langle\langle S \rangle\rangle_\lambda \in \mathcal{D}$, i.e. $\langle\langle S \rangle\rangle_\lambda$ is a Dyck word.*

A.3 Generic case

Given a finite set of positive real numbers S, we says that a real number $\lambda > 1$ is *regular* (with respect to S) if S and λS are disjoint. Otherwise, we say that $\lambda > 1$ is *singular* (with respect to S). This notion was already introduced in (2).

It is easy to check that the number of singular values (corresponding to a finite set S) is finite. In this section we will prove Proposition 3 under the additional hypothesis that λ is regular. The proof that this proposition also holds true for singular values of λ will be deduced from the case for regular values in next section.

Lemma 12. *Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers S. Suppose that λ is regular. Define $L := \{\ln s : \quad s \in S\}$ and $t := \ln \lambda$. The space $\mathcal{T}(L; t)$ is disconnected if and only if $\langle\langle S \rangle\rangle_\lambda$ is a reducible Dyck word.*

Proof. Define $L + t := \{\ell + t : \quad \ell \in L\}$. We have $L \cup (L + t) = \{\ln \mu_i : \quad 0 \leq i \leq k - 1\}$ because λ is regular. Here μ_0, μ_1, ..., μ_{k-1} are the numbers appearing in (A.4). Consider the word $\langle\langle S \rangle\rangle_\lambda = w_0 w_1 ... w_{k-1}$ as defined in (A.2).

Suppose that $\mathcal{T}(L; t)$ is disconnected. In virtue of (A.1), for some $0 \leq j < k - 1$, we have $\ln \mu_j + t < \ln \mu_{j+1}$, i.e., $\lambda \mu_j < \mu_{j+1}$. Indeed, if for any $0 \leq j < k - 1$, we have $\ln \mu_{j+1} \leq \ln \mu_j + t$, then the space $\mathcal{T}(L; t) = [\ln \mu_0, \ln(\mu_{k-1}) + t]$ will be a connected.

So, the list μ_0, μ_1, ..., μ_j contains as many elements from S as elements from λS. It follows from (A.3) that $u := w_0 w_1 ... w_j$ satisfies $|u|_a = |u|_b$. So, u is Dyck word. Therefore, $\langle\langle S \rangle\rangle_\lambda$ is a reducible Dyck word, because its nonempty proper prefix u is a Dyck word.

By Proposition 4, $\langle\langle S \rangle\rangle_\lambda$ is a Dyck word. Suppose that $\langle\langle S \rangle\rangle_\lambda$ is reducible. For some $0 \leq j < k - 1$ we have that the nonempty proper prefix $u := w_0 w_1 ... w_j$ of $\langle\langle S \rangle\rangle_\lambda$ is a Dyck word. The relation $|u|_a = |u|_b$ and (A.3) imply that the list μ_0, μ_1, ..., μ_j contains as many elements from S as elements from λS. So, $\lambda \mu_j < \mu_{j+1}$, i.e. $\ln \mu_j + t < \ln \mu_{j+1}$. Using (A.1) we conclude that $\mathcal{T}(L; t)$ is disconnected. □

Lemma 13. *Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers S. Suppose that λ is regular. Define $L := \{\ln s : \quad s \in S\}$ and $t := \ln \lambda$. The number of connected components of $\mathcal{T}(L; t)$ is $\Omega(\langle\langle S \rangle\rangle_\lambda)$.*

Proof. Let μ_0, μ_1, ..., μ_{k-1} be the numbers appearing in (A.4). Consider the word $\langle\langle S \rangle\rangle_\lambda = w_0 w_1 ... w_{k-1}$ as defined in (A.2). By Proposition 4, $\langle\langle S \rangle\rangle_\lambda$ is a Dyck word. We proceed by induction on the number $c \geq 1$ of connected components of $\mathcal{T}(L; t)$.

Consider the case $c = 1$. Suppose that $\mathcal{T}(L; t)$ is connected. By Lemma 12, $\langle\langle S \rangle\rangle_\lambda$ is irreducible. Then $c = \Omega(\langle\langle S \rangle\rangle_\lambda) = 1$.

Suppose that the number of connected components of $\mathcal{T}(L;t)$ is $\Omega(\langle\langle S\rangle\rangle_\lambda)$, provided that $\mathcal{T}(L;t)$ has at most $c-1$ connected components for some $c > 1$. Assume that $\mathcal{T}(L;t)$ has precisely c connected components. By Lemma 12, $\langle\langle S\rangle\rangle_\lambda$ is reducible. Let p_1, p_2, ..., p_h be irreducible Dyck words satisfying $\langle\langle S\rangle\rangle_\lambda = p_1\, p_2 \ldots p_h$.

For some $0 \leq j < k-1$ we have $p_1 = w_0\, w_1 \ldots w_j$. Notice that $\lambda\, \mu_i \leq \mu_j < \mu_{j+1}$ for all $0 \leq i \leq j$ such that $\mu_i \in S$. Indeed, this follows from the fact that both p_1 is a Dyck word.

Setting $R = \{\mu_0, \mu_1, ..., \mu_j\}$, it follows that $\langle\langle S \backslash R\rangle\rangle_\lambda = p_2\, p_3 \ldots p_h$.

The space $\mathcal{T}(L \backslash \ln(R); t)$, where $\ln(R) := \{\ln s : \quad s \in R\}$, has precisely $c-1$ connected components, because $\ln \mu_j + \ln \lambda < \ln \mu_{j+1}$. Applying the induction hypothesis, $c - 1 = \Omega(\langle\langle S \backslash R\rangle\rangle_\lambda) = h - 1$. Hence, $c = \Omega(\langle\langle S\rangle\rangle_\lambda) = h$.

By the principle of induction, we conclude that the number of connected components of $\mathcal{T}(L;t)$ is $\Omega(\langle\langle S\rangle\rangle_\lambda)$.

A.4 General case

Consider a 3-letter alphabet $\Gamma = \{a, b, c\}$. We define the *Hooley monoid* $\mathcal{C}$ to be the monoid given by the presentation

$$\mathcal{C} := \langle a, b, c | \quad a\, b = \varepsilon, a\, c\, b = a\, b, c\, c = c \rangle .$$

Let $\varphi : \Gamma^* \longrightarrow \mathcal{C}$ be the canonical projection. The *Hooley-Dyck language* $\mathcal{H}$ is the kernel of φ, i.e.

$$\mathcal{H} := \varphi^{-1}(\varphi(\varepsilon)) .$$

Associating the letters a, b and c to the displacements $1+\sqrt{-1}$, $1-\sqrt{-1}$ and 1, respectively, in the complex plane $\mathbb{C}$, it follows that each word $w \in \mathcal{H}$ can be represented by Schröder path, i.e. a lattice path from 0 to $|w|$, using only the above-mentioned steps and always keeping the imaginary part on the upper half-plane $\{z \in \mathbb{C}: \quad \text{Im } z \geq 0\}$. For an example of Schröder path, see Fig A.2.

Notice that the language $\mathcal{H}$ corresponds to all possible Schröder paths having all the horizontal displacements (corresponding to c) strictly above the real axis.

The language of *reducible Hooley-Dyck words* is the submonoid

$$\widetilde{\mathcal{H}} := \{\varepsilon\} \cup \{u\,v: \quad u, v \in \mathcal{H}\backslash\{\varepsilon\}\}$$

of $\mathcal{H}$. The elements of the complement of $\widetilde{\mathcal{H}}$ in $\mathcal{H}$, denoted

$$\mathcal{Q} := \mathcal{H}\backslash\widetilde{\mathcal{H}}$$

are called *irreducible Hooley-Dyck words.*

It is easy to check that $\mathcal{Q}$ freely generates $\mathcal{H}$. So, there is a unique morphism of monoids $\Theta : \mathcal{H} \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the monoid of non-negative integers endowed with the ordinary addition, such that the diagram

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{Q}^* \\ & \overset{\Theta}{\searrow} & \downarrow \\ & & \mathbb{N} \end{array} \tag{A.6}$$

commutes, where $\mathcal{H} \longrightarrow \mathcal{Q}^*$ is the identification of $\mathcal{Q}$ with the free monoid $\mathcal{Q}^*$ and $\mathcal{Q}^* \longrightarrow \mathbb{N}$ is just the length of a word in $\mathcal{Q}^*$ considering each element of the set $\mathcal{Q}$ as a single letter (of length 1).

Lemma 14. *Let $\gamma : \Gamma^* \longrightarrow \Sigma^*$ be the morphism of monoids given by $a \mapsto a$, $b \mapsto b$ and $c \mapsto \varepsilon$. We have that $\gamma(\mathcal{H}) \subseteq \mathcal{D}$.*

Proof. Notice that the diagram

$$\begin{array}{ccc} \Gamma^* & \xrightarrow{\varphi} & \mathcal{C} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Sigma^* & \xrightarrow{\pi} & \mathcal{B} \end{array} \tag{A.7}$$

commutes, where ψ is the morphism of monoids given by $\psi(C) := \gamma(C)$, for each equivalence class $C \in \mathcal{C}$.

Take $w \in \gamma(\ker\varphi)$. By definition, $w = \gamma(v)$ for some $v \in \ker\varphi$. Using the equalities

$$\begin{aligned} \pi(w) &= \pi(\gamma(v)) \\ &= \psi(\varphi(v)) \\ &= \psi(\varphi(\varepsilon)) \\ &= \pi(\varepsilon), \end{aligned}$$

we obtain that $w \in \ker\pi$. Hence, $\gamma(\ker\varphi) \subseteq \ker\pi$, i.e. $\gamma(\mathcal{H}) \subseteq \mathcal{D}$.

Lemma 15. *The morphism γ defined in Lemma 14 satisfies $\gamma(\mathcal{Q}) \subseteq \mathcal{P}$.*

Proof. Take $q \in \mathcal{Q}$. By Lemma 14, we have $\gamma(q) \in \mathcal{D}$. Also, we have $\gamma(q) \neq \varepsilon$, because c^* and $\mathcal{Q}$ are disjoint, where $c^* := \{\varepsilon, c, cc, ccc, ...\}$.

Suppose that $\gamma(q) = u\,v$, for some $u, v \in \mathcal{D}\backslash\{\varepsilon\}$. It follows that $q = \hat{u}\,\hat{v}$ for some $\hat{u}, \hat{v} \in \Gamma^*$ satisfying $\gamma(\hat{u}) = u$ and $\gamma(\hat{v}) = v$. Using the commutative diagram A.7, the fact that ψ is an isomorphism and the equalities,

$$\begin{aligned}\varphi(\hat{u}) &= \psi^{-1}(\pi(\gamma(\hat{u}))) \\ &= \psi^{-1}(\pi(u)) \\ &= \psi^{-1}(\pi(\varepsilon)) \\ &= \varphi(\varepsilon),\end{aligned}$$

we obtain that $\hat{u} \in \ker\varphi = \mathcal{H}$. Similarly, $\hat{v} \in \ker\varphi = \mathcal{H}$. Hence, $q \notin \mathcal{Q}$, contrary to our hypothesis. By *reductio ad absurdum*, $\gamma(\mathcal{Q}) \subseteq \mathcal{P}$.

Lemma 16. *Given $w \in \mathcal{H}$, we have $\Theta(w) = \Omega(\gamma(w))$, where γ is the morphism defined in Lemma 14, Θ is given by diagram (A.6) and Ω is given by diagram (A.5).*

Proof. Notice that the diagram

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{Q}^* \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{P}^* \end{array}$$

commutes, where $\mathcal{D} \longrightarrow \mathcal{P}^*$ is the identification of $\mathcal{D}$ with the free monoid $\mathcal{P}^*$, $\mathcal{H} \longrightarrow \mathcal{Q}^*$ is the identification of $\mathcal{H}$ with the free monoid $\mathcal{Q}^*$, $\mathcal{Q}^* \longrightarrow \mathcal{P}^*$ is the morphism of monoids given by $w \mapsto \gamma(w)$ for all $w \in \mathcal{Q}$ (this function is well-defined in virtue of Lemma 15) and $\mathcal{H} \longrightarrow \mathcal{D}$ is given by $w \mapsto \gamma(w)$ (this function is well-defined in virtue of Lemma 14). It follows that $\Theta(w) = \Omega(\gamma(w))$ holds for each $w \in \mathcal{H}$.

Lemma 17. *Let $\alpha : \Gamma^* \longrightarrow \Sigma^*$ be the morphism of monoids given by $a \mapsto a$, $b \mapsto b$ and $c \mapsto a\,b$. We have that $\alpha(\mathcal{H}) \subseteq \mathcal{D}$.*

Proof. Notice that the diagram

$$\begin{array}{ccc} \Gamma^* & \xrightarrow{\varphi} & \mathcal{C} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\chi} \\ \Sigma^* & \xrightarrow{\pi} & \mathcal{B} \end{array} \tag{A.8}$$

commutes, where χ is the morphism of monoids given by $\chi(C) := \alpha(C)$, for each equivalence class $C \in \mathcal{C}$.

Take $w \in \alpha(\ker\varphi)$. By definition, $w = \alpha(v)$ for some $v \in \ker\varphi$. Using the equalities

$$\begin{aligned} \pi(w) &= \pi(\alpha(v)) \\ &= \chi(\varphi(v)) \\ &= \chi(\varphi(\varepsilon)) \\ &= \pi(\varepsilon), \end{aligned}$$

we obtain that $w \in \ker\pi$. Hence, $\alpha(\ker\varphi) \subseteq \ker\pi$, i.e. $\alpha(\mathcal{H}) \subseteq \mathcal{D}$.

Lemma 18. *The morphism α defined in Lemma 17 satisfies $\alpha(\mathcal{Q}) \subseteq \mathcal{P}$.*

Proof. Take $q \in \mathcal{Q}$. By Lemma 17, we have $\alpha(q) \in \mathcal{D}$. Using the fact that α does not decrease length, we have that $\alpha(q) \neq \varepsilon$, because $\varepsilon \notin \mathcal{Q}$.

Suppose that $\alpha(q) = u\,v$, for some $u, v \in \mathcal{D}\backslash\{\varepsilon\}$. It follows that $q = \hat{u}\,\hat{v}$ for some $\hat{u}, \hat{v} \in \Gamma^*$ satisfying $\alpha(\hat{u}) = u$ and $\alpha(\hat{v}) = v$. Using the commutative diagram A.8, the fact that χ is an isomorphism and the equalities,

$$\begin{aligned}\varphi(\hat{u}) &= \chi^{-1}(\pi(\alpha(\hat{u}))) \\ &= \chi^{-1}(\pi(u)) \\ &= \chi^{-1}(\pi(\varepsilon)) \\ &= \varphi(\varepsilon),\end{aligned}$$

we obtain that $\hat{u} \in \ker\varphi = \mathcal{H}$. Similarly, $\hat{v} \in \ker\varphi = \mathcal{H}$. Hence, $q \notin \mathcal{Q}$, contrary to our hypothesis. By *reductio ad absurdum*, $\alpha(\mathcal{Q}) \subseteq \mathcal{P}$.

Lemma 19. *Given $w \in \mathcal{H}$, we have $\Theta(w) = \Omega(\alpha(w))$, where α is the morphism defined in Lemma 17, Θ is given by diagram (A.6) and Ω is given by diagram (A.5).*

Proof. Notice that the diagram

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{Q}^* \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{P}^* \end{array}$$

commutes, where $\mathcal{D} \longrightarrow \mathcal{P}^*$ is the identification of $\mathcal{D}$ with the free monoid $\mathcal{P}^*$, $\mathcal{H} \longrightarrow \mathcal{Q}^*$ is the identification of $\mathcal{H}$ with the free monoid $\mathcal{Q}^*$, $\mathcal{Q}^* \longrightarrow \mathcal{P}^*$ is the morphism of monoids given by $w \mapsto \alpha(w)$ for all $w \in \mathcal{Q}$ (this function is well-defined in virtue of Lemma 18) and $\mathcal{H} \longrightarrow \mathcal{D}$ is given by $w \mapsto \alpha(w)$ (this function is well-defined in virtue of Lemma 17). It follows that $\Theta(w) = \Omega(\alpha(w))$ holds for each $w \in \mathcal{H}$.

The following construction was previously used in (2).

Definition 2. Given a finite set of positive real numbers S, let ν_0, ν_1, ..., ν_{r-1} be the elements of the union $S \cup \lambda S$ written in increasing order, i.e.

$$S \cup \lambda S = \{\nu_0 < \nu_1 < ... < \nu_{r-1}\}.$$

Consider the word

$$[\![S]\!]_\lambda := u_0\, u_1\, u_2 \ldots u_{r-1} \in \Gamma^*,$$

where each letter is given by

$$u_i := \begin{cases} a & \text{if } \nu_i \in S \setminus (\lambda S), \\ b & \text{if } \nu_i \in (\lambda S) \setminus S, \\ c & \text{if } \nu_i \in S \cap \lambda S, \end{cases}$$

for all $0 \leq i \leq r-1$.

Example 2. The Dyck path corresponding to $\langle\!\langle 126 \rangle\!\rangle_2 = aabaababbabb$ is shown in Fig A.1. The Schröder path corresponding to $[\![126]\!]_2 = acabcaabccabbcabcb$ is shown in Fig A.2.

Lemma 20. *Consider a finite set of positive real numbers S. For any real number $\lambda > 1$ we have $[\![S]\!]_\lambda \in \mathcal{H}$.*

Proof. We proceed by induction on the number of elements of S, denoted $m := \#S$.

For $m = 0$, we have $[\![S]\!]_\lambda = \varepsilon \in \mathcal{H}$.

Given $m > 0$, suppose that for each finite set of positive real numbers S, we have $[\![S]\!]_\lambda \in \mathcal{H}$, provided that $\#S < m$. Take an arbitrary finite set of real numbers S having precisely $\#S = m$ elements. Denote ν_0, ν_1, ν_2,..., ν_{r-1} the elements of $S \cup \lambda S$ written in increasing order. Consider the word $[\![S]\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{r-1}$ as given in Definition 2.

The inequality $\lambda > 1$ implies that there exists at least one integer i satisfying $u_i \neq a$ and $1 \leq i \leq r-1$. Define $j := \min\{i : \quad u_i \neq a \text{ and } 1 \leq i \leq r-1\}$.

Suppose that $u_j = b$. Setting $S' := S\backslash\{\nu_0\}$, we have

$$[\![S']\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{j-2}\, \widehat{u}_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1},$$

where the hat indicates that the corresponding letter is suppressed. Indeed, $\lambda\nu_0 = \nu_j$ and $\nu_0 = \nu_1 = ... = \nu_{j-1} = a$.

By induction hypothesis, $[\![S']\!]_\lambda \in \mathcal{H}$. Hence, $[\![S]\!]_\lambda \in \mathcal{H}$, because it can be transformed into $[\![S']\!]_\lambda \in \mathcal{H}$ using the relation $a\,b = \varepsilon$ from $\mathcal{C}$.

Suppose that $u_j = c$ and $u_{j+1} = b$. Setting $S' := S\backslash\{\nu_0\}$, we have

$$[\![S']\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1}.$$

By induction hypothesis, $[\![S']\!]_\lambda \in \mathcal{H}$. Hence, $[\![S]\!]_\lambda \in \mathcal{H}$, because it can be transformed into $[\![S']\!]_\lambda \in \mathcal{H}$ using the relation $a\,c\,b = a\,b$.

Suppose that $u_j = c$ and $u_{j+1} = c$. Setting $S' := S\backslash\{\nu_0\}$, we have

$$[\![S']\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1}.$$

By induction hypothesis, $[\![S']\!]_\lambda \in \mathcal{H}$. Hence, $[\![S]\!]_\lambda \in \mathcal{H}$, because it can be transformed into $[\![S']\!]_\lambda \in \mathcal{H}$ using the relation $c\,c = c$.

Finally, suppose that $u_j = c$ and $u_{j+1} = a$. Setting $S' := S\backslash\{\nu_0\}$, we have

$$[\![S']\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1}.$$

By induction hypothesis, $[\![S']\!]_\lambda \in \mathcal{H}$. Then using the rewriting rules from $\mathcal{C}$, the word

$$u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1}$$

can be reduced to

$$u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{i_1}\ u_{i_2} \ldots u_{i_h},$$

where $u_{i_1} = b$, and the word obtained after the reduction $u_{j-1}\, u_{i_1} = \varepsilon$,

$$u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{i_2} \ldots u_{i_h},$$

can be reduced to the empty word using the rewriting rules from $\mathcal{C}$. So, using the rewriting rules from $\mathcal{C}$, the original word $[\![S]\!]_\lambda$ can be reduced to

$$u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, u_j\, u_{i_1}\ u_{i_2} \ldots u_{i_h},$$

and the word obtained after the reduction $u_{j-1}\, u_j\, u_{i_1} = a\, c\, b = a\, b = \varepsilon$, can be reduced to the empty word as we mentioned above. Hence, $[\![S]\!]_\lambda \in \mathcal{H}$.

By the principle of induction, we conclude that $[\![S]\!]_\lambda \in \mathcal{H}$ for any finite set of positive real numbers S.

Lemma 21. *Consider a finite set of positive real numbers S. For any real number $\lambda > 1$, we have $\gamma([\![S]\!]_\lambda) = \langle\!\langle S \rangle\!\rangle_\lambda$, where γ is the morphism defined in Lemma 14.*

Proof. In virtue of the identity $(S \cup \lambda S) \backslash (S \cap \lambda S) = S \triangle \lambda S$, the result follows just combining Definition 1 and Definition 2.

Example 3. Lemma 21 can be illustrated by means of Fig A.1 and Fig A.2.

Lemma 22. *Consider a finite set of positive real numbers S. For any real number $\lambda > 1$, the equality $\alpha([\![S]\!]_\lambda) = \langle\!\langle S \rangle\!\rangle_{\lambda'}$ holds for all $\lambda' \in]\lambda, +\infty[$ near enough to λ, where α is the morphism defined in Lemma 17.*

Proof. For any $\lambda' \in]\lambda, +\infty[$, the change from $S \cup \lambda S$ to $S \cup \lambda' S$ keeps fixed the points in S and it displaces the points in λS to the right. This displacement to the right can be made as small as we want just setting λ' near enough to λ. In particular, any point in $S \cap \lambda S$, after this transformation, becomes a pair of different points, one stays at the original position and the other one displaces to the right an arbitrary small distance. Notice that $S \cap \lambda' S = \emptyset$ for all $\lambda' \in]\lambda, +\infty[$ near enough to λ (this guarantees that λ' will be regular). Combining Definition 1 and Definition 2, we conclude that $\alpha([\![S]\!]_\lambda) = \langle\!\langle S \rangle\!\rangle_{\lambda'}$ provided that $\lambda' \in]\lambda, +\infty[$ is near enough to λ.

Example 4. Lemma 22 can be illustrated by means of Fig A.2 and Fig A.3.

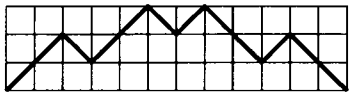

FIGURE A.1 Representation of $\langle\!\langle 126 \rangle\!\rangle_2 = aabaababbabb$.

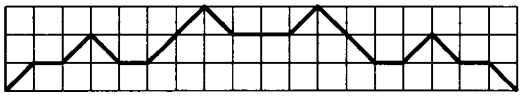

FIGURE A.2 Representation of $[\![126]\!]_2 = acabcaabccabbcabcb$.

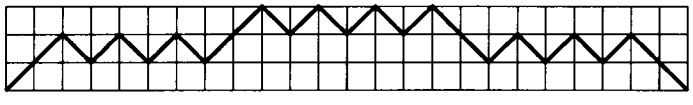

FIGURE A.3 Representation of $[\![126]\!]_{2.001} = \langle\!\langle 126 \rangle\!\rangle_{2.001} = aababababaababababbababababb$.

Lemma 23. *Let S be a finite set of positive real numbers. The step function $]1, +\infty[\longrightarrow \mathbb{N}$, given by $\lambda \mapsto \Omega(\langle\!\langle S \rangle\!\rangle_\lambda)$, is continuous from the right, i.e. given a real number $\lambda > 1$, for each real number $\lambda' \in]\lambda, +\infty[$, we have $\Omega(\langle\!\langle S \rangle\!\rangle_\lambda) = \Omega(\langle\!\langle S \rangle\!\rangle_{\lambda'})$, provided that λ' is near enough to λ.*

Proof. By Lemma 20, $[\![S]\!]_\lambda \in \mathcal{H}$. By Lemma 21, $\gamma([\![S]\!]_\lambda) = \langle\!\langle S \rangle\!\rangle_\lambda$, where γ is the morphism defined in Lemma 14. Using Lemma 16 we obtain $\Theta([\![S]\!]_\lambda) = \Omega(\langle\!\langle S \rangle\!\rangle_\lambda)$. By Lemma 22, $\alpha([\![S]\!]_\lambda) = \langle\!\langle S \rangle\!\rangle_{\lambda'}$ for all $\lambda' \in]\lambda, +\infty[$ near enough to λ, where α is the morphism defined in Lemma 17. Using Lemma 19 we obtain $\Theta([\![S]\!]_\lambda) = \Omega(\langle\!\langle S \rangle\!\rangle_{\lambda'})$ for all $\lambda' \in]\lambda, +\infty[$ near enough to λ. Therefore, $\Omega(\langle\!\langle S \rangle\!\rangle_\lambda) = \Omega(\langle\!\langle S \rangle\!\rangle_{\lambda'})$ for all $\lambda' \in]\lambda, +\infty[$ near enough to λ.

Lemma 24. *Let L be a finite set of real numbers. Consider the step function $f :]0, +\infty[\longrightarrow \mathbb{N}$ such that $f_L(t)$ is the number of connected components of $\mathcal{T}(L;t)$. The function $f_L(t)$ is continuous from the right, i.e. given a real number $t > 0$ we have $f_L(t') = f_L(t)$ for all $t' \in]t, +\infty[$ near enough to t.*

Proof. Let $\ell_0, \ell_1, \ell_2, ..., \ell_{k-1}$ be the elements of L written in increasing order, i.e.

$$L = \{\ell_0 < \ell_1 < \ell_2 < ... < \ell_{k-1}\}.$$

Denote $c := f_L(t)$. In virtue of (A.1), we can write $\mathcal{T}(L;t)$ as the union

$$\mathcal{T}(L,t) = [\ell_{i_1}, \ell_{i_2} + t] \cup [\ell_{i_3}, \ell_{i_4} + t] \cup ... \cup \left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t\right]$$

of the pairwise disjoint sets $[\ell_{i_1}, \ell_{i_2} + t]$, $[\ell_{i_3}, \ell_{i_4} + t]$, ..., $\left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t\right]$, for some subsequence $i_1 < i_2 < i_3 < i_4 < ... < i_{2c-1} < i_{2c}$ of $0, 1, 2, ..., k-1$. So, for all $t' \in]t, +\infty[$, the set $\mathcal{T}(L;t')$ can be expressed as the union

$$\mathcal{T}(L,t') = [\ell_{i_1}, \ell_{i_2} + t'] \cup [\ell_{i_3}, \ell_{i_4} + t'] \cup ... \cup \left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t'\right],$$

where some of sets in the list $[\ell_{i_1}, \ell_{i_2} + t']$, $[\ell_{i_3}, \ell_{i_4} + t']$, ..., $\left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t'\right]$ may overlap among them. Assuming that t' is near enough to t, we guarantee that the sets $[\ell_{i_1}, \ell_{i_2} + t']$, $[\ell_{i_3}, \ell_{i_4} + t']$, ..., $\left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t'\right]$ are pairwise disjoint. Hence,

$f_L(t) = f_L(t')$ for all $t' \in]t, +\infty[$ near enough to t. Therefore, $f_L(t)$ is continuous from the right.

Using the previous auxiliary results, we can prove Proposition 3.

Proof.(Proposition 3) By Lemma 23, the step function $]1, +\infty[\longrightarrow \mathbb{N}$, given by $\lambda \mapsto \Omega(\langle\langle S \rangle\rangle_\lambda)$, is continuous from the right. By Lemma 24, the step function $f_L :]0, +\infty[\longrightarrow \mathbb{N}$ is continuous from the right, where $f_L(t)$ is the number of connected components of $\mathcal{T}(L;t)$. Notice that the step function $]1, +\infty[\longrightarrow \mathbb{N}$, given by $\lambda \mapsto f_L(\ln \lambda) - \Omega(\langle\langle S \rangle\rangle_\lambda)$, is continuous from the right, because the natural logarithm is continuous on $]0, +\infty[$. By Lemma 13, $f_L(\ln \lambda') - \Omega(\langle\langle S \rangle\rangle_{\lambda'}) = 0$ for all $\lambda' \in]\lambda, +\infty[$ near enough to λ (this guarantees that λ' is regular). Hence, $f_L(\ln \lambda) - \Omega(\langle\langle S \rangle\rangle_\lambda) = 0$ follows by continuity from the right. Therefore, the space $\mathcal{T}(L;t)$ has precisely $\Omega(\langle\langle S \rangle\rangle_\lambda)$ connected components.

Proposition 5. *Given a real number $\lambda > 1$, an integer $n \geq 1$ is λ-densely divisible if and only if $\mathcal{T}_\lambda(n)$ is connected.*

Proof. Suppose that n is λ-densely divisible and $\mathcal{T}_\lambda(n)$ is disconnected. In virtue of (A.1), there are two divisors of n, denoted $d < d'$, satisfying

$$\ln d + \ln \lambda < \ln d'$$

and there is no divisor of n on the interval $]d, d'[$. Using the fact that n is λ-densely divisible, there is a divisor of n on the interval $[\lambda^{-1} R, R]$, with $1 \leq R := \lambda(d + \epsilon) < d' \leq n$, for all $\epsilon > 0$ small enough. Notice that $[\lambda^{-1} R, R] \subseteq]d, d'[$. So, there is a divisor of n on the interval $]d, d'[$. By *reductio ad absurdum*, if n is λ-densely divisible then $\mathcal{T}_\lambda(n)$ is connected.

Now, suppose that $\mathcal{T}_\lambda(n)$ is connected and n is not λ-densely divisible. Then there is some $R \in [1, n]$ such that there is no divisor of n on the interval $[\lambda^{-1} R, R]$. It

follows that $R > \lambda > 1$, because 1 is a divisor of n. Let d be the largest divisor of n satisfying $d \leq \lambda^{-1} R$. It follows that $d < n$, because $\lambda^{-1} R \leq \lambda^{-1} n < n$. Let d' be the smallest divisor of n satisfying $\lambda^{-1} R < d'$. Notice that $\lambda^{-1} R < d'$, $\lambda d \leq R$ and there is no divisor of n on the interval $]d, d'[$.

Using the fact that $\mathcal{T}_\lambda(n)$ is connected, we have that

$$[\ln d, \ln d + \ln \lambda] \cap [\ln d', \ln d' + \ln \lambda] \neq \emptyset.$$

It follows that $\ln d' \leq \ln d + \ln \lambda$, i.e. $d' \leq \lambda d$. So, $\lambda^{-1} R < d' \leq \lambda d \leq R$. In particular, $d' \in [\lambda^{-1} R, R]$. By *reductio ad absurdum*, if $\mathcal{T}_\lambda(n)$ is connected then n is λ-densely divisible.

We proceed now with the proof of the main result of this paper.

Proof.(Theorem 6) Statement (i) follows by Proposition 3 taking S to be the set of divisors of n.

Take an integer $n \geq 1$. By Proposition 5, n is λ-densely divisible if and only if $\mathcal{T}_\lambda(n)$ is connected. By Proposition 3, the space $\mathcal{T}_\lambda(n)$ is connected if and only if $\langle\langle n \rangle\rangle_\lambda$ is irreducible. Hence, n is λ-densely divisible if and only if $\langle\langle n \rangle\rangle_\lambda$ is irreducible. Therefore, statement (ii) holds.

A.5 Final remarks

Consider the finite field with q elements, denoted $\mathbb{F}_q$. Let $\mathbb{Z} \oplus \mathbb{Z}$ be the free abelian group of rank 2. For each integer $n \geq 1$, there is a unique polynomial $P_n(q)$ such that for any prime power q, the value of $P_n(q)$ is precisely the number of ideals I of $\mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]$ satisfying that $\mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]/I$ is an n-dimensional vector space over $\mathbb{F}_q$. It was already observed in (2) that, as a consequence of an explicit formula for $P_n(q)$ due to Kassel and Reutenauer (see (3) and (4)), the non-zero coefficients

of the polynomial $(1-q)\,P_n(q)$ are determined by the Dyck word $\langle\langle n\rangle\rangle_2$ (see (2)). Combining these results with Theorem 6, the following result can be easily derived (we leave the details of the proof as an exercise).

Theorem 7. *An integer $n \geq 1$ is 2-densely divisible if and only if all the coefficients of $P_n(q)$ are non-zero.*

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ANNEXE II

MIDDLE DIVISORS AND σ-PALINDROMIC DYCK WORDS

Abstract

Given a real number $\lambda > 1$, we say that $d|n$ is a λ-middle divisor of n if

$$\sqrt{\frac{n}{\lambda}} < d \leq \sqrt{\lambda n}.$$

We will prove that there are integers having an arbitrarily large number of λ-middle divisors.

Consider the word

$$\langle\langle n \rangle\rangle_\lambda := w_1 w_2 ... w_k \in \{a, b\}^*,$$

given by

$$w_i := \begin{cases} a & \text{if } u_i \in D_n \backslash (\lambda D_n), \\ b & \text{if } u_i \in (\lambda D_n) \backslash D_n, \end{cases}$$

where D_n is the set of divisors of n, $\lambda D_n := \{\lambda d : \quad d \in D_n\}$ and $u_1, u_2, ..., u_k$ are the elements of the symmetric difference $D_n \triangle \lambda D_n$ written in increasing order. We will prove that the language

$$\mathcal{L}_\lambda := \{\langle\langle n \rangle\rangle_\lambda : \quad n \in \mathbb{Z}_{\geq 1}\}$$

contains Dyck words having an arbitrarily large number of centered tunnels. We will show a connection between both results.

B.1 Introduction

In order to compute the local zeta function of the Hilbert scheme of n points on a two-dimensional torus, Kassel and Reutenauer introduced the self-reciprocal polynomial

$$P_n(q) := a_{n,0}q^n + \sum_{k=1}^{n-1} a_{n,k}\left(q^{n-1+k} + q^{n-1-k}\right)$$

in (4) and (5), where the coefficients are given by

$$a_{n,k} := \#\left\{d|n : \quad \frac{k+\sqrt{k^2+2n}}{2} < d \leq k+\sqrt{k^2+2n}\right\}.$$

The polynomial $P_n(q)$ is a q-analog of the sum of divisors of n (see (5)). Furthermore, taking $q = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$, for each $m \in \{2,3,4,6\}$, the corresponding arithmetical function $n \mapsto P_n(q)$ can be expressed in terms of classical multiplicative functions via modular forms (see (6)).

We can express $(q-1)\,P_n(q)$ as follows (see (3.2) in (5)),

$$\begin{aligned}(q-1)\,P_n(q) &= q^{n-1} \sum_{\substack{1 \leq k < m,\, mk = 2n \\ k \not\equiv m \pmod 2}} (-1)^{k-1}(q-1)\left[\left[-\frac{m-k-1}{2}, \frac{m-k-1}{2}\right]\right] \\ &= q^{n-1} \sum_{\substack{d|n \\ d \text{ odd}}} \left(q^{(2n/d-d+1)/2} - q^{-(2n/d-d-1)/2}\right),\end{aligned}$$

where $[[a,b]] := q^a + q^{a+1} + q^{a+2} \ldots + q^b$.

Notice that $(2n/d-d+1)/2 = (2n/d'-d'+1)/2$ or $(2n/d-d-1)/2 = (2n/d'-d'-1)/2$ implies that $d = d'$. Hence, the nonzero coefficients of $(q-1)\,P_n(q)$ are -1 and $+1$. So, it is natural to encode the nonzero coefficients of $(q-1)\,P_n(q)$ by a word

over the alphabet $\{a,b\}$. Indeed, in (2) the word [1]

$$\langle\langle n\rangle\rangle_\lambda := w_1 w_2 ... w_k \in \{a,b\}^*,$$

parametrized by a real number $\lambda > 1$, was defined for any integer $n \geq 1$ by means of the expression

$$w_i := \begin{cases} a & \text{if } u_i \in D_n \setminus (\lambda D_n), \\ b & \text{if } u_i \in (\lambda D_n) \setminus D_n, \end{cases}$$

where D_n is the set of divisors of n, $\lambda D_n := \{\lambda d : \quad d \in D_n\}$ and $u_1, u_2, ..., u_k$ are the elements of the symmetric difference $D_n \triangle \lambda D_n$ written in increasing order. For $\lambda = 2$ we recover the non-zero coefficients of $(q-1)\,P_n(q)$.

Interpreting the letters a and b as the parentheses "(" and ")" respectively, it follows that all the parentheses in the word $\langle\langle n\rangle\rangle_\lambda$ are well-matched (see (2)). More technically, $\langle\langle n\rangle\rangle_\lambda \in \mathcal{D}$, where $\mathcal{D}$ is the *Dyck language* over the alphabet $\{a,b\}$, i.e. the smallest subset of $\{a,b\}^*$, with respect to the inclusion, satisfying $\varepsilon \in \mathcal{D}$, $a\,\mathcal{D}\,b \subseteq \mathcal{D}$ and $\mathcal{D}\,\mathcal{D} \subseteq \mathcal{D}$.

Following (1), we say that a word $w \in \{a,b\}^*$ is *σ-palindromic* if $\widetilde{w} = \sigma(w)$, where $\widetilde{w}$ is the mirror image of w. Throughout this paper, σ will be the morphism of monoids given by $a \mapsto b$ and $b \mapsto a$. It was proved in (2) that the word $\langle\langle n\rangle\rangle_\lambda$ is σ-palindromic.

The formal language

$$\mathcal{L}_\lambda := \{\langle\langle n\rangle\rangle_\lambda : \quad n \in \mathbb{Z}_{\geq 1}\}$$

is connected to the theory of divisors of integers (in the sense of (7)) in a nontrivial way. For example, the fact that $\mathcal{L}_e$, where e is Euler's number, contains Dyck words

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1. The operator $*$ is the *Kleene star*. For any set A, the set of all strings over symbols in A, including the empty string ε, is denoted A^*.

having arbitrarily large height [2] is equivalent to the fact that Hooley's Δ-function [3] is unbounded (see (2)).

Consider the set $\mathcal{S} := \{aa, ab, ba, bb\}$ endowed with the binary operation, that we will call *central concatenation*,

$$u \triangleleft v := \varphi^{-1}(\varphi(u)\varphi(v)),$$

where $\varphi : \mathcal{S}^* \longrightarrow \mathcal{S}^*$ is the bijection given by

$$\begin{aligned} \varphi(\varepsilon) &= \varepsilon, \\ \varphi(x\,u\,y) &= (xy)\,\varphi(u), \end{aligned}$$

for all $x, y \in \{a, b\}$ and $u \in \mathcal{S}^*$. It is easy to check that $(\mathcal{S}, \triangleleft)$ is a monoid freely generated by $\mathcal{S}$ and having ε as identity element.

Example 5. For simplicity, we will identify the letters with the parentheses as follows $a \leftrightarrow$ "(" and $b \leftrightarrow$ ")". Consider the following examples of a product in $(\mathcal{S}, \triangleleft)$,

$$((())) \triangleleft (()()) = (((((()))))), \quad (()()) \triangleleft ((())) = (()((()))()).$$

It is easy to check that $(\mathcal{D}, \triangleleft)$ is a submonoid of $(\mathcal{S}^*, \triangleleft)$, which is freely generated by the set of what we will call *centrally irreducible Dyck words*,

$$\mathcal{I} := \mathcal{D}_\bullet \setminus (\mathcal{D}_\bullet \triangleleft \mathcal{D}_\bullet).$$

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2. The height of a Dyck word is the largest number of nested matched parentheses in it.

3. This function, denoted $\Delta(n)$, is the largest number of $d|n$ that can be put simultaneously on an interval of the form $]t, et]$ for all real numbers $t > 1$.

where $\mathcal{D}_\bullet := \mathcal{D}\backslash\{\varepsilon\}$. Let $\mathrm{ct} : (\mathcal{D}, \lhd) \longrightarrow (\mathbb{Z}_{\geq 0}, +)$ be the morphism of monoids given by

$$\mathrm{ct}\,(w) := \begin{cases} 1 & \text{if } w = ab, \\ 0 & \text{if } w \neq ab, \end{cases}$$

for all $w \in \mathcal{I}$. Following (3), $\mathrm{ct}(w)$ will be called the number of *centered tunnels* of w.

Example 6. Consider the following examples of centered tunnels,

$$\underline{(()())}, \quad (\underline{()}\,\underline{()}).$$

Then, $\mathrm{ct}(w) = 2$ for $w = (()())$.

The aim of this paper is to prove the following language-theoretic result, implicitly related to the multiplicative structure of positive integers.

Theorem 8. *For any real number $\lambda > 1$, the language $\mathcal{L}_\lambda$ contains Dyck words having arbitrarily large number of centered tunnels i.e. the set $\mathrm{ct}\,(\mathcal{L}_\lambda)$ is infinite.*

In order to prove Theorem 8, we will generalize the definition of the so-called middle divisors (see (5) and (9)). We say that $d|n$ is a *λ-middle divisor*, where $\lambda > 1$ is a real parameter, if

$$\sqrt{\frac{n}{\lambda}} < d \leq \sqrt{\lambda n}.$$

What is traditionally called a *middle divisor* is nothing but a 2-middle divisors as defined above. Notice that $a_{n,0}$, the central coefficient of $P_n(q)$, counts the middle divisors of n. Vatne (9) proved that the sequence $n \mapsto a_{n,0}$ is unbounded. We will derive Theorem 8 from the following generalization of Vatne's result.

Theorem 9 (Generalized Vatne's Theorem). *For any real number $\lambda > 1$ and any integer $M \geq 1$, there is an integer $n \geq 1$ having at least M λ-middle divisors.*

B.2 Generalized Vatne's Theorem

Lemma 25. *Given a real number $\lambda > 1$ and two positive integers a and b, define*

$$S_\lambda(a,b) := \{(x,y) \in S_\lambda : \quad |x| \leq a \text{ and } |y| \leq b\},$$

where $S_\lambda := \left\{(x,y) \in \mathbb{Z}^2 : \quad -\frac{\ln\lambda}{2} < (\ln 2)\,x + (\ln 3)\,y \leq \frac{\ln\lambda}{2}\right\}$. *The limit*

$$\lim_{\min\{a,b\}\to+\infty} \#S_\lambda(a,b) = +\infty$$

holds.

Proof. In virtue of the Fundamental Theorem of Arithmetic, $\frac{\ln 3}{\ln 2}$ is irrational (an integer $\neq 1$ cannot be simultaneously a power of 2 and a power of 3). Using Equidistribution Theorem we guarantee that any real number can be arbitrarily approximated by $x + \frac{\ln 3}{\ln 2}y$, with $(x,y) \in \mathbb{Z}^2$. So, the set

$$\left\{(\ln 2)\,x + (\ln 3)\,y : \quad (x,y) \in \mathbb{Z}^2\right\}$$

is dense in $\mathbb{R}$. In particular, S_λ is infinite.

The equality

$$S_\lambda = \bigcup_{n\geq 1} S_\lambda(a_n, b_n).$$

holds for any sequence of pairs of positive integers (a_n, b_n) satisfying

$$\min\{a_n, b_n\} \to +\infty.$$

It follows that $\#S_\lambda(a,b) \to +\infty$ as $\min\{a,b\} \to +\infty$. □

We proceed to prove the generalization of Vatne's result.

Proof.(Theorem 9)

Take a real number $\lambda > 1$ and an integer $M \geq 1$. Let $S_\lambda(a,b)$ be the set defined in Lemma 25. Take two integers a and b such that $\#S_\lambda(a,b) > M$. Define

$$n := \left(2^a \cdot 3^b\right)^2.$$

In virtue of the Fundamental Theorem of Arithmetic, for all $(x,y) \in S_\lambda(a,b)$ we have a unique divisor

$$d_{x,y} := 2^{a+x} \cdot 3^{b+y}$$

of n. Notice that the inequalities $-\frac{\ln\lambda}{2} < (\ln 2)\,x + (\ln 3)\,y \leq \frac{\ln\lambda}{2}$ are equivalent to

$$\frac{1}{\sqrt{\lambda}} < 2^x \cdot 3^y \leq \sqrt{\lambda}.$$

So,

$$\underbrace{\frac{1}{\sqrt{\lambda}}\left(2^a \cdot 3^b\right)}_{\sqrt{\frac{n}{\lambda}}} < \underbrace{\left(2^x \cdot 3^y\right)\left(2^a \cdot 3^b\right)}_{d_{x,y}} \leq \underbrace{\sqrt{\lambda}\left(2^a \cdot 3^b\right)}_{\sqrt{\lambda n}}.$$

It follows that d is a λ-middle divisor of n. Therefore, n has at least M λ-middle divisors. □

B.3 Language-theoretic interpretation of middle divisors

B.3.1 Language-theoretic preliminaries

For any $x \in \mathcal{S}$, let

$$\ell_x : (\mathcal{S}^*, \triangleleft) \longrightarrow (\mathbb{Z}_{\geq 0}, +)$$

be the unique morphism of monoids satisfying

$$\ell_x(y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$$

for all $y \in \mathcal{S}$.

Let $(\mathcal{S}^*)^{\sigma\text{-pal}}$ be the set of σ-palindromic words from S^*. It is easy to check that $\left((\mathcal{S}^*)^{\sigma\text{-pal}}, \triangleleft\right)$ is a submonoid of $(\mathcal{S}^*, \triangleleft)$, which is freely generated by $\mathcal{S}^{\sigma\text{-pal}} := \{ab, ba\}$.

We will use the notation $\cdot$ for the concatenation (which is implicit in practice). There is a unique isomorphism of monoids

$$\mathfrak{h} : \left((\mathcal{S}^*)^{\sigma\text{-pal}}, \triangleleft\right) \longrightarrow (\{a, b\}^*, \cdot)$$

such that $w \mapsto \mathfrak{h}_w$, where $\mathfrak{h}_w$ is the unique word satisfying

$$w = \mathfrak{h}_w \, \sigma(\widetilde{\mathfrak{h}_w}).$$

Lemma 26. *For all* $w \in (\mathcal{S}^*)^{\sigma\text{-pal}}$,

$$\ell_{ab}(w) = |\mathfrak{h}_w|_a \text{ and } \ell_{ba}(w) = |\mathfrak{h}_w|_b.$$

Proof. It immediately follows that $\ell_{ab}(\varepsilon) = |\mathfrak{h}_\varepsilon|_a$.

For all $w \in (\mathcal{S}^*)^{\sigma\text{-pal}}$,

$$\begin{aligned} & \ell_{ab}(a\,w\,b) = \ell_{ab}((ab) \triangleleft w) = 1 + \ell_{ab}(w) \\ = \;& 1 + |\mathfrak{h}_w|_a = |a\,\mathfrak{h}_w|_a = |\mathfrak{h}_{awb}|_a, \\ & \ell_{ab}(b\,w\,a) = \ell_{ab}((ba) \triangleleft w) = 0 + \ell_{ab}(w) \\ = \;& 0 + |\mathfrak{h}_w|_a = |b\mathfrak{h}_w|_a = |\mathfrak{h}_{bwa}|_a. \end{aligned}$$

Using the fact that $\left((\mathcal{S}^*)^{\sigma\text{-pal}}, \triangleleft\right)$ is freely generated by $\{ab, ba\}$, we conclude that $\ell_{ab}(w) = |\mathfrak{h}_w|_a$, for all $w \in (\mathcal{S}^*)^{\sigma\text{-pal}}$.

In a similar way, it is easy to prove that, $\ell_{ba}(w) = |\mathfrak{h}_w|_b$, for all $w \in (\mathcal{S}^*)^{\sigma\text{-pal}}$. □

Let $\mathcal{D}^{\sigma\text{-pal}}$ be the set of σ-palindromic words from $\mathcal{D}$. We have that $\left(\mathcal{D}^{\sigma\text{-pal}}, \triangleleft\right)$ is a submonoid of $(\mathcal{D}, \triangleleft)$, which is freely generated by

$$\mathcal{I}^{\sigma\text{-pal}} := \{ab\} \cup \{u\,\sigma(\tilde{u}) : \quad u \in \mathcal{D}_\bullet\}.$$

Lemma 27. *For all $w \in (\mathcal{S}^*)^{\sigma\text{-pal}}$,*

$$|w| = 2\ell_{ab}(w) + 2\ell_{ba}(w).$$

Proof. For all $w \in (\mathcal{S}^*)^{\sigma\text{-pal}}$,

$$\begin{aligned} |w| &= |w|_a + |w|_b \\ &= |\mathfrak{h}_w|_a + \left|\sigma\left(\widetilde{\mathfrak{h}_w}\right)\right|_a + |\mathfrak{h}_w|_b + \left|\sigma\left(\widetilde{\mathfrak{h}_w}\right)\right|_b, \quad \text{by definition of } \mathfrak{h}, \\ &= |\mathfrak{h}_w|_a + |\mathfrak{h}_w|_b + |\mathfrak{h}_w|_b + |\mathfrak{h}_w|_a \quad \text{because } w \text{ is } \sigma\text{-palindromic} \\ &= 2|\mathfrak{h}_w|_a + 2|\mathfrak{h}_w|_b \\ &= 2\ell_{ab}(w) + 2\ell_{ba}(w), \quad \text{by Lemma 26.} \end{aligned}$$

□

Lemma 28. *For all* $w \in \mathcal{D}^{\sigma\text{-}pal}$,

$$\mathrm{ct}\,(w) = \ell_{ab}\,(w) - \ell_{ba}\,(w)\,.$$

Proof. Consider the morphism of monoids $f : \left(\mathcal{D}^{\sigma\text{-pal}}, \triangleleft\right) \longrightarrow (\mathbb{Z}_{\geq 0}, +)$ given by $f\,(w) := \ell_{ab}\,(w) - \ell_{ba}\,(w)$.

Notice that, $f\,(ab) = \ell_{ab}\,(ab) - \ell_{ba}\,(ab) = 1 - 0 = 1$.

Take $w \in \mathcal{I}^{\sigma\text{-pal}} \backslash \{ab\}$. We have that $|\mathfrak{h}_w|_a = |\mathfrak{h}_w|_b$, because $\mathfrak{h}_w$ is a Dyck word for all $w \in \mathcal{I}^{\sigma\text{-pal}} \backslash \{ab\}$. In virtue of Lemma 26,

$$f(w) = |\mathfrak{h}_w|_a - |\mathfrak{h}_w|_b = 0.$$

Using the fact that $\mathcal{D}^{\sigma\text{-pal}}$ is freely generated by $\mathcal{I}^{\sigma\text{-pal}}$, we conclude that $f(w) = \mathrm{ct}(w)$ for all $w \in \mathcal{D}^{\sigma\text{-pal}}$. □

B.3.2 Middle divisors

Lemma 29. *For any integer* $n \geq 1$ *and any real number* $\lambda > 1$, *we have*

$$\ell_{ab}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right) = \#\left\{d|n : \quad d \notin \lambda D_n \text{ and } d < \sqrt{\lambda n}\right\},$$

where D_n *is the set of divisors of* n.

Proof. By definition of $\langle\!\langle n \rangle\!\rangle_\lambda$, the number of $d|n$ satisfying that $d \notin \lambda D_n$ and $d < \lambda \frac{n}{d}$ is precisely $|\mathfrak{h}_w|_a$. The inequality $d < \lambda \frac{n}{d}$ is equivalent to $d < \sqrt{\lambda n}$. So,

$$|\mathfrak{h}_w|_a = \#\left\{d|n : \quad d \notin \lambda D_n \text{ and } d < \sqrt{\lambda n}\right\}.$$

In virtue of Lemma 26, $\ell_{ab}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right) = |\mathfrak{h}_w|_a$. Hence,

$$\ell_{ab}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right) = \#\left\{d|n : \quad d \notin \lambda D_n \text{ and } d < \sqrt{\lambda n}\right\}.$$

□

Lemma 30. *For any integer $n \geq 1$ and any real number $\lambda > 1$, we have*

$$\ell_{ba}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right) = \#\left\{d|n: \quad \frac{n}{d} \notin \lambda D_n \text{ and } d < \sqrt{\frac{n}{\lambda}}\right\},$$

where D_n is the set of divisors of n.

Proof. Notice that $\sqrt{\lambda n} \notin D_n \setminus (\lambda D_n)$, because the equality $d = \sqrt{\lambda n}$, with $d|n$, implies that $d = \lambda \frac{n}{d} \in \lambda D_n$. It follows that

$$\begin{aligned}
& \ell_{ba}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right) \\
= \; & \frac{1}{2}\left|\langle\!\langle n \rangle\!\rangle_\lambda\right| - \ell_{ab}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right), \text{ by Lemma 27,} \\
= \; & \#D_n \setminus (\lambda D_n) - \ell_{ab}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right), \text{ by definition of } \langle\!\langle n \rangle\!\rangle_\lambda, \\
= \; & \#D_n \setminus (\lambda D_n) - \#\left\{d|n: \quad d \notin \lambda D_n \text{ and } d < \sqrt{\lambda n}\right\}, \text{ by Lemma 29,} \\
= \; & \#\left\{d|n: \quad d \notin \lambda D_n \text{ and } d \geq \sqrt{\lambda n}\right\} \\
= \; & \#\left\{d|n: \quad d \notin \lambda D_n \text{ and } d > \sqrt{\lambda n}\right\}, \text{ because } \sqrt{\lambda n} \notin D_n \setminus (\lambda D_n).
\end{aligned}$$

Expressing d as $\frac{n}{d'}$ for some $d'|n$,

(i) the condition $d \notin \lambda D_n$ becomes $\frac{n}{d'} \notin \lambda D_n$,

(ii) the inequality $d > \sqrt{\lambda n}$ becomes $d' < \sqrt{\frac{n}{\lambda}}$.

So, the equality

$$\ell_{ba}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right) = \#\left\{d|n: \quad \frac{n}{d} \notin \lambda D_n \text{ and } d < \sqrt{\frac{n}{\lambda}}\right\}$$

follows. □

Lemma 31. *Let $\lambda > 1$ be a real number. For each integer $n \geq 1$, there is a bijection $A_n \longrightarrow B_n$, where*

$$A_n := \left\{ d|n : \quad d \in \lambda D_n \text{ and } d < \sqrt{\lambda n} \right\},$$
$$B_n := \left\{ d|n : \quad \frac{n}{d} \in \lambda D_n \text{ and } d < \sqrt{\frac{n}{\lambda}} \right\}.$$

Proof. For any $d \in A_n$ we have,

(i) $\frac{d}{\lambda} < \frac{1}{\lambda}\sqrt{\lambda n} = \sqrt{\frac{n}{\lambda}}$, because $d < \sqrt{\lambda n}$,

(ii) $\frac{d}{\lambda}$ is a divisor of n, because $d \in \lambda D_n$,

(iii) $\frac{n}{\frac{d}{\lambda}} = \lambda \frac{n}{d} \in \lambda\, D_n$, because d is a divisor of n.

It follows that the function $f_n : A_n \longrightarrow B_n$, given by $f(d) := \frac{d}{\lambda}$, is well-defined. Also, f is injective, because it is strictly increasing.

For all $k \in B_n$ we have,

(i) $\lambda k < \lambda\sqrt{\frac{n}{\lambda}} = \sqrt{\lambda n}$, because $k < \sqrt{\frac{n}{\lambda}}$,

(ii) $\lambda k \in \lambda D_n$, because k is a divisor of n,

(iii) λk is a divisor of n, because $\lambda k = \frac{n}{\frac{n}{\lambda k}}$ and $\frac{n}{\lambda k}$ is a divisor of n in virtue of the condition $\frac{n}{k} \in \lambda D_n$.

It follows that $\lambda k \in A_n$. So, $f_n(\lambda k) = k$. Hence, f is onto.

Therefore, f_n is a bijection. □

Lemma 32. *Let $\lambda > 1$ be a real number. For any integer $n \geq 1$, the number of centered tunnels in $\langle\langle n \rangle\rangle_\lambda$ is precisely the number of λ-middle divisors of n.*

Proof. Consider the sets A_n and B_n defined in Lemma 31 and satisfying $\#A_n = \#B_n$. The number of middle divisors of n can be expressed as

$$\begin{aligned}
& \#\left\{d|n: \quad \sqrt{\frac{n}{\lambda}} < d \leq \sqrt{\lambda n}\right\} = \#\left\{d|n: \quad \sqrt{\frac{n}{\lambda}} \leq d < \sqrt{\lambda n}\right\} \\
= \; & \#\left\{d|n: \quad d < \sqrt{\lambda n}\right\} - \#\left\{d|n: \quad d < \sqrt{\frac{n}{\lambda}}\right\} \\
= \; & \left(\#\left\{d|n: \quad d < \sqrt{\lambda n}\right\} - \#A_n\right) - \left(\#\left\{d|n: \quad d < \sqrt{\frac{n}{\lambda}}\right\} - \#B_n\right) \\
= \; & \#\left\{d|n: \quad d \notin \lambda D_n \text{ and } d < \sqrt{\lambda n}\right\} \\
& -\#\left\{d|n: \quad \frac{n}{d} \notin \lambda D_n \text{ and } d < \sqrt{\frac{n}{\lambda}}\right\} \\
= \; & \ell_{ab}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right) - \ell_{ba}\left(\langle\!\langle n \rangle\!\rangle_\lambda\right), \quad \text{by Lemmas 29 and 30} \\
= \; & c\left(\langle\!\langle n \rangle\!\rangle_\lambda\right), \quad \text{by Lemma 28.}
\end{aligned}$$

□

We proceed to prove our main language-theoretic result.

Proof.(Theorem 8)

Take a real number $\lambda > 1$ and an integer $M \geq 1$. In virtue of Theorem 9, there is an integer $n \geq 1$ having at least M λ-middle divisors. In virtue of Lemma 32, $\langle\!\langle n \rangle\!\rangle_\lambda \in \mathcal{L}_\lambda$ has at least M centered tunnels. □

B.4 Final remarks

1. For all $n \geq 1$, $\ell_{ab}\left(\langle\!\langle n \rangle\!\rangle_2\right)$ the number of partitions of n into an *odd* number of consecutive parts and $\ell_{ba}\left(\langle\!\langle n \rangle\!\rangle_2\right)$ is the number of partitions of n into an *even* number of consecutive parts. Both results immediately follow combining Lemmas 29 and 30 with the main result from the paper (8).
2. Let c_{00} be the real vector space of sequences of real numbers $\mathbf{x} = (x_k)_{k \geq 1}$

satisfying $x_k = 0$ for all k large enough. Consider the lineal functional $L : c_{00} \longrightarrow \mathbb{R}$ given by

$$L\mathbf{x} := \sum_{k=1}^{\infty} (\ln p_k)\, x_k,$$

where p_k is the k-th prime number (we consider $p_1 = 2$).

Let $c_{00}(\mathbb{Z})$ be the $\mathbb{Z}$-module consisting of $\mathbf{x} = (x_k)_{k\geq 1} \in c_{00}$ for which $x_k \in \mathbb{Z}$ for all $k \geq 1$. The method that we used to prove the Generalized Vatne Theorem may be extended, in order to obtain more precise results, by considering sequences $\left(\mathbf{x}^{(n)}\right)_{n\geq 1}$ of elements in $c_{00}(\mathbb{Z})$ satisfying $\lim_{n\to+\infty} L\mathbf{x}^{(n)} = 0$.

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