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ELEMENTARY EQUIVALENCE AND CODIMENSION IN p-ADIC FIELDS

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We give examples of fields elementarily equivalent to a given finite extension of the p-adic numbers but not containing a subfield of finite codimension elementarily equivalent to the p-adics.

SECTION 0. INTRODUCTION.

It is well known that an algebraically closed field of characteristic zero contains a real closed field of codimension 2. From the point of view of the model theory of real closed fields, this means that a field elementarily equivalent to a finite extension of the real numbers $\mathbb{R}$ contains a subfield of the right codimension elementarily equivalent to $\mathbb{R}$. In this note we show that this is not the case for any field K finite-dimensional over $\mathbb{Q}_p$. Namely, for any finite extension of $\mathbb{Q}_p$ of a given degree $d > 1$, there exists an elementary equivalent field which does not contain a subfield of finite codimension elementarily equivalent to $\mathbb{Q}_p$. So this is a point in which the model theory of $\mathbb{R}$ and $\mathbb{Q}_p$ differ. It is worth noticing that the examples below have the simplest possible value group, namely $\mathbb{Z}$.

From the work of Ax-Kochen and Ershov we know that the elementary theory of $\mathbb{Q}_p$ is the theory of henselian valued fields with residue field $\mathbb{F}_p$, and discretely valued in a $\mathbb{Z}$-group so that $v(p) = 1$. Let pCF be the above theory. Its models are called p-adically closed fields. The model theory of finite extension fields of $\mathbb{Q}_p$ was studied in [PR]. The only model-theoretic fact we shall need is that if K is a finite extension of $\mathbb{Q}_p$ and $K_1 \subseteq K$ a subfield relatively algebraically closed in K then $K_1 \preccurlyeq K$, i.e. the inclusion is elementary.

Our arguments rely on basic algebraic-geometric facts, together with the completeness of $\mathbb{Q}_p$ via Baire's theorem. We can and shall assume everything to take place in a fixed algebraic closure of $\mathbb{Q}_p$. If F is a field then $\tilde{F}$ denotes its algebraic closure. We denote by A_p the field of algebraic p-adic numbers, i.e. the p-adic numbers algebraic over the rationals, and by A^n the affine n-space. If I is a polynomial ideal then $V(I)$ denotes its zero set.

SECTION 1. THE EXAMPLES.

Let K be a finite extension field of $\mathbb{Q}_p$ of degree $d > 1$. We know that $K = \mathbb{Q}_p(\alpha)$ for some algebraic number α. Let $t_0,\dots,t_{d-1} \in \mathbb{Q}_p$ be algebraically independent over A_p and let K_1 be the relative algebraic closure of
$A_p(\alpha, t_0 + t_1\alpha+\dots+t_{d-1}\alpha^{d-1})$ in K. Then K_1 is elementarily equivalent to K and the transcendence degree of K_1 over A_p is 1. If $K_0 \subseteq K_1$ is a subfield of K_1 elementarily equivalent to $\mathbb{Q}_p$ and of codimension d then $\operatorname{tr\,deg} K_0|A_p = 1$.
Moreover:

<u>LEMMA 1</u>. $K_0 = K_1 \cap \mathbb{Q}_p$.

<u>PROOF</u>. K_0 has a unique Henselian valuation, which is the restriction of the unique Henselian valuation on K_1. Since K_1 is a finite extension of K_0, the valuegroup of K_0 is of finite index in that of K_1. Also, $v(p)$ is the least positive value in K_0, hence K_0 is an immediate extension of A_p. Therefore A_p is dense in K_0, so A_p and K_0 have the same (topological) closure in the algebraic closure of $\mathbb{Q}_p$. It follows that $K_0 \subseteq \mathbb{Q}_p$. On the other hand $K_1 \cap \mathbb{Q}_p$ is relatively algebraically closed in $\mathbb{Q}_p$ and so is a model of pCF. Since the extension $K_1 \cap \mathbb{Q}_p|K_0$ is algebraic the equality follows. □

We show below that there exists $t_0,\dots,t_{d-1}$ for which $K_1 \cap \mathbb{Q}_p = A_p$. Such t_i prevent the existence of a suitable K_0 and thus yield our example for K.

Let us point out that an analogous construction for $\mathbb{R}$ does not force $K_0 = K_1 \cap \mathbb{R}$, the reason being that the field K_0 would not necessarily be archimedean and hence not necessarily embeddable in $\mathbb{R}$.

PROPOSITION. There exist $t_0,\ldots,t_{d-1} \in \Phi_p$ which are algebraically independent over $\tilde{A}_p$ and for which $K_1 \cap \Phi_p = A_p$.

PROOF. It suffices to find algebraically independent t_i such that for every irreducible polynomial $f(X,Y) \in A_p(\alpha)[X,Y] \setminus A_p(\alpha)[X]$ and every $x \in \Phi_p$, if $f(x,t_0 + t_1\alpha + \ldots + t_{d-1}\alpha^{d-1}) = 0$ then $x \in A_p$. In fact we find t_i such that $f(x,t_0+t_1\alpha+\ldots+t_{d-1}\alpha^{d-1}) \neq 0$ for all $x \in \Phi_p$, so a fortiori fulfilling the requirement with respect to A_p. Let $f(X,Y)$ be such a polynomial and $\mathcal{C}$ be the affine curve it defines. Set $X = x_0 + x_1\alpha + \ldots + x_{d-1}\alpha^{d-1}$ and $Y = y_0 + y_1\alpha + \ldots + y_{d-1}\alpha^{d-1}$ and let $W = R_{A_p(\alpha)|A_p}(\mathcal{C})$ be the induced Weil restriction of $\mathcal{C}$ for the extension $A_p(\alpha)|A_p$ (see [W]). Let $\sigma_1,\ldots,\sigma_d$ be the d distinct A_p-embeddings of $A_p(\alpha)$ in $\tilde{A}_p$. The affine variety W is isomorphic to the product $\mathcal{C}^{\sigma_1} \times\ldots\times \mathcal{C}^{\sigma_d}$ over the Galois closure of $A_p(\alpha)$ over A_p via the following isomorphism (ibid.)

$$\Phi(x_0,\ldots,x_{d-1},y_0,\ldots,y_{d-1}) = (X_1,Y_1,\ldots,X_d,Y_d)$$

where $X_i = \Sigma x_j\sigma_i(\alpha^j)$ and $Y_i = \Sigma y_j\sigma_i(\alpha^j)$, $j = 0,\ldots,d-1$. Let $M = (a_{ij})$ be the dxd matrix with (i,j)-th entry $a_{ij} = \sigma_i(\alpha^{j-1})$ and let $\underline{x}$, $\underline{y}$, $\underline{X}$, $\underline{Y}$, be the column vectors obtained from the components x_j, y_j, X_i, Y_i, respectively. Then M is invertible and $\underline{X} = M\underline{x}$, $\underline{Y} = M\underline{y}$. The ideal $(f^{\sigma_i}(X_i,Y_i)$, $i = 1,\ldots,d)$ is an ideal of definition for $C^{\sigma_1}\times\ldots\times C^{\sigma_d}$. Let W' be the intersection of W with $x_1 = 0,\ldots,x_{d-1} = 0$ and let J be the ideal generated by the $f^{\sigma_i}(X_i,Y_i)$ and X_1-X_j for $j = 2,\ldots,d$.

LEMMA 2. We have $\Phi W' = V(J)$.

PROOF. The inclusion $\subseteq$ is clear. On the other hand if $P = (X_1,Y_1,\ldots,X_d,Y_d)$ lies in $V(J)$ then it lies in $C^{\sigma_1}\times\ldots\times C^{\sigma_d}$ and $X_1 = X_2 =\ldots= X_d$, Consider the linear system $\underline{X} = M\underline{x}$ with the x_i as unknowns. If $X_1 = 0$ then $x_0 = 0 = x_1 =\ldots= x_{d-1}$ and $\Phi^{-1}P$ is in W'. Otherwise, setting $z_i = X_1^{-1}x_i$, we get the equivalent system $\underline{1} = M\underline{z}$, where $\underline{1}$ denotes the column vector whose entries are all equal to 1. Now the first column of M is equal to $\underline{1}$, so by Cramer $z_0 = 1$ and $z_j = 0$ for $j \geq 1$, whence $\Phi^{-1}P$ lies in W'. □

Using this lemma a straightforward computation of transcendence degree shows that $\dim \Phi W' \leq 1$ and we conclude, via Φ, that $\dim W' \leq 1$.

Let $\pi W'$ be the set theoretic projection of W' on the last d components, i.e. the set of $(y_0,\ldots,y_{d-1})$ for which there exists a x_0 such that $(x_0,0,\ldots,0,y_0,\ldots,y_{d-1})$ is in W'. Since $\dim W' \leq 1$, it follows that the Zariski closure $\overline{\pi W'}^{Z}$ of $\pi W'$ has also dimension ≤ 1. In order to use a Baire argument for $\mathbb{C}_p^d$ to get the t_i we isolate the following fact.

LEMMA 3. Let $V \subseteq A^d$ be an affine variety of dimension $n < d$ defined over $\widetilde{A}_p$. Then $V \cap \mathbb{Q}_p^d$ has empty interior in $\mathbb{Q}_p^d$ for the p-adic topology.

PROOF. For cardinality reasons every ball in $\mathbb{Q}_p^d$ contains a point with components algebraically independent over $\widetilde{A}_p$. This can be seen by proving by induction on d that for any $\beta_1, \ldots \beta_r$ in $\mathbb{Q}_p$ and any ball B in $\mathbb{Q}_p^d$ there is a point P of B whose coordinates are algebraically independent over $\mathbb{Q}(\beta_1, \ldots, \beta_r)$. For $d = 1$ this is a simple cardinality argument. For $d = c + 1$, first choose last coordinate β_{r+1} independent of $\beta_1, \ldots, \beta_r$, and then work in $\mathbb{Q}_p^c$ with $\beta_1, \ldots, \beta_r, \beta_{r+1}$ to get the first c coordinates. □

It follows that $\overline{\pi W'}^z \cap \mathbb{Q}_p^d$ is a nowhere dense subset of $\mathbb{Q}_p^d$ in the p-adic topology, as is $V(g) \cap \mathbb{Q}_p^d$ for any $g \in \widetilde{A}_p[X_1, \ldots, X_d]$. Considering all the $\overline{\pi W'}^z$ thus obtained and all $V(g)$, we conclude by Baire's Theorem that there exists $(t_0, \ldots, t_{d-1}) \in \mathbb{Q}_p^d$ in the complement of all those sets. These are the required t_i and this concludes the proof of the Proposition. □

SECTION 2. CONCLUDING REMARKS.

It is clear that in the above discussion we can replace $\mathbb{Q}_p$ by any of its finite extensions and adjust the arguments accordingly. Let us refer to a field K as having the "codimension property" if any field elementarily equivalent to a finite extension of K contains a subfield elementarily equivalent to K and of the same codimension. The field of rational numbers $\mathbb{Q}$ has, like $\mathbb{R}$, the codimension property, but this time it is related to undecidability. Indeed by Julia Robinson's result, $\mathbb{Q}$ is definable in any fixed finite extension field of itself. This is to be contrasted with the situation of the reals, where both $\mathbb{R}$ and $\mathbb{C}$ are decidable, and the field of the p-adics which, while not having the codimension property, is decidable and has every finite extension decidable.

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