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An Invitation to the Miyaoka-Yau Inequality

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RÉSUMÉ

En 1976, S.T. Yau a observé que la métrique de Kähler-Einstein pouvait être employée pour régler des questions importantes dans la géométrie algébrique. Une des affirmations importantes était l'inégalité entre les nombres de Chern des variétés algébriques. Pour une surface algébrique, S.T.Yau a prouvé $3c_2(M) \geq c_1 2(M)$, une inégalité prouvée indépendamment par Miyaoka employant des techniques algébriques. De plus, S.T. Yau a montré que l'égalité tenait seulement si la courbure sectionnelle holomorphe de M est constante. Nous allons examiner au chapitre un la preuve de S.T. Yau de l'inégalité cidessus en utilisant une approche géométrique différentielle et au chapitre deux la preuve de Y. Miyaoka de l'inégalité à l'aide des outils de la géométrie algébrique.

Mots clefs: Surfaces algébriques de type générale; Variétés Kähler-Einstein; Inégalité de Miyaoka-Yau.

TABLE OF CONTENTS

ABSTRACT

In 1976, S.T. Yau observed that the Kähler-Einstein metric can be used to settle important questions in algebraic geometry. One of the important assertions was the inequality between the Chern numbers for *general* algebraic manifolds (namely manifolds of general type). For such an algebraic surface, S.T. Yau proved $3c_2(M) \geq c_1^2(M)$, an inequality proved independently by Miyaoka using algebraic techniques. Furthermore, S.T. Yau proved that equality holds only if M has constant holomorphic sectional curvature. We are going to examine in chapter one S.T. Yau's proof of the above inequality using a differential geometric approach and in chapter two Y. Miyaoka's proof of the inequality using algebaric-geometric tools.

Key words: Algebraic surfaces of general type; Kähler-Einstein manifolds; Miyaoka-Yau inequality.

INTRODUCTION

As S.T. Yau puts it in his lectures, a fruitful idea to construct geometric structure is to construct metrics that satisfy the Einstein equation. One demands that the Ricci tensor of the metric be proportional to the metric itself. The problem of existence of an Einstein metric is really a very difficult but central problem in geometry.

The traditional approach to construct solutions to the Einstein equation is to assume some global symmetry, such as spherical (rotation) invariance or cylindrical symmetry to simplify the problem by reducing it to a lower dimensional one. The celebrated Schwarzschild solution (Schwarzschild, 1916) is the first example of such a technique. However, if one focuses more on internal symmetries, the ability to fix a gauge, such as holomorphic coordinates (i.e a Kähler structure) is very helpful. The space with such internal symmetry could be a Kähler manifold or more generaly a manifold with a special holonomy group. In particular a class of such manifolds are known as *Kähler-Einstein* manifolds, which are simply Kähler manifolds whose metric satisfy the Einstein equation.

Therefore, a major question one wants to answer is: under what conditions does a Kähler-Einstein metric exist? One of the answers to this question is provided by the Calabi conjecture proved independently by T. Aubin (Aubin, 1976) and by S.T. Yau (Yau, 1977), where one asks whether the necessary condition for the first Chern class to have a definite sign is also sufficient for the existence of such metrics. The importance of Kähler-Einstein metrics reach beyond the physical sciences, they provided S.T.Yau with a very elegant and simple proof of an inequality between the first and second Chern numbers for an algebraic manifold namely the Miyaoka-Yau inequality which is the subject of this mémoire.

Apart from the differential geometric point of view of manifolds, there are the algebraic varieties which are classified according to the map of the manifold into the complex projective space by powers of the canonical line bundle. If the map is an immersion at generic point, the manifold is called an algebraic manifold of general type. This class of manifolds comprises the majority of algebraic manifolds, and can be considered as generalizations of algebraic curves of higher genus. In this case, Miyaoka proved the famous inequality by algebraic geometric techniques and the connection with Kähler-Einstein metrics was realized quite readily since such manifolds have negative first Chern class in the strong sense if one considers them in the birational class of orbifolds.

We see that the Chern number inequalities can therefore be ascertained by both differential geometric means and algebraic-geometric techniques.

In chapter one we are going to focus on the differential geometric approach to the proof due to S.T. Yau (Yau, 1977). First we are going to introduce the necessary geometric tools before proving some useful lemmas and propositions that are used in the final section of chapter one, namely the full differential geometric proof of the Miyaoka-Yau inequality.

In chapter two we focus on the algebraic-geometric approach to the proof due to Miyaoka (Miyaoka, 1977). In a similar organization to chapter one, we first introduce the most fundamental tools of algebraic-geometry, then we move on to prove important lemmas and propositions on which the final proof rests which is itself presented in the last section of the chapter.

A brief comment on the bibliography pertinent to each approach of the proof follows the final section of each chapter.

CHAPTER I

DIFFERENTIAL GEOMETRIC PROOF OF THE MIYAOKA-YAU INEQUALITY

1.1 Motivation

In what follows we will give the differential geometric proof of *Miyaoka-Yau Inequality*, which is a consequence of the existence of Kähler-Einstein metrics due independently to T. Aubin (Aubin, 1976) and S.T. Yau (Yau, 1977). We will start by reviewing some basic notions of complex differential geometry which will serve as well to set up some of the notations to be used throughout this *mémoire*. Most of this material can be found in standard text books on complex differential geometry such as (Kobayashi, 1987) or (Besse, 1987) and the classical (Griffiths and Harris, 1994).

1.2 Preliminaries

In the first sub-section we are going to introduce (omitting the proofs) the important machinery of complex and Kähler geometry which we will use extensively in the subsequent sub-sections under the assumption of the manifold being Kähler, first to prove some specific lemmas concerning the *Chern classes* of a complex vector bundle and then some propositions regarding the relationships of the geometric objects such as the *mean curvature*, the *Ricci tensor* and the *Ricci scalar curvature* to the first and second Chern classes of the given manifold M. Finally, after introducing the notion of an *Einstein-manifold*, we will use those lemmas and propositions to finalize the proof of the

Miyaoka-Yau inequality.

1.2.1 Tools of Complex and Kähler Geometry

We start by recalling the following basic definitions and constructions.

Definition 1.1. *A function $f(z)$ is* **holomorphic** *in a domain $\mathcal{D} \subset C^n$ if $\frac{\partial f}{\partial \bar{z}_i} = 0, \forall\ i$ everywhere in $\mathcal{D}$. This is precisely true if $f(z)$ is analytic in $\mathcal{D}$.*

We have the following propositions for complex vector bundles admitting a holomorphic structure characterized in terms of connections.

Definition 1.2. *An* **n-dimensional complex manifold** *$\mathcal{M}$ is a differentiable manifold admitting an open cover $\{\mathcal{U}_\alpha\}$ and coordinate maps $\varphi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \subset \mathbb{C}^n$ for all α, β.*

Definition 1.3. *A function on an open set $\mathcal{U} \subset \mathcal{M}$ is* **holomorphic** *if for all i, $f \circ \varphi^{-1}$ is holomorphic on $\varphi_i(\mathcal{U}_i \cap \mathcal{U}) \subset \mathbb{C}^n$.*
A collection $z = (z_1, ..., z_n)$ of functions on $\mathcal{U} \subset \mathcal{M}$ is said to be a **(holomorphic) coordinate system** *if both $\varphi_i \circ z^{-1}$ and $z \circ \varphi^{-1}$ are holomorphic on $z(\mathcal{U} \cap \mathcal{U}_i)$ and $\varphi_i(\mathcal{U} \cap \mathcal{U}_i)$ respectively for each i, and z is injective from $\mathcal{U}$ to C^n.*
A **map** *$f : \mathcal{M} \longrightarrow \mathcal{N}$ of complex manifolds* **is holomorphic** *if it is given in terms of local holomorphic coordinates on $\mathcal{N}$ by holomorphic functions.*

Definition 1.4. *We define the* **complexified tangent bundle** *and the* **holomorphic tangent bundle** *as follows:*

$$T_{\mathbb{C}} := T^{\mathbb{R}}_{\mathcal{M}} \otimes \mathbb{C} = T'_{\mathcal{M}} \oplus \overline{T}'_{\mathcal{M}},$$

where the $T^{\mathbb{R}}_{\mathcal{M}}$ stands for the real tangent bundle over the manifold $\mathcal{M}$, while $T'_{\mathcal{M}}$ stands for the holomorphic tangent bundle, and $\overline{T}'_{\mathcal{M}}$ stands for the antiholomorphic tanget bundle. In that context, we define the (p,q)-form to be the C^∞-section of $\bigwedge^p T'_{\mathcal{M}} \otimes \bigwedge^q \overline{T}'_{\mathcal{M}}$.

Definition 1.5. *Similarly, we define the* ***complexified cotangent bundle*** *and the* ***holomorphic cotangent bundle*** *as follows:*

Let $\{dx_1, \cdots, dx_n, dy_1, \cdots, dy_n\}$ *be the dual basis to* $\{\partial_{x^1}, \cdots, \partial_{x^n}, \cdots, \partial_{y^1}, \cdots, \partial_{y^n}\}$. *Then*

$$dz_j = dx_j + i\ dy_j \qquad d\bar{z}_j = dx_j - i\ dy_j$$

$$dx_j = \frac{1}{2}(dz_j + d\bar{z}_j) \qquad dy_j = \frac{1}{2i}(dz_j - d\bar{z}_j).$$

Therefore, we have the following vector bundles on $\mathcal{M}$

$T^*_{\mathcal{M}}(\mathbb{R})$, *the real cotangent bundle, with fiber*

$$T^*_{\mathcal{M},\ x}(\mathbb{R}) = \mathbb{R} < dx_1, \cdots, dx_n, dy_1, \cdots, dy_n > .$$

$T^*_{\mathcal{M}}(\mathbb{C})$, *the complex cotangent bundle, with fiber*

$$T^*_{\mathcal{M},\ x}(\mathbb{C}) = \mathbb{C} < dx_1, \cdots, dx_n, dy_1, \cdots, dy_n > .$$

$T'_{\mathcal{M}}(\mathbb{C})$, *the holomorphic cotangent bundle, with fiber*

$$T'^*_{\mathcal{M},\ x}(\mathbb{C}) = \mathbb{R} < dz_1, \cdots, dz_n > .$$

$\overline{T}'^*_{\mathcal{M}}(\mathbb{C})$, *the anti-holomorphic cotangent bundle, with fiber*

$$\overline{T}'^*_{\mathcal{M},\ x}(\mathbb{C}) = \mathbb{R} < d\bar{z}_1, \cdots, d\bar{z}_n > .$$

We have a canonical injection and a canonical internal direct sum decomposition into complex sub-bundles:

$$T^*_{\mathcal{M}}(\mathbb{R}) \subseteq T^*_{\mathcal{M}}(\mathbb{C}) = T^*_{\mathcal{M}} \bigoplus \overline{T}'^*_{\mathcal{M}}.$$

We give a quick review of vector bundles to shed some light on the above definition.

Definition 1.6. *Let* M *be a differentiable manifold. A (complex)* ***vector bundle*** *of rank* r *is a triple* (E, π, M) *where* E *is a differentiable manifold together with a* C^∞-*surjective map* $\pi : E \longrightarrow M$ *such that we can find an open cover* $\{\mathcal{U}_\alpha\}$ *of* M *and* C^∞-***trivialization maps***

$$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \xrightarrow{\sim} \mathcal{U}_\alpha \times \mathbb{C}^r$$

*together with C^∞-**transition functions***

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow GL(r, \mathbb{C})$$

which are given by

$$g_{\alpha\beta}(z) = \varphi_\alpha \circ \varphi_\beta^{-1}|_{\{z\}\times\mathbb{C}^n}$$

(where $GL(r, \mathbb{C})$ is the general linear group of rank r over the complex number field $\mathbb{C}$ and by "$|$" we mean restriction to the appropriate subsets).
*By definition, these transitions functions must satisfy what are known as the **cocycle conditions***

i) $g_{\alpha\beta} \circ g_{\beta\alpha} = \mathbb{I}_{|\mathcal{U}_\alpha \cap \mathcal{U}_\beta}$

ii) $g_{\alpha\beta} \circ g_{\beta\gamma} = (g_{\alpha\gamma})_{|\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma}$

A set of transition functions can be used to patch or glue together (in a consistent way) local pieces of a bundle if and only if the cocycle conditions are satisfied.

*We call $E_z = \pi^{-1}(z)$ the **fiber** of E over $z \in M$, which is isomorphic to a complex vector space $z \times \mathbb{C}^r$ of dimension r.*

*A rank-1 vector bundle is called a **line bundle**.*

A section σ of a vector bundle E over $\mathcal{U}$ is a C^∞-map $\sigma : \mathcal{U} \longrightarrow E$ such that $\sigma(z) \in E_z$ for all $z \in \mathcal{U}$.

Let M be an n-dimensional complex manifold.

Definition 1.7. *The **canonical line bundle** K_M of M is the top exterior product of the holomorphic cotangent bundle (the dual of the holomorphic tangent bundle)*

$$K_M \equiv \bigwedge^n T^*(M) := det(T^*).$$

Now, let E be a C^∞-complex vector bundle of rank r over M.

Notation 1.8. *We denote by:*

i) $A^{p,q} \equiv$ *the set of* $(p.q)$*-forms over* M.

ii) $A^{p,q}(E) \equiv$ *the set of* (p,q)*-forms over* M *with values in* E.

We give the following useful definitions.

Definition 1.9. *The* ***exterior differential operator*** d ***splits*** *as* $d = d' + d''$, *where*

$$
\begin{aligned}
d' &: A^{p,q} \longrightarrow A^{p+1,q} \\
d'' &: A^{p,q} \longrightarrow A^{p,q+1}
\end{aligned} \tag{1.1}
$$

with the property that $d^2 = 0$.

When M is a real C^∞-manifold, and E a C^∞-complex vector bundle over M, we define the notion of a connection as follows.

Definition 1.10. *A* ***connection*** D *on* E *is a homomorphism over the field of complex numbers* $\mathbb{C}$

$$D : A^0(E) \longrightarrow A^1(E) \tag{1.2}$$

which acts with the following Leibniz rule*:*

$$D(f\sigma) = \sigma df + f \cdot D\sigma, \quad \forall f \in A^0, \ \sigma \in A^0(E), \tag{1.3}$$

where A^r *denotes the space of smooth complex valued* r*-forms over* M *and* $A^r(E)$ *the space of smooth* r*-forms over* M *with values in* E.

Remark 1.11. D *extends naturally to* r*-forms with values in the tensorial combination* F *of* E, $E^\vee$, $\overline{E}$, $\overline{E}^\vee$, *denoted by* A^r.

This is done by extending $D : A^0(E) \longrightarrow A^1(E)$ *to a* C*-linear map*

$$D : A^p(E) \longrightarrow A^{p+1}(E), \ \ p \geq 0 \quad \text{by setting}$$

$$D(\sigma \cdot \phi) = (D\sigma) \wedge \phi + \sigma \cdot d\phi, \quad \forall\, \sigma \in A^0(E), \phi \in A^p.$$

Now, back to the manifold M being complex, we can decompose the **connection** *D as in definition (1.9)*

$$D = D' + D'' \quad with \tag{1.4}$$

$$D' : A^{p,q}(E) \longrightarrow A^{p+1,q}(E) \ \ and \ \ D'' : A^{p,q}(E) \longrightarrow A^{p,q+1}(E)$$

We note as well the following Leibniz rule

$$D'(\sigma\omega) = D'\sigma \wedge \omega + \sigma d'\omega,$$

$$D''(\sigma\omega) = D''\sigma \wedge \omega + \sigma d''\omega, \quad \forall\ \sigma \in A^0(E), \ \ \omega \in A^{p,q}.$$

Definition 1.12. *We define the* **curvature** *R of the connection D to be the map $R = D \circ D : A^0(E) \longrightarrow A^2(E)$.*
This is done using the extended D construction of remark 1.11.

Remark 1.13. *R is known to be A^0-linear. Namely, if $f \in A^0$ and $\sigma \in A^0(E)$, R acts on their product as follows*

$$\begin{aligned} R(f\sigma) &= D^2(f\sigma) = D \circ D(f\sigma) = D(\sigma df + f \cdot D\sigma) \\ &= D\sigma \wedge df + df \wedge D\sigma + fD^2\sigma = fD^2\sigma = fR(\sigma). \end{aligned} \tag{1.5}$$

Therefore R is a 2-form on M with values in $End(E)$. Here, $End(E)$ stands for endomorphism *of the vector bundle E i.e. $End(E) = E \otimes E^*$ where E^* is the dual vector bundle of E.*

Definition 1.14. *When M is a* real *n-dimensional manifold, we define the* **curvature 2-form** *Ω of the connection D with respect to the frame field s by*

$$s\Omega = D^2 s.$$

In terms of the coordinate dependent **connection 1-form** ω, *which we write in matrix notation as* $\omega = (w_i^j)$ *of* D *where* $\omega_i^j \in A^1|_{U\subset M}$ *(* U *is the open subset of* M *for the local coordinate), defined by* $Ds = s\cdot\omega, \forall\ s \in A^0|_U$, *we have*

$$\begin{aligned} s\Omega &= D(s\cdot\omega) = Ds\wedge\omega + sd\omega \\ &= s\cdot\omega\wedge\omega + sd\omega = s(\omega\wedge\omega + d\omega) \end{aligned} \tag{1.6}$$

using the properties of the connection D *outlined in definition 1.10. Therefore,*

$$\Omega = \omega\wedge\omega + d\omega. \tag{1.7}$$

Definition 1.15. *We note also that the* ***exterior differential*** *of* ω *is given by*

$$d\Omega = \Omega\wedge\omega - \omega\wedge\Omega.$$

When M *is a* complex *manifold, the curvature* R *in view of formula (1.4) can be decomposed as*

$$R = D'\circ D' + (D'\circ D'' + D''\circ D') + D''\circ D'',$$

where

$$D'\circ D' \in A^{2,0}(End(E)),\ \ D'\circ D'' + D''\circ D' \in A^{1,1}(End(E)),\ \ D''\circ D'' \in A^{0,2}(End(E)).$$

In the same manner, the connection 1-form ω and the curvature form ω split as

$$\omega = \omega^{1,0} + \omega^{0,1}$$

$$\Omega = \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}.$$

We now come to complex vector bundles admitting a holomorphic structure.

Definition 1.16. *By a* ***holomorphic structure*** *on a rank-r* C^∞*-complex vector bundle* E *we mean, a collection of* ***holomorphic local trivializations***, *i.e. a collection of local trivializations* $\Psi_\alpha : E|_{U_\alpha} \longrightarrow \mathbb{C}^r \otimes U_\alpha$ *such that the transition maps,*

$$g_{\alpha\beta} := \Psi_\beta(q)\circ\Psi_\alpha^{-1}(q) : U_{\alpha\beta} \longrightarrow GL(r,\mathbb{C}) \subset \mathbb{C}^{r^2}$$

are holomorphic (here $GL(r,\mathbb{C})$ is the general linear group of rank r over the complex field $\mathbb{C}$, and (U_α) is an open cover of U).

Definition 1.17. *A* ***holomorphic vector bundle*** *is a pair (vector bundle, holomorphic structure).*

Definition 1.18. *Two holomorphic structures $\Psi = (\Psi_\alpha, g_{\alpha\beta} = \Psi_\beta \circ \Psi_\alpha^{-1})$ and $\omega = (\omega_\alpha, h_{\beta\alpha} = \omega_\beta \circ \omega_\alpha^{-1})$ are said to be* ***isomorphic*** *if there exist holomorphic maps $T_\alpha : U_\alpha \longrightarrow GL(r,\mathbb{C})$ such that*

$$h_{\beta\alpha} = T_\beta g_{\beta\alpha} T_\alpha^{-1} \tag{1.8}$$

Remark 1.19. *The transition functions in these definitions must satisfy the gluing lemma i.e. the cocycle or compatibility condition on the intersection of the appropriate open sets $U_\alpha \cap U_\beta$.*

Proposition 1.20. *Let E be a holomorphic vector bundle (as defined in definition 1.17) over a complex manifold M. Then there is a connection D such that,*

$$D'' = d''.$$

For such a connection, the $(0,2)$-component $D'' \circ D''$ of the curvature R vanishes.

Proposition 1.21. *Let E be a C^∞-complex vector bundle over a complex manifold M. If D is a connection on E, such that, $D'' \circ D'' = 0$, then there is a unique holomorphic vector bundle structure on E such that $D'' = d''$.*
For a proof see (Kobayashi, 1987 p.9-10).

Definition 1.22. *Let E be a C^∞-vector bundle over a (real or complex) manifold M. An* ***Hermitian structure*** *or* ***Hermitian metric*** *h on E is a C^∞-field of* ***Hermitian*** *inner products on the fibers of E. Thus,*

i) $h(\zeta,\eta)$ is linear in ζ, $\forall\ \zeta,\eta \in E_x$

ii) $h(\zeta, \eta) = \overline{h(\eta, \zeta)}$

iii) $h(\zeta, \zeta) > 0, \ \forall \, \zeta \neq 0$

iv) $h(\zeta, \eta)$ *is a* C^{∞}*-function if both* ζ *and* η *are* C^{∞}*-sections.*

The couple (E, h) *is called a* ***Hermitian vector bundle.***

One also view h *as an element of* $E^{\vee} \otimes \overline{E}^{\vee}$ *and write*

$$h(\zeta, \eta) = h(\zeta \otimes \overline{\eta}).$$

Notation 1.23. *Given a local frame field* $s_u = (s_1, ..., s_n)$ *of* E *over* U*, we set*

$$h_{i\bar{j}} \equiv h(s_i, s_j),$$

where $i, j = 1, ..., r$.

Remark 1.24. *For a complex manifold* M*, one can give the above hermitian condition for the holomorphic tangent bundle* $T'M$ *in terms of the complex structure* $\mathcal{J}$ *acting on the real tangent space* $T_{\mathbb{R}}M$*,* $\mathcal{J} \in C^{\infty}(End(T_{\mathbb{R}}M))$ *with* $\mathcal{J}^2 = -I$*. Here* $T'M$ *is canonically identified with the* $T_{\mathbb{R}}M$ *and the condition for the metric* g *to be hermitian would read*

$$g(\mathcal{J}w_{(1)}, \mathcal{J}w_{(2)}) = g(w_{(1)}, w_{(2)}),$$

for $w_{(1)}$ *and* $w_{(2)}$ *lying in the complexified tangent space* $T_{\mathbb{C}}M$*.*

We have the hermitian analog of proposition 1.20:

Proposition 1.25. *Given a Hermitian structure* h *on a holomorphic vector bundle* E*, there is a unique* D *such that* $D'' = d''$ *and* $Dh = 0$*. This connection is called the* ***Hermitian connection*** *of the holomorphic vector bundle* (E, h)*.*

Together with proposition 1.21, this implies that the curvature R, of such a connection has neither $(0,2)$-components nor $(2,0)$-components, i.e. R is a $(1,1)$-form with values in $End(E)$. From definition 1.14, the connection 1-form $\omega = \omega^i_j$ is of type $(1,0)$ while the curvature form ω is equal to the $(1,1)$-component of equation (1.6). Therefore, we obtain that

$$\Omega = d''\omega \tag{1.9}$$

Definition 1.26. *Let $\mathcal{M}$ be an n-dimensional complex manifold endowed with a hermitian metric g and let (E,h) be a hermitian vector bundle of rank r over $\mathcal{M}$. Let $(s_i)_{i=1,..,r}$ be a local unitary frame fields for (E,h) and $(\theta^\alpha)_{\alpha=1,..n}$ a local unitary frame fields for the cotangent bundle T^*M of (M,g). Then we write*

$$\Omega^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

*The **mean curvature** K of (E,h) is defined by*

$$K^i_j = \sum g^{\alpha\bar{\beta}} R^i_{j\alpha\bar{\beta}} \text{ and } K_{j\bar{k}} = \sum h_{i\bar{k}} K^i_j.$$

In terms of the Kähler form ω_g and R,

$$K\omega_g^n = \sqrt{-1} R \wedge \omega_g^{n-1}.$$

*The **Ricci tensor** $\rho = (R_{k\bar{l}})$ is defined by contracting the curvature tensor R via*

$$R_{k\bar{l}} = \sum R^i_{ik\bar{l}} = \sum g^{i\bar{j}} R_{i\bar{j}k\bar{l}}.$$

*The **scalar curvature** is defined by:*

$$\sigma = \sum R^i_{i\alpha\bar{\alpha}} = K^i_i = \sum R_{\alpha\bar{\alpha}} \tag{1.10}$$

Consider a hermitian manifold (M,g) of dimension n with local coordinates $(z_1, ..., z_n)$.

Definition 1.27. *Associated to the Hermitian metric g on the tangent bundle TM is the **fundamental** 2-**form** defined by*

$$\omega_g = \sqrt{-1} \sum g_{i\bar{j}} \, dz^i \wedge d\bar{z}^j.$$

We note that:

$$\omega_g^n = (\sqrt{-1})^n n!\ det(g_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^n$$

Remark 1.28. *The real* $(1,1)$*-form* ω_g*, may or may not be* ***closed****.*

Definition 1.29. *If the* $(1,1)$*-form* ω_g *is closed i.e.* $d\omega_g = 0$*, then the metric* g *is called a* ***Kähler metric*** *and the couple* (M, g) *a* ***Kähler manifold****. We also say that the hermitian metric* g *is* ***Kähler*** *if the complex structure operator* $\mathcal{J}$ *is parallel for the* Levi Civita *connection* D*, i.e.* $D\mathcal{J} = 0$ *that is*

$$D_X \mathcal{J} Y = D_Y \mathcal{J} X, \quad \forall X, Y \in T_x M.$$

In view of notation (1.23) we can write

$$g = \sum g_{i\bar{j}} dz^i d\bar{z}^j \quad \text{where } g_{i\bar{j}} = g(\partial/\partial z^i, \partial/\partial z^j) = g(\partial/\partial z^i \otimes \partial/\partial \bar{z}^j).$$

Let D be the Hermitian connection of g and R its curvature. By proposition 1.25 and the discussion that followed it, $Dg = 0$ and $D" = d"$ and $R \in A^{1,1}(End(TM))$. In terms of the frame field $(\partial/\partial z^1, \dots, \partial/\partial z^n)$ and its dual $(dz^1, \dots dz^n)$ we can express both R and Ω as

$$R = \sum \Omega^i_j dz^i \otimes \partial/\partial z^i, \quad \text{where } \Omega^i_j = \sum R^i_{jk\bar{l}} dz^k \wedge d\bar{z}^l \tag{1.11}$$

Expressing equation (1.9) in terms of local coordinates, we have

$$R^i_{ja\bar{b}} = -\sum g^{i\bar{k}} \frac{\partial^2 g_{j\bar{k}}}{\partial z^a \partial \bar{z}^b} + \sum g^{i\bar{k}} g^{l\bar{m}} \frac{\partial g_{l\bar{k}}}{\partial z^a} \frac{\partial g_{j\bar{m}}}{\partial \bar{z}^b}$$

and by **lowering** $R^i_{ja\bar{b}}$ with $g_{a\bar{b}}$ (i.e. $R_{j\bar{k}a\bar{b}} = \sum g_{i\bar{k}} R^i_{ja\bar{b}}$) we get

$$R_{j\bar{k}a\bar{b}} = -\sum \frac{\partial^2 g_{j\bar{k}}}{\partial z^a \partial \bar{z}^b} + \sum g^{l\bar{m}} \frac{\partial g_{l\bar{k}}}{\partial z^a} \frac{\partial g_{j\bar{m}}}{\partial \bar{z}^b}.$$

If (M, g) is Kähler, then

$$R_{i\bar{j}km} = R_{kmi\bar{j}}.$$

By further contracting indices of the curvature tensor we can get the following geometric quantities:

Proposition 1.30. *Again, let s and θ be as in definition 1.26, we have the following useful relations*

$$\begin{aligned}
\omega_g &= \sum_\alpha \sqrt{-1}\, \theta^\alpha \wedge \overline{\theta}^\alpha & (1.12)\\
||R||^2 &= \sum |R^i_{j\alpha\bar\beta}|^2 = \sum |R_{ij\alpha\bar\beta}|^2 & (1.13)\\
||K||^2 &= \sum |K^i_j|^2 = \sum |K_{ji}|^2 = \sum |R^i_{j\alpha\bar\alpha}|^2 & (1.14)\\
||\rho||^2 &= \sum |R_{\alpha\bar\beta}|^2 = \sum |R^i_{i\alpha\bar\beta}|^2. & (1.15)
\end{aligned}$$

Where once again (as in definition 1.26) K denotes the ***mean curvature*** *of (E,h), ρ the* ***Ricci tensor*** *which is the curvature of the determinant bundle $det(E) = \bigwedge^k E$ and σ the* ***scalar curvature*** *of the Hermitian manifold M.*

Definition 1.31. *Let $e(x)$ denotes the field of unitary frames defined for x in an open subset of M, and let $\zeta = \sum_i \zeta_i e_i$ be a tangent vector at x, then the* ***holomorphic sectional curvature*** *in the direction of ζ is defined to be*

$$R(x,\zeta) = \frac{2}{(\sum \zeta_i \bar\zeta_i)^2} \sum R_{ij\gamma\delta}\zeta_i\zeta_\gamma\bar\zeta_j\bar\zeta_\delta$$

From the symmetries of R (i.e. $R_{ij\gamma\delta} = \overline{R_{ij\gamma\delta}}$) we see that $R(x,\zeta)$ is a real quantity and determines the whole Kähler curvature.

We introduce the closed $2k$-form γ_k and the Chern classes via the following definition.

Definition 1.32. *The k-th* ***Chern Class*** *$c_k(E,h) \in H^{2k}(M,\mathbb{Z})$ of a hermitian vector bundle (E,h) is represented by the closed $2k$-form γ_k defined by*

$$\begin{aligned}
det(I_r - \frac{1}{2\pi i}\omega_g) &= 1 + \gamma_1 + \gamma_2 + \dots + \gamma_r & (1.16)\\
\gamma_k &= \frac{(-1)^k}{(2\pi i)^k k!} \sum \delta^{j_1\dots j_k}_{l_1\dots l_k} \Omega^{l_1}_{j_1} \wedge \dots \wedge \Omega^{l_k}_{j_k} & (1.17)
\end{aligned}$$

Theorem 1.33. *The class $c_k(E,h)$ is independent of h. Hence we may set*

$$c_k(E) = c_k(E,h).$$

We also write $c_k(M)$ for $c_k(TM)$.

We note in particular that

$$\begin{aligned}\gamma_1 &= \frac{-1}{2\pi i}\sum \Omega^j_j \\ \gamma_2 &= \frac{-1}{8\pi^2}\sum(\Omega^j_j \wedge \Omega^k_k - \Omega^j_k \wedge \Omega^k_j)\end{aligned} \tag{1.18}$$

For example the first Chern class $c_1(E,h)$ is represented by

$$c_1(E,h) = \frac{-1}{2\pi i}\sum \Omega^k_k = \frac{-1}{2\pi i}\sum R_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$$

in terms of a local holomorphic coordinate $z = (z_1, ..., z_n)$ and is independent of h.

1.2.2 En Route Towards the Proof of Miyaoka-Yau Inequality

We are now in a position to apply the tools developed in the previous sub-section to prove some lemmata and propositions that will be used in the proof of the *Miyaoka-Yau Inequality*.

Lemma 1.34. (Apte, 1955) *The first and second Chern classes $c_1(E,h)$ and $c_2(E,h)$ satisfy the following two relations respectively*

i) $c_1(E,h)^2 \wedge \omega_g^{n-2} = \frac{1}{4\pi^2 n(n-1)}(\sigma^2 - ||\rho||^2)\omega_g^n = \gamma_1^2 \wedge \omega_g^{n-2}$

ii) $c_2(E,h) \wedge \omega_g^{n-2} = \frac{1}{8\pi^2 n(n-1)}(\sigma^2 - ||\rho||^2 - ||K||^2 + ||R||^2)\omega_g^n = \gamma_2 \wedge \omega_g^{n-2}$

Proof. — From the definition of the γ_1 and γ_2 in formula (1.18) we have,

$$\begin{aligned}\gamma_1^2 &= \left(\frac{i}{2\pi}\right)^2 \sum \Omega^l_l \wedge \Omega^j_j = \frac{-1}{4\pi^2}\sum \Omega^l_l \wedge \Omega^j_j \\ \gamma_2 &= \gamma_1^2 + \frac{1}{8\pi^2}\sum \Omega^j_k \wedge \Omega^k_j\end{aligned}$$

Therefore we need only to prove the following relations

i') $n(n-1)\sum \Omega^l_l \wedge \Omega^j_j \wedge \omega_g^{n-2} = -(\sigma^2 - ||\rho||^2)\omega_g^n$

ii') $n(n-1)\sum \Omega^k_j \wedge \Omega^j_k \wedge \omega_g^{n-2} = -(||K||^2 - ||R||^2)\omega_g^n$

For i') we have,

$$\begin{aligned} n(n-1)\sum \Omega^l_l \wedge \Omega^j_j \wedge \omega_g^{n-2} &= n(n-1)\sum (R_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta) \wedge (R_{\gamma\bar{\delta}}\theta^\gamma \wedge \bar{\theta}^\delta) \wedge \omega_g^{n-2} \\ &= n(n-1)\sum R_{\alpha\bar{\beta}}R_{\gamma\bar{\delta}}\theta^\alpha \wedge \bar{\theta}^\beta \wedge \theta^\gamma \wedge \bar{\theta}^\delta \wedge \omega_g^{n-2} \\ &= -n(n-1)\sum (R_{\alpha\bar{\alpha}}R_{\gamma\bar{\gamma}} - R_{\alpha\bar{\gamma}}R_{\gamma\bar{\alpha}})\omega_g^n \end{aligned}$$

We have used the fact that the local unitary frames θ^μ and $\bar{\theta}^\mu$ appear in pairs in the above equations with

i) $n(n-1)\sum_{\alpha=\beta\neq\gamma=\delta} \theta^\alpha \wedge \bar{\theta}^\beta \wedge \theta^\gamma \wedge \bar{\theta}^\delta \wedge \omega_g^{n-2} = +\omega_g^n$

ii) $n(n-1)\sum_{\alpha=\delta\neq\gamma=\beta} \theta^\alpha \wedge \bar{\theta}^\beta \wedge \theta^\gamma \wedge \bar{\theta}^\delta \wedge \omega_g^{n-2} = -\omega_g^n$

iii) 0 otherwise.

By formulae (1.15) and (1.10),

$$n(n-1)\sum \Omega^l_l \wedge \Omega^j_j \wedge \omega_g^{n-2} = -(\sigma^2 - ||\rho||^2)\omega_g^n$$

which proves i').

ii') follows similarly:

$$\begin{aligned} n(n-1)\sum \Omega^k_j \wedge \Omega^j_k \wedge \omega_g^{n-2} &= \sum R_{k\bar{j}\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta \wedge R_{j\bar{k}\gamma\bar{\delta}}\theta^\gamma \wedge \bar{\theta}^\delta \wedge \omega_g^{n-2} \\ &= -\sum (R_{k\bar{j}\alpha\bar{\alpha}}R_{jk\gamma\bar{\gamma}} - R_{k\bar{j}\alpha\bar{\gamma}}R_{jk\gamma\bar{\alpha}})\omega_g^n \\ &= -(||K||^2 - ||R||^2)\omega_g^n \end{aligned}$$

where in the last step we have used formulae (1.13) and (1.14). □

We have the following lemma relating $||R||^2$ and $||\rho||^2$.

Lemma 1.35. *Let (E, h) be an Hermitian vector bundle of rank r over a compact Hermitian manifold (M, g) of dimension n. Then $||R||^2$ and $||\rho||^2$ satisfy the following inequality:*

$$r||R||^2 - ||\rho||^2 \geq 0$$

and equality holds if and only if

$$rR^i_{j\alpha\bar{\beta}} = \delta^i_j R_{\alpha\bar{\beta}}.$$

Proof. — Set $T^i_{j\alpha\bar{\beta}} = R^i_{j\alpha\bar{\beta}} - \frac{1}{r}\delta^i_j R_{\alpha\bar{\beta}}$, then

$$\begin{aligned} 0 \leq ||T||^2 &= \sum |R^i_{j\alpha\bar{\beta}} - \frac{1}{r}\delta^i_j R_{\alpha\bar{\beta}}|^2 \\ &= \sum |R^i_{j\alpha\bar{\beta}}|^2 - \frac{2}{r}\sum |R_{\alpha\bar{\beta}}|^2 + \frac{1}{r}\sum |R_{\alpha\bar{\beta}}|^2 \\ &= \sum |R^i_{j\alpha\bar{\beta}}|^2 - \frac{1}{r}\sum |R_{\alpha\bar{\beta}}|^2 \end{aligned}$$

By formulæ (1.13) and (1.15) we get our desired result, namely

$$0 \leq ||T||^2 = ||R||^2 - \frac{1}{r}||\rho||^2$$

. ∎

Definition 1.36. *When $R^i_{j\alpha\bar{\beta}} = \frac{1}{r}\delta^i_j R_{\alpha\bar{\beta}}$ holds, we say that (E, h) is* ***projectively flat****.*

We are now in a position to introduce the notion of **Einstein manifolds** via the **Hermitian Einstein condition** for vector bundles (Kobayashi,1987).

Definition 1.37. *Let (E, h) be a holomorphic Hermitian vector bundle of rank r over an Hermitian manifold (M, g). We say that (E, h) satisfies the* ***weak Einstein condition*** *(with factor φ) if the mean curvature K satisfies*

$$K = \varphi \mathbb{I}_E \ \ i.e. \ \ K^i_j = \varphi \delta^i_j \tag{1.19}$$

where φ is a real function defined on M. If φ is a constant, we say that (E, h) satisfies the ***Einstein condition*** *and (E,h) is then called an* ***Einstein vector bundle*** *over (M, g).*

If further $E = TM$ and $h = g$ we call the couple (M, g) an ***Hermitian Einstein manifold****. Furthermore, if the $(1,1)$-form ω_g is closed i.e. satisfied the condition in definition 1.29 we call (M, g) a* ***Kähler-Einstein manifold****.*

Making use of definition 1.26 and proposition 1.30, we have the following lemma.

Lemma 1.38. *For any n-dimensional Kähler manifold M, we have always*

$$||R||^2 \geq \frac{2}{n+1}||K||^2$$

and equality holds if and only if M is a space of constant holomorphic sectional curvature.

Proof. — In the Kählerian case we have

$$R_{i\bar{j}k\bar{m}} = R_{k\bar{m}i\bar{j}} \quad \text{so that} \quad K_{i\bar{j}} = R_{i\bar{j}}$$

and from formulæ (1.14) et (1.15) we get in particular that

$$||K||^2 = ||\rho||^2 \tag{1.20}$$

Now, in the Einstein-Kähler case, we have that $K^i_j = \frac{\sigma}{n}\delta^i_j$ where σ is the scalar curvature of the n-dimensional manifold M, therefore

$$||K||^2 = \frac{\sigma^2}{n} = ||\rho||^2 \tag{1.21}$$

Now set

$$T_{i\bar{j}k\bar{m}} = R_{i\bar{j}k\bar{m}} - \frac{\sigma}{n(n-1)}(\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj}).$$

Then

$$\begin{aligned} 0 \leq ||T||^2 &= \sum |R_{i\bar{j}k\bar{m}} - \frac{\sigma}{n(n-1)}(\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj})|^2 \\ &= ||R||^2 - \frac{2\sigma^2}{n(n-1)}. \end{aligned}$$

This implies that

$$||R||^2 \geq 2\sigma^2 n(n-1)$$

and equality holds if and only if $T = 0$. Therefore from equation (1.21),

$$||R||^2 \geq \frac{2}{n+1}||K||^2$$

as required. $\square$

1.3 Proof of the Miyaoka-Yau Inequality (Differential Geometric)

Finally, we are in a position to give the differential geometric proof of the Miyaoka-Yau inequality, the algebraic counterpart of which will be given in the next chapter.

Theorem 1.39. (Apte, 1955) *Let (M, g) be a compact Kähler-Einstein manifold of dimension n. Then*

$$\int_M \{n\ c_1(M)^2 - 2(n+1)\ c_2(M)\} \wedge \omega_g^{n-2} \leq 0$$

and equality holds if and only if (M, g) is of constant holomorphic sectional curvature.

Proof. — We denote $c_i(TM, g)$ (the i^{th} chern class of the tangent bundle TM) by $c_i(M, g)$ and set $E = TM$ and $g = h$ so we can apply the results of lemma 1.34.
By i) in lemma 1.34 we have

$$\begin{aligned} c_1(M,g)^2 \wedge \omega_g^{n-2} &= \frac{1}{4\pi^2 n(n-1)}(\sigma^2 - ||\rho||^2)\omega_g^n \\ &= \frac{1}{4\pi^2 n(n-1)}(n||\rho||^2 - ||\rho||^2)\omega_g^n \\ &= \frac{1}{4\pi^2}||\rho||^2\frac{\omega_g^n}{n} = \frac{1}{4\pi^2}||K||^2\frac{\omega_g^n}{n} \quad \text{(by equation (1.21)).} \end{aligned}$$

By ii) in lemma 1.34 and using equation (1.21) we have

$$\begin{aligned} c_2(M,g) \wedge \omega_g^{n-1} &= \frac{1}{8\pi^2 n(n-1)}(\sigma^2 - ||\rho||^2 - ||K||^2 + ||R||^2)\omega_g^n \\ &= \frac{1}{8\pi^2 n(n-1)}((n-2)||K||^2 + ||R||^2)\omega_g^n. \end{aligned} \tag{1.22}$$

Hence,

$$\begin{aligned}
&\int_M \{2(n+1)c_2(M) - nc_1(M)^2\} \wedge \omega_g^{n-2} \\
= &\ \frac{1}{4\pi^2 n(n-1)} \int_M ((n+1)(n-2)||K||^2 + (n+1)||R||^2 - n(n-1)||K||^2) \wedge \omega_g^n \\
= &\ \frac{1}{4\pi^2 n(n-1)} \int_M (-2||K||^2 + (n+1)||R||^2) \wedge \omega_g^n.
\end{aligned}$$

We therefore get

$$\int_M \{2(n+1)c_2(M) - nc_1(M)^2\} \wedge \omega_g^{n-2} = \frac{1}{4\pi^2 n(n^2-1)} \int_M (||R||^2 - \frac{2||K||^2}{n+1}) \wedge \omega_g^n.$$

By lemma 1.38 we have that

$$||R||^2 \geq \frac{2}{n+1}||K||^2$$

which implies that the right hand side is positive and therefore,

$$\int_M \{n\ c_1(M,g)^2 - 2(n+1)c_2(M,g)\} \wedge \omega_g^{n-2} \leq 0.$$

Equality holds if

$$||R||^2 = \frac{2}{n+1}||K||^2$$

i.e. when the holomorphic sectional curvature is constant.

$\square$

In particular for $n = 2$ we have that

$$c_1(M)^2 \leq 3\ c_2(M)$$

which is the Miyaoka-Yau inequality for a compact Kähler-Einstein manifold M of dimension two.

1.4 Discussion and Conclusion

We have thus proved the geometric equivalence of the Miyaoka-Yau inequality for a compact Kähler-Einstein manifold using tools from complex differential geometry. It is remarkable that the proof is very simple and is quite short (modulo the Einstein condition) compared to its algebraic-geometric analog as we will see in chapter two. The main ingredients of the differential geometric proof were the fact that the manifold was *Kähler* (Definition 1.29) and *Einstein* (definition 1.37). Those conditions simplified tremendously the relationships between the *mean curvature* the *Ricci tensor* and the *scalar curvature* of proposition 1.30. Although the Kähler-Einstin condition is a very strong one, we were able with minimal assumptions (we did not need any on the Chern class for example, to be compared with the algebraic-geometric proof) to arrive at our desired result.

1.4.1 Remarks (for non-mathematicians) on the Bibliography

Readers who are not familiar with the abstract mathematical literature would find the following books very helpful as an introduction to the ideas and techniques of *Fiber bundles*, (Isham, 1999) and (Nash and Sen, 1992). The latter especially has a very nice and comprehensive discussion of *Chern classes* with examples. At a more advanced level I would recommend (Spivak, 1979) especially volume 5 for a detailed discussion of *Chern classes*, *Gauss Bonnet Theorem* etc....

For the readers interested in *Kähler Geometry*, I recommend the book by Kobayashi (Kobayashi, 1987) and parts of (Nicolaescu, 2000).

For those who would like a quick introduction to all of the above (yet superficial) including applications to physics such as in gravitation or in gauge-theory, a good pedagogical review is (Egushi, Gilkey and Hanson, 1980), it treats even *Index theorems with and without boundaries.*

Finally, all interested readers should consider (Griffiths and Harris, 1994) to bridge the gap between this chapter on complex differential geometry and the next chapter on algebraic geometry.

CHAPTER II

ALGEBRAIC-GEOMETRIC PROOF OF THE MIYAOKA-YAU INEQUALITY

2.1 Motivation

Chapter one was devoted to the differential geometric approach to the proof of Miyaoka-Yau inequality. In this chapter we give the algebraic-geometric proof due to Miyaoka (Miyaoka, 1977) following (Barth, Hulek, Peters, and Van De Ven, 2004). It is in algebraic geometry that the inequality has been most useful. In fact, in the classification theory (see (Ueno, 1975), (Hartshorne, 1977) and (Friedman, 1998)) of *(minimal) surfaces of general type* the Miyaoka-Yau inequality plays an essential role. This role is mostly played in what is known as the **geography** of surfaces of general type. The main question posed there is that given a pair (c_1^2, c_2) of numerical invariants allowed by the Miyaoka-Yau inequality, can one find a minimal surface of general type that realizes this pair of integers? And if so, what can we say about the geometry (e.g. the curvature, connection, ...) of the *moduli* space?

The answer to these questions are still open, but to fully appreciate it let us now turn to the proof itself which will not only shed some light on the result but on the tools used to study such questions as well.

2.2 Preliminaries

Following the same organization of chapter one, we start in the first subsection by introducing the necessary tools of algebraic-geometry that we will need for the proof of the Miyaoka-Yau inequality. It will be brief and not intended to be complete.

In the subsequent sections we will give some lemmas and propositions which will be the building blocks of the final proof.

2.2.1 Tools of Algebraic Geometry

Following (Yang, 1991) we introduce the following notions

Sheaves

Definition 2.1. *Let X be a topological space. A* ***Presheaf*** *$\mathcal{P}$ of Abelian groups X is given by two pieces of information:*

a) For every open set $U \subset X$ we are given an Abelian group $\mathcal{P}(U)$

b) For every pair of open sets $V \subset U$ of X there is a homomorphism, called the restriction map, $\rho_{VU} : \mathcal{P}(U) \longrightarrow \mathcal{P}(V)$ such that
$\rho_{UU} = id.$, $\rho_{WU} = \rho_{WV} \circ \rho_{VU}$ whenever $W \subset V \subset U$.

We also write ρ_{UV} as s_V.

Definition 2.2. *Given a presheaf $\{\mathcal{P}(U) : U \text{open in } X\}$ on X. We fix a point $x \in X$. Then we define the* ***stalk*** *of $\mathcal{P}$ at $x \in X$ to be the inverse limit,*

$$\mathcal{P}_x := \varinjlim_{x \in \mathcal{U}} \mathcal{P}(U).$$

An element of $\mathcal{P}_x$ is called a germ of sections of $\mathcal{P}$ at x.

Definition 2.3. *Let X be a topological space. A* ***sheaf*** *of Abelian groups over X is a topological space $\mathcal{S}$ with a map $\pi : \mathcal{S} \longrightarrow X$ such that,* *a)* π *is a local homeomorphism.*

b) For every $x \in X$, $\pi^{-1}(x) = \mathcal{S}_x$ *is an Abelian group.*

c) The group operations are continuous, where on $\mathcal{S} \times \mathcal{S}$ *we use the product topology.*

A sheaf gives rise to a presheaf in a natural way by considering local sections:

Definition 2.4. *A* ***section*** *of* $\mathcal{S}$ *over an open set* $U \subset X$ *is defined to be a continuous map* $f : U \longrightarrow \mathcal{S}$ *with* $\pi \circ f = id$.
We denote by $\mathcal{S}(U)$ *the set of all sections over* U.
The presheaf $\{\mathcal{S}(U) : U \text{ open in } X\}$ *is called the* ***presheaf associated to*** $\mathcal{S}$, *while the stalk at* $x \in X$ *of the associated sheaf is precisely* $\pi^{-1}(x) = \mathcal{S}_x$.

Example 2.5. *Let* $X = M$ *be a* C^∞*-manifold. We have*

$$C^\infty \longrightarrow M,$$

the sheaf of germs of smooth functions on M. The presheaf $C^\infty(U)$ *consists of smooth functions on U, while the stalk at* $x \in M$ *is the set of germs of smooth functions defined in a neighborhood of* x.

Definition 2.6. *A* ***morphism of presheaves*** $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ *is a collection where we associate, for each open set* U *a morphism of groups*

$$\phi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

such that for $\sigma \in \mathcal{F}(U)$ *and* $V \subset U$, *we have*

$$\phi_U(\sigma)|_V = \phi_V(\sigma|_V).$$

Lemma 2.7. *For every presheaf* $\mathcal{F}$ *over* X, *there exists a unique sheaf* $\mathcal{F}_f$ *over* X *satisfying the following conditions:*

i) There exist a morphism of presheaves

$$\phi : \mathcal{F} \longrightarrow \mathcal{F}_f.$$

ii) For every morphism of presheaves

$$\psi : \mathcal{F} \longrightarrow \mathcal{G}$$

where $\mathcal{G}$ is a sheaf, there exists a unique morphism *of sheaves $\chi : \mathcal{F}_f \longrightarrow \mathcal{G}$ such that $\psi = \chi \circ \phi$.* (See (Voisin, 2002) p.86-87 for a proof.)

Remark 2.8. *If $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves i.e. $\mathcal{F}$ and $\mathcal{G}$ are sheaves and ϕ is a morphism of presheaves, then ϕ induces a morphism of abelian groups*

$$\phi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

at each point x.

Definition 2.9. *The morphism ϕ is* ***injective*** *(resp.* ***surjective****) if for every $x \in X$ the morphism ϕ_x is* ***injective*** *(resp.* ***surjective****).*

To understand better what sheafs and stalks are, we can think of them as follows

Remark 2.10. *A sheaf over X is* ***a fiber bundle*** *and gives rise to a presheaf so that the stalks of the presheaf are the fibers.*

Notation 2.11. *Let $\mathcal{M}$ be a complex manifold. Then*

$$\mathcal{O} \longrightarrow M$$

denotes the sheaf of germs of holomorphic functions on M, also known as the structure sheaf of M. Meanwhile

$$\mathcal{O}^* \longrightarrow M$$

denotes the multiplicative sheaf of germs of nowhere zero holomorphic functions on M.

We also have

$$\mathcal{M} \longrightarrow M, \;\; \mathcal{M}^* \longrightarrow M,$$

which denote respectively, the sheaf of germs of meromorphic functions and the multiplicative sheaf of not identically zero meromorphic functions on M.

Remark 2.12. *Script letters such as $\mathcal{M}$ will be reserved for sheaves while ordinary capital letters such as M will denote spaces or surfaces unless otherwise specified.*

Line Bundles and Divisors

In view of definition 1.4 of chapter one on vector bundles we state the following definitions

Definition 2.13. *A complex rank $r = 1$-vector bundle over a smooth manifold M is called a* ***complex-line bundle****.*

Definition 2.14. *If M is a complex manifold and if $L \longrightarrow M$ is a complex-line bundle admitting trivializations $(U; \varphi_a)$ with holomorphic transition functions (c.f. chap.1, def. 1.3 and def 1.4) $\{g_{ab} : U_a \cap U_b \to GL(1, \mathbb{C}) = \mathbb{C}^*\}$ (where $\mathbb{C}^* = \mathbb{C}/\{0\}$), the line bundle is called a* ***holomorphic line bundle****.*

Notation 2.15. *Let X be a topological space and $\mathcal{S}$ a sheaf of groups on X.*
We shall denote by $H^i(X, \mathcal{S})$ or $H^i(\mathcal{S})$ the i-th cohomology group of X with coefficients in $\mathcal{S}$. For the group of sections we shall also write $H^0(X, \mathcal{S})$ or $\Gamma(X, \mathcal{S})$.
If $\mathcal{S}$ is a sheaf of real or complex vector spaces, then $H^i(X, \mathcal{S})$ is also a real or complex vector space.
Further if $\dim H^i(X, \mathcal{S})$ is finite then we denote by $h^i(X, \mathcal{S}) = h^i(\mathcal{S})$ this dimension.

Remark 2.16. *i) We will often use the same notation for a holomorphic vector bundle and its sheaf of sections.*

ii) A characteristic property of cohomology of sheaves is that a short exact sequence of sheaves over X gives rise to a long exact sequence of cohomogies over X with values in these sheaves. That is given the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

of sheaves over a manifold X, we have the associated long exact sequence of coho-

mology

$$\begin{aligned} 0 &\longrightarrow H^0(X,\mathcal{F})\longrightarrow H^0(X,\mathcal{G})\longrightarrow H^0(X,\mathcal{H}) \\ &\longrightarrow H^1(X,\mathcal{F})\longrightarrow H^1(X,\mathcal{G})\longrightarrow H^1(X,\mathcal{H})\longrightarrow \cdots \\ \cdots &\longrightarrow H^q(X,\mathcal{F})\longrightarrow H^q(X,\mathcal{G})\longrightarrow H^q(X,\mathcal{H})\longrightarrow \cdots . \end{aligned}$$

(See (Voisin, 2002) for a detailed discussion of the cohomology of sheaves.)

Proposition 2.17. *The collection (isomorphism classes) of complex line bundles over M is canonically identified with $H^1(M,\mathcal{A}^*)$ where $\mathcal{A}^*$ is the multiplicative sheaf of germs of no where zero complex valued smooth functions on M.*
For a proof of this see (Yang, 1991) p.53.

Replacing $\mathcal{A}^*$ by $\mathcal{O}^*$ we have the following important proposition

Proposition 2.18. *The collection of all holomorphic line bundles over a complex manifold is naturally identified with $H^1(M,\mathcal{O}^*)$.*

Remark 2.19. *As a consequence a holomorphic line bundle $L\longrightarrow M$ can be thought of as an element of the cohomology group $H^1(M,\mathcal{O}^*)$. Under this identification the* group operations *in $H^1(M,\mathcal{O}^*)$ are given by:*

1) $L+L' = L\otimes L'$

2) $-L = L^*$, where $L^*\longrightarrow M$ denotes the **dual** *of L.*

Definition 2.20. *A* ***divisor*** *D on a compact complex manifold M, is a finite (integral) sum $D = \sum_i a_i V_i$ with $a_i \in \mathbb{Z}$, where the V_i's are irreducible analytical hypersurfaces of M (i.e. the V_i's are algebraic subvarieties of codimension 1). If all the $a_i^{'}s$ are non-negative i.e all $a_i \geq 0, \forall\ i$, we say that D is* ***effective*** *and we write $D \geq 0$.*

Remark 2.21. *Under addition the set of all divisors on M, denoted by $Div(M)$, forms a free Abelian group.*

Definition 2.22. *Let V be an irreducible subvariety of codimension 1 of M. By a* **local defining function** *f at $x \in V$ we mean an element of $\mathcal{O}_x$ vanishing along V such that if another germ in $\mathcal{O}_x$ vanishes along V, it has to be a multiple of f in $\mathcal{O}_x$).*

Definition 2.23. *Given $g \in \mathcal{O}(U)$, $U \ni x$, we define the* **order of g** *along V at x by the maximal a such that $g = f^a h$, for some $h \in \mathcal{O}_x$. We denote it by $ord_{V,x}(g)$. This is a notion independent of the point $x \in V \cap U$ since V is connected, and therefore we talk about the $ord_V(g)$ at any $x \in V$. Since $g \in \mathcal{M}$ implies $g = g_1/g_2$ as germs at x for g_1, $g_2 \in \mathcal{O}_x$, we can define*

$$ord_{V,x}(g) = ord_{V,x}(g_1) - ord_{V,x}(g_2).$$

Definition 2.24. *The* **divisor** *of f is defined by*

$$(f) = \sum ord_V(f).V, \quad \forall \text{ irreducible } V \subset M \text{ as in definition 2.20}$$

.

Definition 2.25. *A divisor D on M is called a* **principal divisor** *if there exists a meromorphic function f such that $(f) = D$.*

Definition 2.26. *Two divisors D_1 and D_2 are said to be* **linearily equivalent** *if $D_1 - D_2$ is principal.*

We arrive at the following important proposition.

Proposition 2.27. *Let M be a compact complex manifold. Then:*

a) *$Div(M)$ are naturally isomorphic to $H^0(M, \mathcal{M}^*/\mathcal{O}^*)$*

b) *There exists a canonical homomorphism*
   $\varphi : Div(M) \longrightarrow H^1(M, \mathcal{O}^*), \quad D \mapsto \mathcal{L}_D$
   with $\ker(\varphi) =$ *{normal subgroup of principal divisors} and for all divisors D, there exist a meromorphic section of $\mathcal{L}_D$ such that $(s) = D$.*
   We call $H^1(M, \mathcal{O}^*)$ the **Picard group** *and denote it by* **$Pic(M)$**.
   For a proof see (Yang, 1991) p.65-67.

Remark 2.28. *We note that given a divisor D on X, the line bundle L_D is also written as $L = \mathcal{O}_X(D)$. If L has holomorphic sections s and $D' = (s)$ the effective divisor associated to s, then $L = \mathcal{O}_X(D')$.*

Proposition 2.29. *Let D be a divisor on the compact complex manifold M. Denote by $\mathcal{L}(D)$ a space of meromorphic functions f on M such that $D + (f) \geq 0$ and by $|D|$ the set of all effective divisors linearly equivalent to D. We have the following isomorphism*

$$|D| \cong \mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}(H^0(M, \mathcal{O}([D])).$$

Proof. — Let s_0 be a global meromorphic section of $[D]$ with $(s_0) = D$, then for any global holomorphic section s of $[D]$, the quotient $f_s = \frac{s}{s_0}$ is a meromorphic function on M with

$$(f_s) + D = (s) - (s_0) + D = (s) \geq 0,$$

which means that $f_s \in \mathcal{L}(D)$. Meanwhile, for any $f \in \mathcal{L}(D)$ the section $s = f.s_0$ of $[D]$ is holomorphic. Thus we have established the identification

$$\mathcal{L}(D) \xrightarrow{\otimes s_0} H^0(M, \mathcal{O}([D])).$$

For every $D_0 \in |D|$, there exists an $f \in \mathcal{L}(D)$ such that $D_0 - D = (f)$ and any two such functions f_1 and f_2 differ by a non zero constant. This establishes $|D| \cong \mathbb{P}(\mathcal{L}(D))$. $\square$

Definition 2.30. *Let M be a compact complex manifold with a holomorphic line bundle L. A linear subspace of $\mathbb{P}(H^0(M, \mathcal{O}([L]))$ is called a* ***linear system*** *of divisors.*
A ***complete linear system*** *is a linear system of the form $|D|$ for some divisor D. The dimension of the linear system is $\dim H^0(M, \mathcal{O}(L)) - 1$.*

Chern Classes revisited

Following (Voisin, 2002) we outline the construction of the Chern classes from a different (yet related) point of view than chapter one.
Let X be a topological or differentiable manifold, and let $E \longrightarrow X$ be a (topological or differentiable) complex vector bundle of rank r.

Definition 2.31. *We construct the* **Chern classes** $c_i(E) \in H^{2i}(X,\mathbb{Z})$, $1 \leq i \leq$, *with the convention that* $c_0(E) = 1$ *and* $c_i(E) = 0$ *for* $i > r$, *by introducing the* **Chern polynomial**

$$c(E) = \sum_i c_i t^i \ \in \ H^*(X,\mathbb{Z})[t].$$

Now consider the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{C}^0 \xrightarrow{\exp} \mathcal{C}^{0*} \longrightarrow 0,$$

where $\mathcal{C}$ *is the sheaf of continuous complex functions and* $\mathcal{C}^{0*}$ *is the sheaf of everywhere non-zero functions. By the associated long exact sequence, it gives the isomorphism*

$$c_1 : H^1(X,\mathcal{C}^{0*}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$$

(the map is an isomorphism since $H^1(X,\mathcal{C}^0) = H^2(X,\mathcal{C}^{0*}) = 0$ *because the topological manifold admits partitions of unity subordinate to open covers.)*

Remark 2.32. *i) The group* $H^i(X,\mathcal{C}^{0*})$ *is the group of isomorphism classes of complex line bundles over* X, *with the group structure given by the tensor product.*

ii) If a line bundle L *is endowed with a hermitian metric, we remark that it is not difficult to show that* $c_1(L)$ *is represented by its curvature form.*

Theorem 2.33. (See (Hirtzebruch, 1966)) *There exists a unique* Chern class *map* c, *which associates to a complex vector bundle* E *over* X *an element*

$$c(E) \in H^*(X,\mathbb{Z})[t] \quad i.e. \ c(E) = \sum_i c_i(E)t^i, \quad \text{with each } c_i(E) \in H^{2i}(X,\mathbb{Z})$$

satisfying the following conditions:

i) If rank $E = 1$, *then* $c(E) = 1 + tc_1(E)$.

ii) The Chern class *map satisfies the following functoriality conditions: if* $\phi : Y \longrightarrow X$ *is a continuous (or differentiable) map, then*

$$c(\phi^* E) = \phi^*(c(E)),$$

where $\phi^* : H^{2i}(X,\mathbb{Z}) \longrightarrow H^{2i}(Y,\mathbb{Z})$ *is the pullback map.*

iii) **(Whitney's formula)** *if E is the direct sum of two complex bundles F and G, then*

$$c(E) = c(F)c(G),$$

where the ring structure on $H^*(X, \mathbb{Z})[t]$ is used.

To close this brief exposition on Chern classes we give the celebrated *splitting principle* via the following lemma.

Lemma 2.34. *Let $E \longrightarrow X$ be a complex vector bundle. Then there exists a continuous map $\phi : Y \longrightarrow X$ satisfying:*

i) The pullback maps $\phi^* : H^k(X, \mathbb{Z}) \longrightarrow H^k(Y, \mathbb{Z})$ are injective.

ii) The pullback $\phi^* E$ is a direct sum of line bundles.

Remark 2.35. *i) The splitting principle provides us with the curvature Whitney formula for hermitian bundles which are metrically direct sums of line bundles and thus gives us the connection with the curvature definition of the previous chapter.*

ii) We also want to point out that the fundamental class *in analytic and algebraic duality differs by a factor of $2\pi i$. If X is a compact algebraic manifold of dimension n, then under the identification*

$$H^n_{alg}(X, \Omega^n) = H^n_{an}(X, \Omega^n)$$

we have

$$algebraic\ fund.\ class = (2\pi i)^n\ analytic\ fund.\ class.$$

Riemann-Roch Theorem, and Further Tools

Following (Barth, Hulek, Peters, and Van De Ven, 2004) we further introduce the following theorems and propositions.

Notation 2.36. *Let X be an n-dimensional complex manifold, then we shall denote by*

i) $\mathcal{T}_X$: *the (holomorphic) tangent bundle of X, while its dual we will denote by* $\mathcal{T}_X^\vee$.

ii) Ω_X^i: *the sheaf of germs of holomorphic i-forms on X, i.e. the sheaf of sections in the bundle* $\bigwedge^i \mathcal{T}_X^\vee$ $(i \geq 1)$

iii) $\mathcal{O}_X$: *the structure sheaf of X.*

iv) $\mathcal{K}_X$: *the canonical line bundle on X, i.e. the holomorphic 1-vector bundle* $\bigwedge^n \mathcal{T}_X^\vee$

v) $\mathcal{N}_{Y/X}$: *the* ***normal bundle*** *of the complex submanifold Y in X defined by the* ***normal bundle sequence***

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0,$$

where by "|" we denote the analytic restriction.

vi) $c_i(X)$: *the i-th Chern class of X, that is $c_i(\mathcal{T}_X)$.*
(In particular in the case where X is compact we have that $[c_n(X)] = e(X)$ where $e(X)$ denotes the ***Euler number*** *of X.)*

Remark 2.37. *By a common abuse of notation, we often use the same symbole for a bundle and its sheaf of sections.*

Definition 2.38. *In particular if we set $\Omega_X^0 = \mathcal{O}_X$ we can define:*

$$\begin{aligned} h^{p,q}(X) &= h^q(\Omega_X^p) \\ q(X) &= h^{0,1}(X), \quad \text{the } \textbf{\textit{irregularity}} \text{ of } X \\ p_g(X) &= h^{0,n}(X), \quad \text{the } \textbf{\textit{geometric genus}} \text{ of } X \end{aligned}$$

We note as well the ***Betti number*** *$b_i(X)$ which is a topological invariant of the surface X and is related by* Hodge theory *to the irregularity $q(X)$ and the geometrical genus*

$p_g(X)$ respectively by:

$$b_1(X) = 2q(X) \tag{2.1}$$

$$b_2(X) = 2p_g(X) + h^{1,1}(X) \tag{2.2}$$

Definition 2.39. *A sheaf of $\mathcal{O}_X$-modules $\mathcal{S}$ is is said to be* **coherent,** *if locally there always is some exact sequence of sheaves of $\mathcal{O}_X$-modules*

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathcal{S} \longrightarrow 0$$

The complex vector spaces $H^i(X, \mathcal{S})$ are finite dimensional provided that X be compact and $\mathcal{S}$ a coherent sheaf on X. Therefore we have that $h^i(X, \mathcal{S})$ is finite for such a sheaf. It vanishes unless $0 \leq i \leq n$ by Grothendieck vanishing theorem (c.f. (Hartshorne, 1977) Thm. 2.7 in Section III, p.208).

As a consequence the **Euler characteristic** is well defined

$$\chi(X, \mathcal{S}) = \sum_{i=0}^{n} (-1)^i h^i(\mathcal{S}) \tag{2.3}$$

One of the cornerstones of algebraic geometry is the following theorem due to Serre, known as *Serre's duality theorem* for manifolds.

Theorem 2.40. *(Serre's Duality Theorem)*

Let X be a compact, connected complex n-dimensional manifold, and $\mathcal{V}$ **a holomorphic vector bundle** *on X. Then*

$$h^i(\mathcal{V}) = h^{n-i}(\mathcal{V}^\vee \otimes \mathcal{K}_X)$$

As a special case we find that $p_g(X) = h^{0,n}(X) = h^{n,0}(X)$ as we noted at the end of definition 2.25. ($\mathcal{V}^\vee$ here denotes the dual vector bundle of $\mathcal{V}$).

We will also need the celebrated **Riemann-Roch Theorem** which we now state in the following form.

Theorem 2.41. *(Hirzebruch-Atiyah-Singer Riemann-Roch theorem)*

Let $\mathcal{V}$ be a holomorphic vector bundle on a compact, connected n-dimensional complex *manifold X. Then*

$$\chi(X, \mathcal{V}) = t_{2n}(Todd(X).ch(\mathcal{V}))$$

In particular when $\mathcal{V}$ is the ***trivial line bundle*** $\chi(X) = \chi(\mathcal{O}_X) = t_{2n}(Todd(X))$, *where the righthand side is a homogeneous polynomial with rational coefficients in the Chern classes of X, while the lefthand side is called the* ***Todd genus*** *of X and denoted by $T(X)$.*

Remark 2.42. *We briefly describe what the* ***Chern character*** *and* ***Todd Class*** *are. Let E be a complex vector bundle over X of rank r. The* ***Chern character*** $ch(E)$ *of E is defined as follows by means of the factorization of the total* Chern class*:*

$$\text{If} \quad \sum c_i(E)x^i = \prod(1 + t_i x), \quad \text{then} \quad ch(E) = \sum \exp t_i.$$

While the Chern class $c(E)$ is in $H^*(X; \mathbb{Z})$, the Chern character $ch(E)$ is in $H^*(X; \mathbb{Q})$. The Chern character satisfies (Hirzebruch, 1966)

$$ch(E \oplus E') = ch(E) + ch(E')$$

$$ch(E \otimes E') = ch(E) \bigcup ch(E'),$$

where $\bigcup$ denotes the cup product. The first few terms of the Chern character $ch(E)$ can be expressed in terms of Chern classes as follows:

$$ch(E) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots,$$

where r is the rank of E.

Now if our multiplicative sequence T is given by the series

$$f(t) = \frac{t}{(1 - e^{-t})}$$

then for our complex bundle E the class defined by

$$\tau(E) = T(c(E)) \in H^*(X(E))$$

is called the ***Todd class*** *of E. So the Todd class and the Chern character are related through their dependence on the total Chern class.*

As a consequence we have the following computational theorem

Theorem 2.43. *(Todd-Hirzebruch Formula).*

If X is any compact, connected complex manifold, then

$$\chi(X) = T(X)$$

For $n = 1$ this gives that $q(X) = g(X)$, where $g(X)$ is the ***topological genus*** *of X.*
For $n = 2$ we find the ***Noether's formula***

$$1 - q(X) + p_g(X) = \frac{1}{12}(c_1^2(X) + c_2(X)). \tag{2.4}$$

Applying theorem 2.41 to a line bundle $\mathcal{L}$ on a compact, connected smooth curve X, we get

$$h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}) = c_1(\mathcal{L}) - g(X) + 1$$

we also call $c_1(\mathcal{L})$ the ***degree*** *$deg(\mathcal{L})$ of $\mathcal{L}$, denoted sometimes by $\delta(\mathcal{L})$ as well (this is* ***the classical Riemann-Roch for curves****).*
For $n = 2$, rank $\mathcal{V}$=1:

$$h^0(X, \mathcal{V}) - h^1(X, \mathcal{V}) + h^2(X, \mathcal{V}) = \frac{1}{2}\, c_1(\mathcal{V})(c_1(\mathcal{V}) - c_1(K)) + T(X) \tag{2.5}$$

combining the above formula with Serre duality we get:

$$h^0(X, \mathcal{V}) - h^1(X, \mathcal{V}) + h^0(X, \mathcal{K}_X \otimes \mathcal{V}^\vee) = \frac{1}{2}c_1(\mathcal{V})(c_1(\mathcal{V}) - c_1(K)) + T(X). \tag{2.6}$$

Finally, for $dimX = n$, and rank $\mathcal{V} = 1$:

$$h^0(X, \mathcal{V}) - h^1(X, \mathcal{V}) + \ldots + (-1)^n h^n(X, \mathcal{V}) = \frac{c_1^n(\mathcal{V})}{n!} + \ldots. \tag{2.7}$$

where "..." stands for terms containing lower powers of $c_1(\mathcal{V})$.

Theorem 2.44. *Let X be a complex manifold, and $\mathcal{V}$ a holomorphic vector bundle on X. Let $Y = \mathbb{P}(\mathcal{V}^\vee) = \mathbb{P}(\mathcal{V})$* (the projectivization of $\mathcal{V}^\vee$ and $\mathcal{V}$ respectivelly) *and let $p : Y \longrightarrow X$ be the projection. We denote by $\mathcal{L}^\vee$ the* ***tautological*** *line bundle on Y.*

Then for every coherent sheaf $\mathcal{S}$ on X and for all $n \geq 1$ there are natural isomorphisms of $\mathcal{O}_X$-modules:

$$\begin{aligned} p_*(p^*(\mathcal{S})) &\xrightarrow{\sim} \mathcal{S} \\ p_*(\mathcal{L}^n \otimes p^*(\mathcal{S})) &\xrightarrow{\sim} S^n\mathcal{V} \otimes \mathcal{S} \\ p_{*i}(\mathcal{L}^n \otimes p^*(\mathcal{S})) &= 0, \ \forall\, i \geq 0. \end{aligned} \tag{2.8}$$

Where by $S^n\mathcal{V}$ we denote the n-th symmetric product of the holomorphic vector bundle $\mathcal{V} \longrightarrow X$.

By the Leray spectral sequence (see (Bott and Tu, 1982) chap. 3), or (Griffiths and Harris, 1994) chap. 5 *we have the following cohomological isomorphisms*

$$\begin{aligned} H^i(Y, p^*(\mathcal{S})) &\xrightarrow{\sim} H^i(X, \mathcal{S}) \\ H^i(Y, \mathcal{L}^n \otimes p^*(\mathcal{S})) &\xrightarrow{\sim} H^i(X, S^n\mathcal{V} \otimes \mathcal{S}), \ \ \forall i \geq 1 \end{aligned} \tag{2.9}$$

Lemma 2.45. *(Grothendieck, 1958) Let $\mathcal{F}$ be a locally free sheaf of rank r over a complex manifold X, and $Y = \mathbb{P}(\mathcal{F})$. Denote by $\mathcal{L}^\vee$ the tautological line sub-bundle of $p^*\mathcal{F}$ on Y, where $p : Y \longrightarrow X$ is the projection.*

We have the following exact sequence

$$0 \longrightarrow \mathcal{L}^\vee \longrightarrow p^*(\mathcal{F}) \longrightarrow \mathcal{L}^\vee \otimes \mathcal{W} \longrightarrow 0.$$

Together with the functoriality of the Chern classes, we get

$$p^*(c(\mathcal{F})) = c(\mathcal{L}^\vee \otimes \mathcal{W}).c(\mathcal{L}^\vee) \tag{2.10}$$

and by eliminating the Chern classes of $\mathcal{W}$ we have the following identity in the cohomology group $H^{2r}(\mathbb{P}(\mathcal{F}), \mathbb{Z})$

$$\sum_{j=0}^{r} c_1^j(\mathcal{L}^\vee) p^*(c_{r-j}(\hat{\mathcal{F}})) = 0 \tag{2.11}$$

where by $\hat{\mathcal{F}}$ we mean the dual sheaf to $\mathcal{F}$.

Upon expending and specifically when $r = 2$ we get

$$c_1^2(\mathcal{L}) + p^*(c_1(\mathcal{F})).c_1(\mathcal{L}) + p^*(c_2(\mathcal{F})) = 0 \tag{2.12}$$

where we have switched to $\mathcal{L}$ and $\mathcal{F}$ instead of their duals, which does not alter the above result.

Remark 2.46. *Equation (2.12) should be understood as addition and multiplication in the cohomology ring and therefore each term is in the class of $H^4(\mathbb{P}(\mathcal{F}), \mathbb{Z})$.*

Notation 2.47. *We write L as $O_{\mathbb{P}(\mathcal{F})}(1)$ and also as $H_{\mathcal{F}}$.*

In view of the above, keeping in mind that $\mathcal{F}$ denotes a rank-two locally free subsheaf, we can derive the following expressions:

Intersection Formulas ($r = 2$, $n = 2$)

If we multiply equation (2.12) with $p^*(c_1(\mathcal{F}))$ we get

$$c_1^2(\mathcal{L}).p^*(c_1(\mathcal{F})) + (p^*(c_1(\mathcal{F})))^2.c_1(\mathcal{L}) + p^*((c_1(\mathcal{F})).c_2(\mathcal{F})) = 0.$$

The last term is zero, since it comes from $H^6(X, \mathbb{Z}) = 0$.
While on the other hand, the second term

$$(p^*(c_1(\mathcal{F})))^2.c_1(\mathcal{L}) = p^*(c_1^2(\mathcal{F})).c_1(\mathcal{L}) = \left[c_1^2(\mathcal{F}))\right]$$

Therefor the above equation simplifies to

$$p^*(c_1(\mathcal{F})).c_1^2(\mathcal{L}) = -c_1^2(\mathcal{F}). \tag{2.13}$$

Now, if we multiply equation (2.12) with $c := c_1(\mathcal{L})$ we get:

$$c^3 + c^2.p^*(c_1(\mathcal{F})) + c.p^*(c_2(\mathcal{F})) = 0$$

By noting that the last term is equal to $[c_2(\mathcal{F})]$ and with the help of equation (2.13). we arrive at the very simple looking relation:

$$c^3 = c_1^2(\mathcal{F}) - c_2(\mathcal{F}) \tag{2.14}$$

for $c = c_1(\mathcal{L}) = c_1\left(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)\right)$.

Covering Tricks

Theorem 2.48. *(Branched covering trick)*

Given a holomorphic $\mathbb{P}_1$-bundle over an irreducible, complex space X, with total space B and projection $p : B \longrightarrow X$.

If S is any irreducible divisor on B meeting a general fiber F in n points, then there exists a complex manifold Y, a generically surjetive map $f : Y \longrightarrow X$ and n effective divisors $S_1, S_2, ..., S_n$ on **the fiber product** *$B' = B \times_X Y := \{(x, v) \mid p(x) = f(v)\} \subset B \times Y$, all meeting the general fiber $f^{-1}F$ of $B' \longrightarrow Y$ in one point, such that for the projection $g : B' \longrightarrow B$ we have $g^*(S) = S_1 + ... + S_n$.*

Proof. — For $n = 1$, $g^*(S) = S_1$ and there is nothing to prove.

For $n \geq 2$, we consider the desingularization on $\bar{S}$ of S and we put $B_1 = \bar{S} \times_X B$, and denoting by $g_1 : B_1 \longrightarrow B$ the natural projection, we have in a canonical way $g^*(S) = S_1 + S'$, where S_1 meets a general fiber F of $B_1 \longrightarrow \bar{S}$ in one point, while S' meets F in $n - 1$ points.

We can continue this procedure for $\bar{S}$ and so on, till we reach the desired result. □

As a consequence we have the following theorem:

Theorem 2.49. *Let X be a compact, connected complex manifold and $\mathcal{L}$ a holomorphic line bundle on X with $h^0(\mathcal{L}^{\otimes n}) \geq 2$ for some $n \geq 1$. Then there exists a compact complex manifold Y and a generically finite-to-one map $f : Y \longrightarrow X$, such that $h^0(f^*(\mathcal{L})) \geq 2$.*

Theorem 2.50. *Unbranched covering trick*

Let X be a connected complex manifold.

i) If $b_1(X) \neq 0$, then X admits unbranched coverings of any order.

ii) If $H_1(X, \mathbb{Z})$ contains k-torsion,then X has an unbranched covering of order k.

We will mainly be concerned with part i) of the theorem and therefore we outline its proof.

Proof. — i) If $b_1(X) \neq 0$, then $H_1(X, \mathbb{Z})$ is infinite and therefore admits cyclic quotient groups of any order. As a consequence the fundamental group $\pi_1(X)$ of X admits such quotients, and so there exist unbranched cyclic coverings of X of arbitrary order. $\square$

Finally we cite the following important theorem and definitions.

Theorem 2.51. *(Adjunction formula)*
If Y is a complex submanifold of codimension 1 of a complex manifold X, then

$$K_Y = K_X \otimes \mathcal{O}_X(Y)|_Y$$

where by "|" we denote the analytic restriction, and $\mathcal{O}_X(Y)|_Y \cong \mathcal{N}_{Y/X}$ the normal bundle introduced earlier (c.f. notation 2.4 v).

Definition 2.52. *Let X be a connected compact Kähler manifold and let $\{\omega_1, ...\omega_g\}$ be a basis for $H^0(\Omega_X)$ (where $g = h^{1,0}(X)$). The* **Albanese map** *α is defined by:*

$$\alpha = \left(\int_{z_0}^{z} \omega_1, ..., \int_{z_0}^{z} \omega_n\right) : X \longrightarrow \mathbb{C}^n.$$

It is well defined up to the period Λ given by

$$\Lambda = \left\{ \left(\int_{\gamma} \omega_1, ... \int_{\gamma} \omega_n\right) |\gamma \in \ H_1(\mathbb{Z}) \right\}$$

by homological invariance of the definition with respect to the path γ between z and z_0.

Definition 2.53. *We define the* **Kodaira dimension** *of a surface X denoted $Kod(X)$ by:*

$$Kod(X) = \lim_{m \to \infty} \frac{\log h^0\left(K_X^{\otimes m}\right)}{\log m}.$$

Having introduced some of the language, terminology and tools of algebraic geometry we turn now to more specific lemmas and propositions that we will use in the final proof. We refer the reader to (Griffiths and Harris, 1994), (Ueno, 1995), (Yang 1991) as well as (Friedman, 1998) and (Barth, Hulek, Peters, and Van De Ven, 2004) for an introduction to algebraic geometry and some of the concepts introduced so far.

2.2.2 En Route Towards Miyaoka-Yau Inequality

We know that to classify projective algebraic surfaces, we need only consider the *minimal models* (c.f. definition 2.63 bellow) of such surfaces, which follows from the following two theorems.

Theorem 2.54. *Every non singular surface with $Kod(X) \geq 0$ has a minimal model.*

Theorem 2.55. *If X is a nonsingular connected surface with $Kod(X) \geq 0$, then all minimal models of X are isomorphic.*

For a proof of the above theorems see (Barth, Hulek, Peters, and Van De Ven, 2004) p.99.

Definition 2.56. *By $P_m(X)$ we denote the* **m-th plurigenus** *of X which is equal to $h^0(\mathcal{K}_X^{\otimes m}), \forall\, m \geq 1$.*

Notation 2.57. *By $\mathcal{K}_X^{\otimes m}$ we mean the k-th tensor product of the canonical line bundle $\mathcal{K}_X$. We denote by K_X a divisor associated to $\mathcal{K}_X$. Hence, mK_X is associated to $\mathcal{K}_X^{\otimes m}$.*

The Kodaira Dimension

We have the following important theorem which relates the behavior of $P_m(X)$ for large m to the Kodaira dimension of X.

Theorem 2.58. *Let X be a compact connected complex manifold. Then:*

i) $Kod(X) \in \{-\infty, 0, ..., n\}$.

i) $Kod(X) = -\infty$ *if and only if* $P_m(X) = 0$ *for all* $m \geq 1$

iii) $Kod(X) = 0$ *if and only if* $P_m(X) = 0$ *or* 1 *for* $m \geq 1$, *but not always* 0

iv) $Kod(X) = k$, *for* $1 \leq k \leq \dim X \iff$ *there exists real constants* $\alpha > 0$, $\beta > 0$, *such that for* m *large enough (i.e.* $m \gg 0$*) we have*
$\alpha m^k < P_m(X) < \beta m^k$.
So for $k \geq 1$, *we have that* $P_n(X)$ *grows like* m^k.

We refer to (Ueno, 1997) for this result.

We note also the following two properties of the Kodaira dimension.

Theorem 2.59. *If* X_1 *and* X_2 *are compact connected complex manifolds, then*

$$Kod(X_1 \times X_2) = Kod(X_1) + Kod(X_2).$$

Theorem 2.60. *Let* X *and* Y *be compact, connected complex manifolds of the same dimension. If there exists a generically finite holomorphic map from* X *onto* Y, *then* $P_m(X) \geq P_m(Y)$ *for* $n \geq 1$, *hence* $Kod(X) \geq Kod(Y)$.
Furtheremore, if the map is an unramified covering, then $Kod(X) = Kod(Y)$.

Definition 2.61. *A line bundle* L *on a projective manifold* X *is called* **nef** *if for all curve* C *in* X *one has*

$$(L.C) \geq 0$$

where $(L.C)$ *stands for the intersection product of* $c_1(L) \in H^2(X)$ *and* $[C] \in H_2(X)$.

Notation 2.62. $Kod(X)$ *is sometimes written as* $\kappa(X)$.

Definition 2.63. *A non-singular projective surface* X *is called a* **minimal model** *if the canonical bundle* K_X *is nef.*

Notation 2.64. *Let* D *be a divisor on a surface* X. *We will set* $D^2 = c_1^2([D])$.

Theorem 2.65. (Serre, 1955) ***Serre's Criterion for Ampleness, Serre's Vanishing theorem:***

Let X *be a nonsingular projective surface. A divisor* A *on* X *is said to be* **ample** *if and only if any of the conditions below hold:*

i) For any coherent sheaf $\mathcal{F}$ on X

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(nA)) = 0, \quad i > 0, \quad \forall\, n \gg 0$$

ii) For any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{O}_X(nA)$ is generated by its global sections.

iii) For any line bundle $L \longrightarrow X$, $L \otimes \mathcal{O}_X(nA)$ is (very) ample for $n \gg 0$.

Remark 2.66. *Another condition for ampleness is the* ***Kleiman criterion*** *(Kleiman, 1966). Let X be a projective variety and D a Cartier divisor on X. Then D is ample if and only if there exist an $\epsilon > 0$ such that*

$$D.C \geq \epsilon\, ||C||$$

for all curve C in the real vector space of curves in X endowed with a fixed norm $||.||$.

In what follows we consider X to be a **minimal surface of general type** i.e. of $Kod(X) = 2$ and we work with algebraic varieties defined over the algebraic field $\mathbb{K} = \mathbb{C}$ of characteristic zero.

We arrive at one of the main pillars in the proof of the Miyaoka-Yau inequality, namely the following lemma.

Lemma 2.67. ***Kodaira's Lemma and the Positivity of $c_1^2(X)$***
Let X be a minimal model of dimension 2. Then,

$$Kod(X) = dim X = 2 \quad \textit{if and only if } K_X^2 > 0$$

Proof. — Suppose $Kod(X) = 2$.
Choose an $n_0 \in \mathbb{N}$ and $\alpha, \beta > 0$ such that as in theorem 2.58 .
Let A be a very ample divisor on X as characterized in Serre's theorem (theorem 2.65), and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(n_0 m K_X - A) \longrightarrow \mathcal{O}_X(n_0 m K_X) \longrightarrow \mathcal{O}_A(n_0 m K_X) \longrightarrow 0$$

which gives rise to the exact cohomology sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X(n_0mK_X - A)) \longrightarrow H^0(X, \mathcal{O}_X(n_0mK_X)) \xrightarrow{\eta} H^0(A, \mathcal{O}_A(n_0mK_X)) \longrightarrow \ldots .$$

We have by theorem 2.58 that $h^0(X, \mathcal{O}_X(n_0mK_X)) \geq \alpha m^{\kappa(X)} \quad \forall\, m \gg 0$.
Together with $h^0(A, \mathcal{O}_X(n_0mK_X)) \approx m.deg_A(n_0K_X|_A)$, we deduce that

$$0 \neq \ker(\eta) \subset H^o(X, \mathcal{O}_X(n_0mK_X - A)) \quad \text{for some} \quad m \in \mathbb{N}$$

and thus we can represent $n_0mK_X - A$ by a divisor D such that $D \approx l_0K_X$ where $D \in |n_0mK_X|$ is an effective divisor and we set $l_0 := n_0m$. Here by $|l_0K_X|$ we understand the linear system which is the set of all effective divisors linearly equivalent to D.
Now by Kleiman's criterion (c.f. remark 2.66)

$$l_0K_X^2 = K_X.(D + A) \geq K_X.A = \frac{1}{l_0}(D + A).A > 0$$

and hence

$$K_X^2 = c_1^2(X) > 0.$$

We now suppose that $K_X^2 > 0$.
By Riemann-Roch we get

$$\begin{aligned}
\chi(\mathcal{O}_X(mK_X)) &= h^0(\mathcal{S}, \mathcal{O}_X(mK_X)) - h^1(\mathcal{S}, \mathcal{O}_X(mK_X)) + h^2(\mathcal{S}, \mathcal{O}_X(mK_X)), \\
&= \frac{1}{2}(mK_X - K_X).mK_X + \chi(\mathcal{O}_X) \\
&\leq h^0(\mathcal{S}, \mathcal{O}_X(mK_X)) + h^2(\mathcal{S}, \mathcal{O}_X(mK_X))
\end{aligned}$$

and by Serre's duality for $m \gg 0$ we have that

$$h^2(\mathcal{S}, \mathcal{O}_X(mK_X)) = h^0(\mathcal{S}, \mathcal{O}_X(K_X - mK_X))$$

which must vanish since otherwise there would exists an effective divisor $G \in |(1-m)K_x|$ and therefore by Kleiman's criterion

$$0 > K_X(1 - m).A = (K_X - mK_X).A = G.A > 0, \quad \forall\, m \gg 0$$

a contradiction. Thus $h^0(\mathcal{S}, \mathcal{O}_X(mK_X)) \approx \frac{1}{2}m^2K_X^2 \;\; \forall\, m \gg 0$ and by theorem 2.58 we conclude that $Kod(X) = 2$. □

Definition 2.68. *A singular rational curve C with self intersection -1 (i.e. $(C.C) = C^2 = -1$) is called an* ***exceptional curve*** *or a* (-1)-***curves***.

We have the following two propositions on such curves (see (Barth, Hulek, Peters, and Van De Ven, 2004) p.91 for a proof of propostions 2.69 and 2.70 and p.270 for prop. 2.73).

Proposition 2.69. *An irreducible curve $C \subset X$ is a (-1) curve if and only if*

$$C^2 < 0, \quad \text{and } (K_X.C) < 0$$

Proposition 2.70. *Let X be a smooth compact, connected surface with $Kod(X) \geq 0$ and D an effective divisor on X such that $(K_X.D) < 0$, then D contains (-1)-curves (called* ***exceptional curves****).*

Definition 2.71. *A smooth rational curve with self-intersection -2 is called a (-2)-* ***curve***.

An important corollary of Kodaira's lemma (lem.2.67) is the following

Corollary 2.72. *Let X be a minimal surface of general type and C an irreducible curve on X. Then*

$$K_X.C \geq 0 \text{ and } K_X.C = 0$$

if and only if C is a (-2)-curve.

Proposition 2.73. *If there exists on an algebraic surface X an algebraic system of effective divisors, of dimension at least 1, such that the general member is a (possibly singular)* rational *or* elliptic curve, *then $Kod(X) \leq 1$.*

We give some inequalities involving the *Hodge numbers* via the following propositions and corollaries.

Proposition 2.74. *Let $f : X \longrightarrow S$ be a fibration and X_{gen} a nonsingular fibre, which we denote sometimes by F. Then*

i) $e(X_s) \geq e(X_{gen})$ *for all fibre* X_s

ii) If X is compact, then

$$e(X) = e(X_{gen}).e(S) + \sum_{s \in S}(e(X_s) - e(X_{gen})).$$

(See (Barth, Hulek, Peters, and Van De Ven, 2004) p.118 for a proof.)

Corollary 2.75. *If X is a compact surface and $F : X \longrightarrow S$ a fibration with fibre genus g_1 and base genus g_2, then*

$$e(X) \geq 4(g_1 - 1)(g_2 - 1).$$

The following useful proposition is known as the *DeFranchis-Severi-Castelnuovo lemma*.

Proposition 2.76. *If on a compact surface X, there are two linearly independent holomorphic 1-forms ω_1 and ω_2 with $\omega_1 \wedge \omega_2 \equiv 0$, then there exists a smooth curve C of genus $g(C) \geq 2$, a connected holomorphic map $k : X \longrightarrow C$ and 1-forms α_1, α_2, such that: $\omega_1 = k^*(\alpha_1)$ and $\omega_2 = k^*(\alpha_2)$ (where ω_1, $\omega_2 \in H^0(\Omega_X)$ and α_1, $\alpha_2 \in H^0(\Omega_C)$).*

Remark 2.77. *A map is called* connected *if its fibers are.*

Its corollary include the following inequality of Hodge numbers.

Proposition 2.78. *If the compact surface X does not admit a holomorphic map onto a curve of genus $g \geq 2$, then*

$$h^{2,0}(X) \geq 2h^{1,1}(X) - 3.$$

Remark 2.79. *If X is Kählerian (c.f. definition 1.16), then we can write the inequality in propostion 2.74 as*

$$p_g(X) \geq 2q(X) - 3.$$

Proposition 2.80. *If the compact surface X with $h^{1,0}(X) \geq 2$ does not admit any holomorphic map onto a curve of genus $g \geq 2$, then*

$$h^{1,1}(X) \geq 2h^{1,0}(X) - 1.$$

Corollary 2.81. *If the compact Kähler surface X does not admit any connected fibration with base genus $g \geq 2$ then*

$$h^{1,1}(X) \geq 2h^{1,0}(X) - 1.$$

(See (Barth, Hulek, Peters, and Van De Ven, 2004) p.158 for a proof of propositions 2.9 and 2.10.)

We now arrive at the second pillar in the proof of the Miyaoka-Yau inequality, namely

Proposition 2.82. *If X is any surface of general type (i.e. $Kod(X) = 2$), then*

$$c_2(X) > 0.$$

Proof. — There are two cases.

Case I If X **admits** a connected holomorphic map π onto a curve of genus $g(C) \geq 2$, then the general fibre F must have $g(F) \geq 2$

Proof. — We are given $g(C) \geq 2$ and $K_X = \bigwedge^2 \Omega_X$ together with $\pi : X \longrightarrow C$. We have the short exact sequence ($F = \pi^{-1}(x)$ is the generic fibre)

$$0 \longrightarrow \pi^* \Omega_C|_x \xrightarrow{\pi^*} \Omega_X|_F \longrightarrow \Omega_F \longrightarrow 0$$

and the corresponding

$$0 \longrightarrow \mathcal{T}_F \longrightarrow \mathcal{T}_X \xrightarrow{d\pi} \pi^* \mathcal{T}_C|_x \longrightarrow 0$$

As $\pi^*(\Omega_C|_x) = \mathcal{O}_F = \pi^*(\mathcal{T}_C|_x)$ this implies that $K_X^{\otimes m}|_F = \Omega_F = K_F$. Thus, $g(X) = g(F) \geq 2$ $\square$

From proposition 2.74 or corollary 2.75 we have that

$c_2(X) = e(X) \geq 4(g(C)-1)(g(F)-1) \geq 4$

Case II If X does **not admit** such a map. From the *Hodge-diamond*:

$$\begin{array}{lccccccccccc}
H^4 & & & h^4 & & & & & & h^0 & & \\
H^3 & & h^{3,0} & & h^{0,3} & & & & h^{3,0} & & h^{3,0} & \\
H^2 & h^{2,0} & & h^{1,1} & & h^{0,2} & \xrightarrow{Serre's\ duality} & h^{2,0} & & h^{1,1} & & h^{2,0} \\
H^1 & & h^{1,0} & & h^{0,1} & & & & h^{1,0} & & h^{1,0} & \\
H^0 & & & h^0 & & & & & & h^0 & &
\end{array}$$

Putting in the known numbers, as well as Pincaré duality $h^1 = h^3$

$$\begin{array}{lccccc}
H^4 & & & 1 & & \\
H^3 & & h^{1,0} & & h^{1,0} & \\
H^2 & h^{2,0} & & h^{1,1} & & h^{2,0} \\
H^1 & & h^{1,0} & & h^{1,0} & \\
H^0 & & & 1 & &
\end{array}$$

$$c_2(X) = e(X) = 2(1) - 4h^{1,0} + 2h^{2,0} + h^{1,1}$$

with $h^{2,0} := p_g(X)$ and $h^{1,0}(X) := q(X)$ (from definition 2.38), we have

$$c_2(X) = e(X) = 2 - 4q(X) + 2p_g(X) + h^{1,1}(X)$$

By proposition 2.78 and corollary 2.81 we see that $e(X) > 0$ unless one of the following two cases holds:

a) $q(X) = 1$; $p_g(X) = 0$

b) $q(X) = 2$; $p_g(X) = 1$.

We have two ways of finishing the proof.

First let us consider the following facts:

In **case a**), since $q(X) = 1$ and $p_g(X) = 0$, the albanese map α maps X onto an elliptic curve. Therefore we have by equations (2.1) and (2.2) that

$$\begin{aligned} h^{1,1}(X) &= e(X) + 2b_1(X) - 2 \\ &= e(X) + 2 \end{aligned}$$

which implies that

$$e(X) = h^{1,1}(X) - 2 = 0$$

and hence $c_2(X)$=0.

By Noether's formula we would thus get

$$c_1^2(X) + c_2(X) = 12(1 - q(X) + p_g(X)) = 0$$

which would force $c_1^2(X)$ to be zero, in contradiction with lemma 2.67.

In **case b)**, with $q(X) = 2$ and $p_g(X) = 1$, the Albanese map α is either a map from X onto a 2-torus T or a map onto a curve of genus at least 2 by proposition 2.76 which is excluded here.

In the case $\alpha : X \longrightarrow C^{q(X)}/\Gamma$ where here $q(X) = 2$, (i.e. the target space defines a 2-torus) , we have that:

$$\begin{aligned} c_2(X) = e(X) &= 2h^0(X) - 4h^{1,0}(X) + 2h^{2,0}(X) + h^{1,1}(X) \\ &= 2(1) - 4(2) + 2(1) + h^{1,1}(X) \\ &= -4 + h^{1,1}(X) \end{aligned}$$

Since $h^{1,1}(X) \geq h^{1,1}(T)$ we have that

$$c_2(X) = -4 + h^{1,1}(X) \geq -4 + h^{1,1}(T) = e(T) = 0$$

Since $c_2(X) = 0$ is excluded by Noether's formula we conclude that $c_2(X) > 0$ as desired.

The second way of finishing up the proof is by considering the following:

We are given $K_X^2 > 0$ from lemma 2.67, let us assume that there exists a surface X such that $q(X) > 0$ and $c_2(X) \leq 0$. The positivity of K_X^2 forces already $c_2(X) < 0$

(by Noether's formula) and hence $b_1(X) = 2q(X) \neq 0$. We have by i) in theorem 2.50 and by theorem 2.60, that X must admit unbranched covering (all of general type) of any order. We pick one, say Y, with $c_2(Y) < -3$. From case I above, this Y cannot admit a connected holomorphic map onto a curve of genus $g(C) \geq 2$, otherwise $c_2(Y)$ would be at least 4.

But by proposition 2.80 this would imply that $c_2(Y) \geq -3$, which is in contradiction with our original assumption. Thus, such a surface X does not exist and therefore $c_2(X)$ can only be positive.

□

Remark 2.83. *If one assumes **Bogomolov's theorm** (Bogomolov, 1979) on the instability of rank 2-vector bundles (and its generalization) which states that:*

a) *Given X an algebraic surface and H an ample divisor on X. Suppose that $\mathcal{V}$ is an H-stable vector bundle of rank 2 (generalized to rank r) on X. Then:*

$$c_1(\mathcal{V})^2 \leq 4c_2(\mathcal{V})$$

$$(r-1)c_1^2(\mathcal{V}) \leq 2rc_2(\mathcal{V}) \quad \textit{(generalized version)}.$$

b) *The cotangent bundle Ω_X of X is H-stable for some H.*

We see immediately, from lemma 2.67 that the positivity of $c_2(\Omega_X)$ is automatically satisfied.

We will need the following theorem, again a consequence of proposition 2.80.

Proposition 2.84. *If on an algebraic surface X there exists a line bundle $\mathcal{L}$ with*

$$h^0(\mathcal{L}^\vee \otimes \Omega_X^1) \neq 0,$$

then there exists a constant c such that

$$h^0(\mathcal{L}^{\otimes k}) \leq ck, \;\; \forall\, k \geq 1.$$

Proof. — We assume that $h^0(\mathcal{L}^{\otimes k_0}) \geq 2$ for some $k_0 \geq 1$; otherwise the result is trivial. We distinguish between two cases.

Case I: Suppose $k_0 = 1$. Take $s_1, s_2 \in H^0(\mathcal{L})$ to be linearly independent sections and let $h : \mathcal{L} \longrightarrow \Omega^1_X$ be a homomorphism, with $h \neq 0$, then $h(s_1)$ and $h(s_2)$ are linearly independent 1-forms on X, with $h(s_1) \wedge h(s_2) = 0$. Thus, we are in a position to apply proposition 2.76 and consequently, there exists a holomorphic map $f : X \longrightarrow C$ where C is a smooth curve, such that both $h(s_1)$ and $h(s_2)$ are pull-backs of 1-forms on C. It follows that the vanishing of s_1 on a curve D, implies that this curve is contained on some of the fibres of f and hence by remark 2.28 $\mathcal{L} \simeq \mathcal{O}_X(D)$. (In the case that $D = 0$, we take for s_1 the constant function 1).

Since $(D - nF).A < 0$ for some $n \gg 0$, A ample and F any given fibre, there are no non-zero divisors on X which are homologous to $k(D - nF)$ where $k \in \mathbb{N}$, $n \gg 0$. If we denote by F_k the divisor which consists of nk-general (smooth) fibres of f, we have the following standard exact sequence:

$$0 \longrightarrow \mathcal{O}_X(kD - F_k) \longrightarrow \mathcal{O}_X(kD) \longrightarrow \mathcal{O}_{F_k}(kD) \longrightarrow 0.$$

(Recall that we have $\mathcal{L} \simeq \mathcal{O}_X(D)$ and $\mathcal{O}_x(kD) \simeq \mathcal{L}^{\otimes k}$.) We find that:

$$h^0(\mathcal{L}^{\otimes k}) \leq h^0(\mathcal{O}_{F_k}(\mathcal{L}^{\otimes k})) \leq ck, \quad \forall\, k \geq 1.$$

Case II: Now let $k_0 > 1$ for the general case.

From theorem 2.49 there exists an algebraic surface Y and a holomorphic surjective map $\alpha : Y \longrightarrow X$ such that, $\alpha^*(\mathcal{L})$ has two independent sections.

Since

$$h^0(\mathcal{L}^\vee \otimes \Omega^1_X)) \cong h^0(\mathcal{H}om(\mathcal{L}, \Omega^1_X)) \neq 0$$

implies that

$$h^0(\mathcal{H}om(\alpha^*(\mathcal{L}), \Omega^1_Y) \neq 0,$$

we can apply to Y and $\alpha^*(\mathcal{L})$ the result of *Case I* above with $k_0 = 1$, i.e. there exists a constant c, such that for all $k \geq 1$ the inequality

$$h^0((\alpha^*(\mathcal{L}))^{\otimes k})) \leq ck$$

holds. But since

$$h^0(\mathcal{L}^{\otimes k}) \leq h^0((\alpha^*(\mathcal{L}))^{\otimes k}) \leq ck,$$

we are done.

□

As a final step towards the proof of Miyaoka-Yau inequality we will need the following proposition together with its generalization.

Proposition 2.85. *Let X be an algebraic surface, $\mathcal{O}_X(D)$ a line bundle on X, and $\mathcal{F}$ a locally free, rank-two subsheaf of Ω^1_X, such that:*

(i) $c_1(\mathcal{F}).S \geq 0$ *for every* effective *divisor S on X.*

(ii) $h^0(\mathcal{H}om(\mathcal{O}_X(D), \mathcal{F})) \neq 0$.

Then $c_1(\mathcal{F}).D \leq \max(c_2(\mathcal{F}), 0)$.

Proof. — We note that since

$$H^0(\mathcal{H}om(\mathcal{O}_X(D), \mathcal{F})) \cong H^0(\mathcal{F} \otimes \mathcal{O}_X(-D))) \neq 0,$$

there is a non-negative divisor S on X, such that $\mathcal{F} \otimes \mathcal{O}_X(-D-S)$ admits a section with at most isolated zeros.

Applying Todd-Hirzebruch (theorem 2.43), equation (2.5) for $dimX = 2$, and rank $\mathcal{V} = 1$:

$$c_2(\mathcal{F} \otimes \mathcal{O}_X(-D-S)) = (-1)^2(D+S)^2 - (D+S)c_1(\mathcal{F}) + c_2(\mathcal{F}) \geq 0.$$

We have that

$$c_1(\mathcal{F}).D \leq (D+S)^2 + c_1(\mathcal{F}).S + c_2(\mathcal{F}).$$

But by assumption (i) in the hypothesis, $c_1(\mathcal{F}).S \geq 0$ and therefore

$$c_1(\mathcal{F}).D - (D+S)^2 \leq c_2(\mathcal{F}).$$

If $(D+S)^2 \leq 0$, we are done.

On the other hand, if $(D+S)^2 > 0$, then applying Riemann-Roch with Serre's duality (i.e. $h^0(\mathcal{O}_X(n(D+S))+h^2(\mathcal{O}_X(n(D+S)) = h^0(\mathcal{O}_X(n(D+S))+h^0(\mathcal{O}_X(K_X-n(D+S)))$ together with theorem 2.58, there exists a $d>0$ such that for $n \gg 0$ we have

$$h^0(\mathcal{O}_X(n(D+S))+h^0(\mathcal{O}_X(K_X-n(D+S))>dn^2$$

So we either have

$$h^0(\mathcal{O}_X(n(D+S))>\frac{1}{2}dn^2$$

or

$$h^0(\mathcal{O}_X(K_X-n(D+S))>\frac{1}{2}dn^2 \ for\ infinite\ number\ of\ n.$$

Since $h^0(\mathcal{H}om(\mathcal{O}_X(D),\mathcal{F}))\neq 0$ implies that $h^0(\mathcal{H}om(\mathcal{O}_X(D+S),\Omega^1_X)\neq 0$, we can apply proposition 2.84 and therefore there exists a c such that $h^0(n(D+S)\leq cn$ for all $n\geq 1$ excluding the first possibility.

In the second possibility, we have by (i)

$$c_1(\mathcal{F}).K_X - c_1(\mathcal{F}).n(D+S)\geq 0.$$

Rearranging the terms,

$$c_1(\mathcal{F}).\frac{K_X}{n}\geq c_1(\mathcal{F}).(D+S)$$

As the left hand side goes to zero when n tends to infinity,

$$c_1(\mathcal{F}).(D+S)\leq 0,$$

which tells us that

$$c_1(\mathcal{F}).D > -c_1(\mathcal{F}).S \leq 0$$

and therefore that

$$c_1(\mathcal{F}).D\leq 0.$$

□

We generalize proposition 2.85 by considering the n-th symmetric power of the locally free subsheaf $\mathcal{F}$ as follows:

Proposition 2.86. *Let X be as in proposition 2.85 and consider the n-th symmetric power of the locally free subsheaf $\mathcal{F}$ such that assumption (ii) of proposition 2.85 reads: (ii') $h^0(\mathcal{H}om(\mathcal{O}_X(D), S^n\mathcal{F})) \neq 0$ together with $c_1(\mathcal{F}).S > 0$, Then*

$$c_1(\mathcal{F}).D \leq \max(nc_2(\mathcal{F}), 0).$$

Proof. — Let Z be the projectivization of $\mathcal{F}$, i.e. $Z = \mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{F}^\vee)$ and $p : Z \longrightarrow X$ the projection, then by theorem 2.44, there exists a divisor class $H = H_{\mathcal{F}}$ on Z such that by applying equation (2.9) we have a canonical isomorphism between $H^0(\mathcal{O}_Z(nH + p^*(E)))$ and $H^0(S^n\mathcal{F} \otimes \mathcal{O}_X(E))$ for any given divisor E on X. In our case there exists an effective divisor G on Z, such that

$$\mathcal{O}_Z(G) = \mathcal{O}_Z(nH - p^*(D)).$$

Furthermore, the *Branched Covering Trick* (c.f. theorem 2.48) tells us that there exists an algebraic surface Y, together with a surjective map $f : Y \longrightarrow X$ with $deg(f) = k$, such that under the induced bundle map from $\mathbb{P}(f^*(\mathcal{F}))$ into $\mathbb{P}(\mathcal{F})$ and $q : \mathbb{P}(f^*(\mathcal{F})) \longrightarrow Y$ the projection, we have that the pull-back of G decomposes into a sum of effective divisors representing $H_{q^*(\mathcal{F})} - q^*(D_i)$. The D_i's need not be effective, but we have that

$$D = \sum_i D_i.$$

By the canonical isomorphism of theorem 2.44 and assumption (ii') in our hypothesis, we have that

$$h^0(\mathcal{H}om(\mathcal{O}_Y(D_i), f^*(\mathcal{F}))) \neq 0.$$

Together with the functoriality of the Chern classes we have that

$$c_1(f^*(\mathcal{F})).P = c_1(\mathcal{F}).f_*(P) \geq 0, \tag{2.15}$$

for every effective divisor P on Y.

Observe that equation (2.15) is the equivalent of the assumption (i) in proposition 2.85. Therefore from proposition 2.85 we have:

$$c_1(f^*(\mathcal{F})).D_i \leq \max(c_2(\mathcal{F}), 0).$$

By summing up over i and with the above observation we get

$$f^*(c_1(\mathcal{F}).D) \leq \max(nc_2(f^*(\mathcal{F})), 0)$$

$$kc_1(\mathcal{F}).D \leq k\max(nc_2(\mathcal{F}), 0)$$

which implies the required result. □

2.2.3 Proof of the Miyaoka-Yau Inequality (Algebraic-Geometry)

Finally, we are in a position to prove Miyaoka-Yau inequality from the algebraic geometric point of view using the tools we have developed in the first section while relying on the lemmas and propositions we have discussed in the second section.
We will state briefly once again the assumptions behind our approach and then go on directly to prove the theorem.

Since **blowing up** (a point on a complex manifold M, consists in replacing a point p of M by the set of (complex) tangent directions around the point, leaving unchanged the remainder of M) increases the Euler number (i.e. $c_1^2(X)$ goes down while $c_2(X)$ goes up) we may assume together with theorem 2.54 and proposition 2.55 that our surface X is a minimal surface of general type (i.e. of $Kod(X) = 2$). We will work as well, with algebraic varieties defined on the closed field $\mathbb{C}$ of complex numbers of characteristic zero.

Theorem 2.87. *(Main Theorem)*
For every surface of general type X, the inequality $c_1^2(X) \leq 3c_2(X)$ holds.

Proof. —(A slight simplification of (Miyoka, 1977))
We set

$$\alpha := \frac{c_2(X)}{c_1^2(X)} < \frac{1}{3} \tag{2.16}$$

and derive a contradiction based on that assumption.
Let β be given by

$$\beta = \frac{1}{4}(1 - 3\alpha) \tag{2.17}$$

such that

$$\alpha + \beta = \frac{1}{4}(\alpha + 1)$$

and let n be a natural number such that $n(\alpha + \beta) \in \mathbb{Z}$.

We consider the vector bundle $\mathcal{V}_n$ that we set to:

$$\mathcal{V}_n := S^n\Omega^1_X \otimes \mathcal{O}_X(-n(\alpha + \beta)K_X), \tag{2.18}$$

where with the same notation as in equation (2.8), $S^n\Omega^1_X$ denotes the n-th symmetric product of the cotangent bundle.

We note in passing that the dual of this vector bundle is

$$\begin{aligned}
\mathcal{V}_n^\vee &= \left(S^n\Omega^1_X \otimes \mathcal{O}_X(-n(\alpha + \beta)K_x)\right)^\vee \\
&= \left(S^n(\mathcal{T}^1_X \otimes K_X^{-1}) \otimes \mathcal{O}_X(n(\alpha + \beta)K_X + K_X)\right) \\
&= \left(S^n\mathcal{T}^1_X \otimes K_X^{-n} \otimes \mathcal{O}_X(K_X^{n(\alpha+\beta)+1})\right).
\end{aligned} \tag{2.19}$$

Here we used the fact that given a locally free sheaf $\mathcal{F}$ of rank-2 with $\det \mathcal{F} := K$, $\mathcal{F}^\vee \otimes K = \mathcal{F}$ (K_X the canonical divisor of X in the case $\mathcal{F} = \Omega^1_X$).

This is true via the *splitting principle* so that we can write:

$$\mathcal{F} = \mathcal{L} \oplus \mathcal{L}',$$

while

$$K = \mathcal{L} \otimes \mathcal{L}'$$

and verify that

$$\det \mathcal{F} = K$$

as stated above and

$$\begin{aligned}
\mathcal{F}^\vee \otimes K &= (\mathcal{L} \oplus \mathcal{L}')^\vee \otimes K \\
&= \mathcal{L}'^{\vee} \otimes K \oplus \mathcal{L}^\vee \otimes K \\
&= \mathcal{L}'^{\vee} \otimes (\mathcal{L}' \otimes \mathcal{L}) \oplus \mathcal{L}^\vee \otimes (\mathcal{L} \otimes \mathcal{L}') \\
&= \mathcal{L} \oplus \mathcal{L}' = \mathcal{F}
\end{aligned}$$

where in the last step we have used the fact that $A \otimes B = -B \otimes A$ and its dual.

Let us look at the Euler characteristic of our vector bundle $\mathcal{V}_n$ of equation (2.18). From equation (2.3) we can write

$$\chi(\mathcal{V}_n) = h^0(\mathcal{V}_n) - h^1(\mathcal{V}_n) + h^2(\mathcal{V}_n) \tag{2.20}$$

We will specialize to $\mathcal{F} = \Omega^1_X$ and our divisor $D = n(\alpha+\beta)$ and write:

$$h^0(S^n\mathcal{F} \otimes \mathcal{O}_X(-D)) = h^0(S^n\Omega^1_X(-n(\alpha+\beta)K_X)).$$

(We can work with $\mathcal{F}$ and K instead of Ω^1_X and K_X, the final result and proof are identical with minor changes.)

From Serre's duality (c.f. theorem 2.40) we have:

$$\begin{aligned} h^2(\mathcal{V}_n) &= h^0(\mathcal{V}_n^\vee \otimes K_x) \\ &= h^0(S^n\Omega^1_X \otimes \mathcal{O}_X((n(\alpha+\beta-1)+1)K_X)) \equiv h^0(S^n\mathcal{F} \otimes \mathcal{O}_X((n(\alpha+\beta-1)+1)K)) \\ &= h^0(S^n\mathcal{F} \otimes \mathcal{O}_X(K^{(n(\alpha-3)+4)/4})) \end{aligned}$$

We **claim** that: *i)* $h^0(\mathcal{V}_n) = 0$ and *ii)* $h^2(\mathcal{V}_n) = 0$

i) $h^0(\mathcal{V}_n) = 0$

Proof. —

$$\begin{aligned} h^0(\mathcal{V}_n) &= h^0(S^n\mathcal{F} \otimes \mathcal{O}_X(-n(\alpha+\beta)K)) = h^0(\mathcal{H}om(\mathcal{O}_X(n(\alpha+\beta)K, S^n\mathcal{F})) \\ &= h^0(S^n\mathcal{F} \otimes \mathcal{O}_X(-n(\alpha+1)K/4)) \end{aligned}$$

where in the last step we have used the fact that $\alpha+\beta = \frac{1}{4}(\alpha+1)$.

Let us consider a divisor $D = \frac{n}{4}(\alpha+1)K$ and $c_1(\mathcal{F}) \equiv K$.

We compute the intersection

$$c_1(\mathcal{F}).D = \frac{n}{4}(\alpha+1)K.K = \frac{n}{4}(\alpha+1)K^2.$$

By lemma 2.67, $c_1^2(\mathcal{F}) \equiv K^2 > 0$, thus we have that

$$c_1(\mathcal{F}).D = \frac{n}{4}\left(\frac{c_2(\mathcal{F})}{c_1^2(\mathcal{F})} + 1\right) K^2 = \frac{n}{4}(c_2(\mathcal{F}) + c_1^2(\mathcal{F})).$$

This implies that

$$c_1(\mathcal{F}).D > \frac{n}{4} c_2(\mathcal{F}) > 0$$

(on account of $c_2(\mathcal{F}) > 0$ proposition 2.82).

By proposition 2.86 we thus have that

$$h^0(\mathcal{S}^n\mathcal{F} \otimes \mathcal{O}_X(K^{-n(\alpha+\beta)}) = 0 = h^0(\mathcal{V}_n).$$

□

ii) $h^2(\mathcal{V}_n) = 0$

We proceed in a similar manner:

Proof. — We look at $h^0(\mathcal{S}^n\mathcal{F} \otimes \mathcal{O}_X((n(\alpha+\beta-1)+1)K))$ and proceed as before. We compute the intersection of $c_1(\mathcal{F}) = K$ and the divisor $\widetilde{D}$ given by

$$\widetilde{D} = -[n(\alpha+\beta-1)+1]K.$$

$$\begin{aligned} c_1(\mathcal{F}).\widetilde{D} &= -K^2\ [n(\alpha+\beta-1)+1] \\ &= -K^2\ [\frac{n}{4}(\alpha-3)+1] \\ &= \frac{K^2}{4}\ [n(3-\alpha)-4] \\ &\geq \frac{K^2}{4} n(3-\alpha). \end{aligned}$$

Now

$$\begin{aligned} \frac{K^2}{4} n(3-\alpha) &= \frac{K^2}{4} n \left(3 - \frac{c_2(\mathcal{F})}{c_1(\mathcal{F})}\right) \\ &= \frac{K^2}{4c_1^2(\mathcal{F})} n(3c_1^2(\mathcal{F}) - c_2(\mathcal{F})) \\ &= \frac{1}{4} n(3c_1^2(\mathcal{F}) - c_2(\mathcal{F})) \end{aligned}$$

By our assumption in equation (2.16), $c_1^2(\mathcal{F}) > 3c_2(\mathcal{F})$ which implies that

$$3c_1^2(\mathcal{F}) > 9c_2(\mathcal{F})$$

and therefore,

$$c_1(\mathcal{F}).\widetilde{D} > \frac{1}{4}n(9c_2(\mathcal{F}) - c_2(\mathcal{F})) = 2nc_2(\mathcal{F}).$$

This gives

$$c_1(\mathcal{F}).\widetilde{D} > 2nc_2(\mathcal{F}), \quad for\ all\ n \gg 0.$$

As $c_2(\mathcal{F}) > 0$ we arrive at

$$c_1(\mathcal{F}).\widetilde{D} > \max(nc_2(\mathcal{F}), 0).$$

Therefore on account of proposition 2.86 we conclude that:

$$h^0(\mathcal{S}^n\mathcal{F} \otimes \mathcal{O}_X((n(\alpha+\beta-1)+1)K)) = 0.$$

$\square$

Continuing with the proof of theorem 2.87, the fact that $h^0(\mathcal{V}_n) = h^2(\mathcal{V}_n) = 0$, implies using equation (2.20) that

$$\chi(\mathcal{S}^n\mathcal{F} \otimes \mathcal{O}_X(-n(\alpha+\beta)K) < 0 \quad \forall\, n \gg 0. \tag{2.21}$$

On the other hand, we can compute $\chi(\mathcal{S}^n\mathcal{F}\otimes\mathcal{O}_X(-n(\alpha+\beta)K)$ from the Todd-Hirzebruch formula (c.f. theorem 2.43 equation (2.7)) and we get the following result:

$$\begin{aligned}
\chi(\mathcal{V}_n) &= \chi(\mathcal{S}^n\mathcal{F} \otimes \mathcal{O}_X(-n(\alpha+\beta)K) \\
&= \chi(\mathcal{L}^n \otimes p^*(\mathcal{O}_X(-n(\alpha+\beta)K))) \quad \text{(by theorem 2.44, equation two in (2.8)).} \\
&= c_1^3(\mathcal{L} \otimes p^*(\mathcal{O}_X(-(\alpha+\beta)K)))\frac{n^3}{3!} + \gamma n^2 + \delta n + \epsilon. \qquad (2.22)
\end{aligned}$$

We finally **claim** that $c_1^3(\mathcal{V}_n) > 0$,

Proof. Our claim is that

$$c_1^3(\mathcal{V}_n) \sim c_1^3(\mathcal{L} \otimes p^*(\mathcal{O}_X(-(\alpha+\beta)K)))\frac{n^3}{3!} > 0$$

$$\begin{aligned} c_1^3(\mathcal{L} \otimes p^*(\mathcal{O}_X(-(\alpha+\beta)K))) &= (c-(\alpha+\beta)p^*(-c_1(\mathcal{F})))^3, \\ &= c^3+3(\alpha+\beta)c^2.p^*(c_1(\mathcal{F}))+3(\alpha+\beta)^2c.[p^*(c_1(\mathcal{F})]^2. \end{aligned}$$

Recall that $c_1^3(\mathcal{F}) \in H^6(\mathbb{P}(\mathcal{F}), \mathbb{Z} = 0$. Where we put $c := c_1(\mathcal{L})$. Therefore the equation simplifies to:

$$c_1^3(\mathcal{L} \otimes p^*(\mathcal{O}_X(-(\alpha+\beta)K))) = c^3 + 3(\alpha+\beta)c.[cp^*(c_1(\mathcal{F})) + (\alpha+\beta)p^*(c_1^2(\mathcal{F}))].$$

Rewriting equation (2.14) as:

$$c^3 = (1-\alpha)c_1^2(\mathcal{F})$$

and recalling that $c_2(\mathcal{F}) = \alpha\ c_1^2(\mathcal{F})$, we get:

$$\begin{aligned} c_1^3(\mathcal{L} \otimes p^*(\mathcal{O}_X(-(\alpha+\beta)K))) &= c^3+3(\alpha+\beta)c.[-c^2-p^*(c_2(\mathcal{F}))+(\alpha+\beta)p^*(c_1^2(\mathcal{F}))] \\ &= c^3+3(\alpha+\beta)c.[-c^2-\alpha p^*(c_1^2(\mathcal{F}))+\alpha p^*(c_1^2(\mathcal{F}))+\beta p^*(c_1^2(\mathcal{F}))] \\ &= c^3+3(\alpha+\beta)[-c^3+\beta c.p^*(c_1^2(\mathcal{F}))] \end{aligned}$$

By the intersection formulas equations (2.13) and (2.14) we get:

$$\begin{aligned} c_1^3(\mathcal{L} \otimes p^*(\mathcal{O}_X(-(\alpha+\beta)K))) &= c_1^2(\mathcal{F})[(1-\alpha)+3(\alpha+\beta)(-(1-\alpha)+\beta)] \\ &= \frac{c_1^2(\mathcal{F})}{16}[16-16\alpha+3(1+\alpha)(\alpha-3)] \\ &= \frac{c_1^2(\mathcal{F})}{16}[16-16\alpha+3\alpha-9+3\alpha^2-9\alpha] \\ &= \frac{c_1^2(\mathcal{F})}{16}[3\alpha^2-22\alpha+7] \\ &= \frac{c_1^2(\mathcal{F})}{16}[(3\alpha-1)(\alpha-7)] > 0 \qquad (2.23) \end{aligned}$$

The last step follows from the fact that $c_1^2(\mathcal{F}) > 0$ and our assumption equation (2.16).

□

which would imply that

$$\chi(\mathcal{V}_n) > 0.$$

But this is in contradiction with our previous calculations of $\chi(\mathcal{V}_n)$ from its definition, therefore our assumption that $\alpha \leq 1/3$ is false and our main theorem is thus proved.

That is:

For every surface X of general type, the inequality $c^2(\mathcal{F}) \leq 3c_2(\mathcal{F})$ holds for $\mathcal{F}$ a subsheaf of Ω^1_X. $\square$

2.3 Discussion and Conclusion

We have arrived at our aim, mainly to prove the Miyaoka-Yau inequality using the algebraic geometric approach putting all the developed tools, lemmas and propositions at use to accomplish this aim. We have tried to remain as transparent and complete as possible in writing the proof, leaving very little out for the reader to figure out. All the steps even the trivial ones were worked out completely as this presentation is meant to be pedagogical in its approach benefiting the experts and the non-expert equally.

The baggage needed for algebraic geometry is so broad and abstract that one might find it difficult to understand what algebraic geometry is all about. The need for such a vast preparation arises from the way the subject was developed. Algebraic geometry was essentially developed to put on firm grounds the works of Monge's *Géometrie* (1795), Möbius, Plücker, and Cayley's projective geometry, Bernhard Riemann's birational geometry, as well as the works of Gauss, Euler and Abel to site just a few of the pioneers and fathers of this subject. Rigorous constructions of the theory were needed to avoid paradoxes arising from naive intuition and by the way algebraic geometers borrowed from every branch of mathematics to patch together the skeleton of their theory (see (Dieudonné, 1985) for the history of algebraic geometry. In the next section, we suggest some references at various levels to the interested readers for further consultation.

2.4 Remarks on the Bibliography

Readers with a good background in algebra and seek an introduction to algebraic-geometry should consult one of the following (Ueno, 1995) or/and (Yang, 1991) which provide a good introduction to the abstract language and techniques of sheaves, and pre-sheaves as well as giving a very clear exposition of the notion of divisors with very good examples and explanation.

For more advanced readers, I would recommend (Griffiths and Harris, 1994) for both the differential geometric content and the algebraic-geometric material. A masterpiece to consult is (Hartshorne, 1977) which is a classic on algebraic geometry and a prerequisite to more advanced texts such as (Matsuki, 2002) and (Barth, Hulek, Peters, and Van De Ven, 2004).

A personal favourite of mine is (Yang, 1991) cited above, both for its style, material and clarity and is a very good place to learn the ABC of the Riemann-Roch theorem for curves paving the way for understanding the higher dimensional generalization.

FURTHER DISCUSSION AND CONCLUSION

The Kähler-Einstein condition for a manifold provided us with a very simplifying tool to prove the Miyaoka-Yau inequality following the footsteps of S.T.Yau's differential geometric approach as well as away to understand its complex structure by metrics. The argument also showed when the equality holds, and this fact turned out to be useful for proving the Severi conjecture (that $\mathbb{CP}^2$ has only one complex structure, namely the obvious one; this is sort of like a complex analogue of the Poincaré conjecture).
The inequality is optimal as the equality it is achieved by quotient of the complex ball. However, it is still an open question to prove *analytically* that the equality $3c_2(M) = c_1^2(M)$ implies that either M is $\mathbb{C}P^2$ or quotient of the ball.

Kähler-Einstein metrics with cosmological constant zero (i.e. Ricci-flat Kähler metrics) are also used in algebraic geometry and string theory, for instance in establishing various versions of Torelli's theorem. The Kähler-Einstein metric seems to serve in some sense as a concrete "witness" of the fact that a certain bundle is stable (i.e. there exits a link between these metrics and the algebraic-geometric stability of the underlying manifold).

To bridge the gap between chapter one and chapter two, we point out the paper by (R. Kobayashhi, 1985) which generalizes the Miyaoka-Yau *result* by considering a compact normal surface X with only *quotient* singularities. The canonical divisor bundle for such a surface still makes sense up to a multiple and its ampleness guarantees by the same analysis as that of Yau to produce an orbifold Kähler-Einstein metric and thus the same inequality of Miyaoka-Yau. Now for surfaces of general type, its canonical model is obtained by contracting the (-2)-curves ending in a normal surface with only ordinary double points as singularities (these are just the order 2 quotient singularities). Hence Yau's result in this setting implies Miyaoka's result for a surface of general type.

We would like to point out as well that the Miyaoka-Yau *inequality* has been generalized to higher dimensions (see for example (Lu and Miyaoka 1998) for such a generalization) which could be of importance in String theory. We leave such a realization of the inequality in String Theory as an important question to study, together with finding possible bounds on the Hodge numbers in a higher dimensional set-up.

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