AN ACTIVITY ENTAILING EXACTNESS AND APPROXIMATION OF ANGLE MEASUREMENT IN A DGS

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We describe here an activity seeking a coordination between geometry and arithmetic. It starts from a geometric manipulation with GeoGebra – the iterate rotation of an isosceles triangle forming a regular polygon – and gradually leads students to consider the divisors of 360 and to reflect upon approximations and exact representations of rational numbers related to angle measurements. Through the issue of measurement in geometry, the activity takes place in the 'geometer-physicist' paradigm proposed by Tanguay & Geeraerts (2012) and instilled by Jahnke (2007). We propose an a priori *analysis of the activity, set within the theoretical frame of Kuzniak (2010, 2013) on* Espace de Travail Mathématique (ETM).

Keywords : linking arithmetic and geometry, exactness, approximation, measurement, dynamic geometry software, geometer-physicist's paradigm

In his most recent contribution to the symposium *Espace de Travail Mathématique* (ETM 3, Université de Montréal, October 2012), A. Kuzniak (2013) reflects, from a teaching and learning standpoint, on the role expected from bridging different mathematical fields when students are involved in a problem-solving process requiring a back and forth between two fields [1]. As a typical example, we may think of these problems in which one must optimize the area of a polygon subject to inscriptibility constraints, and where geometry is bridged to the algebraic/functional field. In the present article, we propose the *a priori* analysis (Artigue, 1988) of an activity mingling geometrical work and arithmetical work with GeoGebra. We will set it in a theoretical framing based on the work of Kuzniak (2010, 2013) on ETMs. By doing so, some aspects of the frame will be examined.

PHYSICS, BETWEEN EVERYDAY THINKING AND MATH THINKING

In a 2007 article, H. N. Jahnke compares generally valid statements depending on whether they come from mathematics, from physics or from everyday life. General statements from everyday life and physics share their empirical basis and the fact that the set of all conditions limiting their scope of validity is virtually unattainable. In everyday life, searching for completeness as regards these conditions is more often irrelevant. For instance in a given specific context, one may examine some precise conditions invalidating the statement "every evening, Johnny comes back from work around 18:00", but trying to figure out all possible misfortune in the world would be foolish. By contrast in physics, we try to relate each general statement to the most accurate domain of validity, even if we know that the theory will always remain subject to falsification by new observed phenomena (e.g. Popper, 1991). In established mathematics, determining the domain of validity of a general statement is not only possible but essential: it is indeed what accounts for the way mathematics operate, according to which the set of conditions is fixed and closed by the building of a (preferably axiomatized) theory.

In this respect, physics could be situated at the passage between the two others: as in mathematics, general statements in physics are connected by hypothetico-deductive developments that integrate them into a network and build them up as a theory. Yet the empirical bases of physics are not disqualified but rather enriched: any experimental verification about a statement not only corroborates it, but also increases the conviction that all other statements connected to it in the network are true. These considerations bring Jahnke (2007) to advance that in the classroom, dealing with mathematics as in physics would provide a more harmonious transition between everyday life thinking and mathematical thinking, and would also lay out a stronger epistemological foundation for teaching proof, with respect to its aim and to the type of certainty it brings.

A TRANSITION PARADIGM RECONCILING MEASURES AND PROOF

Following Jahnke, we propose in Tanguay & Geeraerts (2012, 2013), for the first years of secondary school, an approach of synthetic geometry [2] modelled after experimental physics. We speak of it in terms of *paradigm*, namely the *geometer-physicist paradigm*. We use the term 'paradigm' first to put forward its role as an articulation between two paradigms, the GI paradigm referring to the geometry of perception and intuition (Houdement & Kuzniak, 2006) and the GII paradigm referring to classical euclidean geometry. But also to stress the importance, regarding this approach, of considering the class (the school group) as a scientific community (Wenger, 1998) where are decided and assumed the motivations, the premises, the (didactical) contracts and prescriptions at the basis of such an experimental practice: it appears to us an essential condition for engaging students into progressively moving this practice towards the building up of a theory. Finally, because according to the historical/epistemological analysis of Jahnke (2010), such an approach would have been the one followed by the Pre-Socratic geometers, thus being at the very source of the developments that led to Euclid's *Elements* as a culmination.

With this paradigm, experimentation and empirical validations are brought back into the fold. In geometry, experimentation is mainly conveyed through construction and measurement: with a ruler, a protractor, a compass, or with specific functionalities of dynamic geometry software packages (DGSs) such as *Cabri-Géomètre* or *GeoGebra*. Researchers and teachers have often blamed measurement for engendering a hindrance to proof and proving : why prove something that can be verified simply by measuring? The measurement tools of DGSs magnify this effect (e.g. Boclé, 2008) because of their precision, but also because the dragging functionality allows an efficient investigation of examples. In the approach that we propose, each tackled statement is written and diagrammatically illustrated on a card (24 cm × 16 cm). Statements verified experimentally and statements proved deductively are systematically distinguished, and the two different statuses are made apparent on the cards. These are classified in a ring binder that each student has always at hand in his working space. The binder concretely accounts for the theoretical frame of reference. For more details, see Tanguay & Geeraerts (2012, §3). Empirical verification does not come into conflict with proving to the extent that every experimentally validated result keeps an hypothetical status which is explicitly stated and exhibited, and that the certainty of any deductively proved result remain dependent on results the proof uses as 'rules of inferences' (Duval, 1991 or Tanguay, 2007), in particular on the hypothetical results among them :

> The epistemological motivation of proof is not to be founded on the idea that proofs in contrast to measurement provide absolute certainty, but on the idea that proofs open new and more complex possibilities of empirical corroboration. In short, in an empirical environment *proofs do not replace measurements but make them more intelligent*. (Jahnke, 2007, p. 83, italics from the original text)

MEASUREMENTS, APPROXIMATIONS AND NUMBERS

To consider experimentation with measurement as would do a physicist in his lab, one must assume that measuring, even with DGSs, provides nothing more than approximations (see Tanguay & Geeraerts, 2013). In our opinion, neither textbooks nor ministerial programs deal adequately with this issue, leaving in limbo the epistemological status of measurement in geometry : « It is for us symptomatic that institutional teaching resources rebuke equalities such as $\frac{4}{3} = 1{,}33$ or $\sqrt{2} = 1{,}414$, but in the same time agree without a murmur with equalities such as $mes[AB] = 5$ cm, in contexts where inferred measures and measures obtained with the geometry tools are blithely combined » (Tanguay & Geeraerts, 2012, p. 21; our translation).

Besides, discussing about measurements as approximations allows discussions about the ideal character of the measured objects : points without dimension, line segments and lines without thickness, angles sprawling at infinity, with edges of zero measurement… These discussions then lead to institutionalizations whose content is no more the sole responsibility of the teacher. For instance, we may well imagine a class-discussion about the following GeoGebra display :

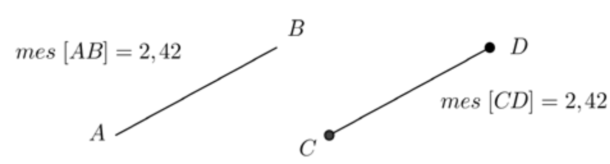

Fig 1 : what is the 'thickness' of point *A* ? of point *C* ?

If we now analyse how measurement can be situated into the *mathematical working space* (Kuzniak, 2010, 2013), using Kuzniak's two planes model, we observe that measurement contributes to, and is part of, both *figural genesis* and *instrumental genesis*. The latter is of course related to the measuring tools. As for figural genesis, the outcome follows from the fact that while putting a ruler down on a line segment with the aim of measuring it, the student tracks it down and isolates it as the side of a given figure, whose *dimensional deconstruction* (Duval, 2005) is thus sparked off : the sides must be located as the boundaries of the shape, the « corners » as endpoints of the sides, the endpoints being matched with the graduations of the ruler. This dimensional deconstruction is certainly resulting from a form of visualization but also refers to a set-theoretical modelling of the plane, with points as (atomic) elements and the sides as subsets of the figure. In this sense, through the cognitive activity of measuring, there is indeed a projection into the theoretical frame of reference, here the one pertaining to GII. Recall that the theoretical frame of reference is in Kuzniak or in Coutat & Richard (2011) one of the two poles of *discursive genesis*.

Besides, the effort of visualization on the measured figure – the figure as a '*representamen*' or 'signifier' in the *epistemological plane* of Kuzniak (2013) – is built on an effort of coordination with other signifiers, namely the numbers resulting from measurement : these numbers constitute a 'property' of the measured objects but *are not* these objects themselves. So, it is not a coordination between registers of representation in Duval's sense (1993) because the *signified* are not the same. Moreover, they belong to two different fields, the field of (synthetic) geometry and the field of arithmetic. In the perspective of Kuzniak's theoretical framing, it is as if the epistemological plane had been split in two, with a plane in each field and round trips between the two via the cognitive plane. Through these considerations, one can evaluate the complexity of what is involved in a measuring activity, be it conducted for construction, validation or experimentation purposes.

Regarding complexity, the subject is indeed not exhausted. In a geometric problem-solving context, it happens frequently that the task also deals with *computed* measures, for example when some measures are inferred from Pythagoras' or Thales' theorems, thus giving rise to irrational numbers or rational numbers with infinite decimal expansion. The representations of all the related numbers may then come from several registers [3] : the register of finite decimal or repeating decimal (with the period overlined), the register of quotients written in the form p/q, the register of representations using the root symbol, etc. There is then a need for a *coordination between registers*, as the one brought up by Duval. But in order for this coordination to be 'scientifically coherent', the non exactness of the decimal numbers resulting from measurement must be fully taken into account. For an example of a problem lacking in such a coherence, see the problem *Marie et Charlotte* in Kuzniak & Rauscher (2011) or in Kuzniak (2013).

In sum, if measuring in geometry belongs mainly to mathematical activities relevant to figures, it also resorts to numbers, so to a field that is not synthetic geometry and has its own representations and theoretical references. We insist that in that instance, the duality exactness-approximation should be a teaching goal and issue. Then almost inevitably, the issue will echo with the representations of rational and irrational numbers and their approximation by decimal numbers, and in parallel with the problem of (visually) representing ideal geometrical objects. In the proposed activity, the measurements are not carried out with the usual geometrical tools but rather, are obtained from the ad hoc functionalities of GeoGebra. Then, students' relationship to the displayed numbers gets more intricate (Tanguay & Geeraerts, 2013). The issue of exactness, for both the measurements and the associated numerical representations, is directly linked to the possibility of 'closing' (or not) the regular polygon to be produced. We put forward the hypothesis that regarding these issues, the discussions and reflections thus prompted will be rich and significant.

A SITUATION PLACING AT ITS CENTRE EXACTNESS OF MEASUREMENT AND OF NUMERICAL REPRESENTATIONS

Description

The following teaching situation has been designed by a group of researchers from Quebec and Mexico, working in collaboration. We started from a situation designed by L. Guerrero, in which working on regular polygons with GeoGebra was planned. The situation was intended for students from the beginning of secondary school (12-14 years old). Revising it brought us to extend the task towards arithmetic, via the measures of the angles involved. The arithmetic topics singled out – mainly divisors and divisibility – and the study of regular polygons are prescribed subjects from the secondary curricula of Quebec and Mexico.

The activity provokes a back and forth between geometry and arithmetic by considering the GeoGebra displays of the measures related to a well chosen angle. It can easily fill up two lessons. The first lesson involves the decomposition of the *n*-sided regular polygon in *n* isosceles triangles grouped around the centre, the decomposition being linked to the divisibility of 360 by the degree measurement of the central angle. The students are thus brought to examine, within a geometrical context, the list of divisors of 360. They work in teams of two, one team per each computer terminal. The instructions are open :

> The triangle ΔABC visible at the screen is isosceles, with [AC] ≅ [BC]. Form all the regular polygons you can by rotating ΔABC around point C. You may vary ∠ACB either with the slider α or by typing directly α, the measure of ∠ACB, in the given box. The slider *n* allows you to change the number of triangles obtained as images of ΔABC by repeated rotations around centre C and of angle α. Do you know the name of each polygon you formed ? As you go, fill in the following table. You can add as many lines as you want.

Measure α of angle ACB	Number of triangles, images obtained by rotating ΔABC	Name of the regular polygon you formed
	n =	
	n =	
	n =	
	n =	

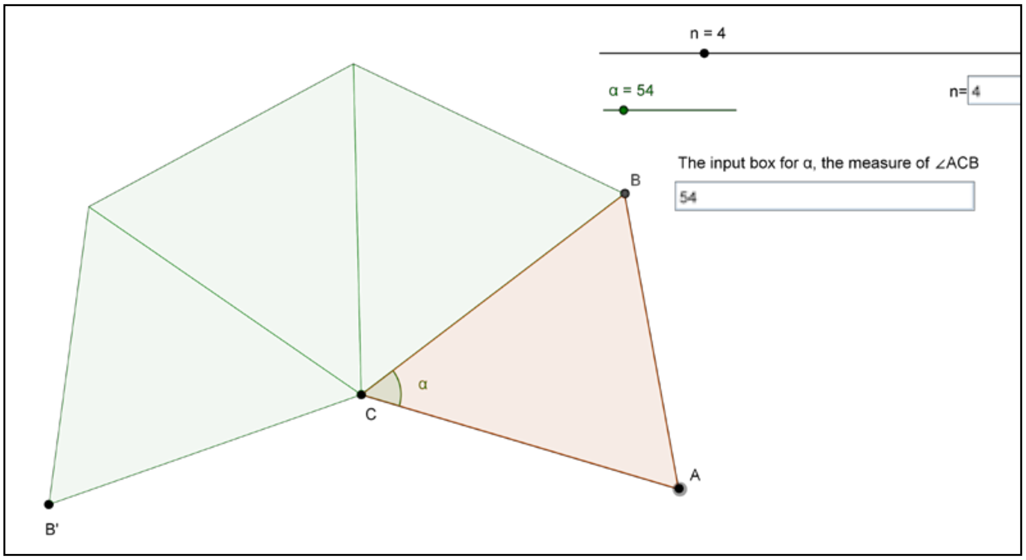

Fig 2 : The GeoGebra screen given from the start

To do the task, students lean on the fact that any regular polygon can be decomposed in as many isosceles triangles as there are sides. These triangles share a common vertex, the centre of the circumscribed circle. Some knowledge elements are thus activated here, that the students could have previously met or as well, that they may be discovering through the activity.

A priori analysis : the divisors of 360

The students can either fix a value for α and modify *n* afterwards, trying then to 'close' the figure; or keep *n* fixed and modify the angle, by exploring with the slider or by entering directly a value for α. We assume that exploration by trial and error with the two sliders will spontaneously be initiated, but will remain relatively ineffective : because of the imprecise control of the angle with the slider, students will probably obtain nothing more than the usual polygons met in elementary school, with standard values for α : the square, the hexagon, the octagon, possibly also the equilateral triangle. It should be noticed that the colour of ΔABC differs from the colour of its images, enabling an easy perception of overlaps. In this respect, it is important to provide the students with a slider for *n* [4] and not only a box where the number is entered. Indeed, seeing the triangles unfurl like a fan when dragging the slider allows to realize that the value for n is not the right one when ΔABC and its images coincide perfectly, e.g. when $n = 5$ and $\alpha = 90°$.

We assume that through this trial and error process, the link between the correct value of n for a given α, and the fact that $n\alpha$ must be equal to 360°, will emerge. By seeking for a better control of their trials, students will explore this link in a more systematic way, by considering the divisors of 360 and by using the input box for α.

We put forth the hypothesis that while engaging a whole-group discussion at about two-thirds of the first lesson, some teams will invoke the divisors of 360 (of course in their own words). In his/her institutionalization, the teacher clarifies the underlying reasoning and establishes, with the whole class, the list of all divisors of 360. Complying with the strategies he/she observed from his/her students, he/she can construct this list by considering the possible values for n (along an increasing list starting at 3) or by considering the values for α along a decreasing list, beginning with $\alpha = 120$. For each divisor, he/she ask the class what is the associated value, of α or n, respectively. He/she shows onto the screen the corresponding polygon and gives its name. He/she supplies the list of divisors with 1, 2, 180 and 360. He/she asks the class if an associated regular polygon could be considered for each of these four values. It appears to us important that the teacher carries the discussion to its ending, namely that depending on the adopted strategy, we won't keep the same divisors to construct the regular polygons : either we keep 1 and 2 as the degree measure of the central angle and we set aside 180° and 360°, or we set aside 1 and 2 as the number of sides and we construct polygons with 180 and 360 sides. At this point, the teacher may suggest that fractional angle measures are possible, for example by considering the regular polygon with 48 sides, and central angle measuring $15° \div 2 = 7{,}5°$. The status of these 'special cases' must then be explicitly clarified with respect to the notion of *divisor*.

A priori analysis : the heptagon

The goal of the following lesson is to tackle the notion of approximation, both from a numerical standpoint (the written representation, exact or not, of a rational number) and a geometrical standpoint (the status of the figure).

The starting point is brought up by the teacher, who comes back to the cases where the division of 360 by n does not result in an exact decimal value. He/she asks students to investigate these cases further. In principle, the case of the heptagon ($n = 7$) should be the first to turn up. Research (e.g. Krikorian, 1996) shows that spontaneously, secondary students don't make use of fractional notation. We anticipate that students will enter in the α box the approximation of 360/7 they'll get from their pocket calculator. GeoGebra then displays 51.43, its standard round up. The teacher must watch for teams who got to that point, first to ask them for a verification that the polygon is well closed, using the zoom [5]; and once they acknowledge that the polygon *is not* well closed, to show them how to get the maximum of 15 decimals into the menu bar (→ Option → Rounding). Then the teacher gives the following instruction : “add one by one the decimals of 360/7 in the input box for α, and at each step, zoom in to check if the figure is well closed”. The students must then find a way to obtain the decimals beyond the scope of their pocket calculator.

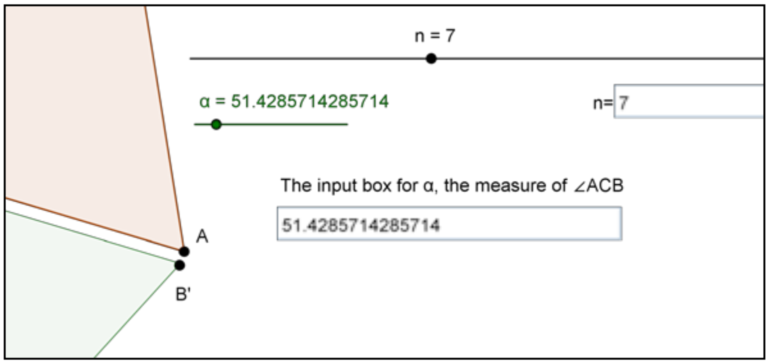

Fig 3 : a close zoom, for an approximation of 360/7 with 13 decimals

Even with 13 decimals, a sufficiently close zoom shows a figure which is not well closed. Entering 15 decimals or more, GeoGebra (version 4.0.41.0) rounds up at 14 decimals and displays 51,42857142857143. Then as close as we get with the zoom, we see a polygon *seemingly* closed. When the teams have reached that point, a whole-class discussion is called upon. The issue of the exact value of 360/7 is raised : “what is it ? Is it 51,42857142857143 ? If I multiply this number by 7, do I get 360 back ? Compute the product by hand.” In the same time, the teacher enter the multiplication into the input bar (at the bottom of the screen), and 360 is displayed in the *Algebra View*. “Are these computations by GeoGebra exact ? The polygon appears to be closed but can we rely on GeoGebra here ? It seems that we have reached the limit of the software ! So, this polygon with seven sides, we can close it or not ?”

In accordance with our theoretical framing, the idea is here to bring students to reflect on the ideal character of the heptagon regardless of what is produced and seen on the screen, and then to extend this ideal character to the already produced polygons and ultimately, to any geometrical object. Going back to arithmetic, students reflect upon what exact angle should be produced (and hence measured) to be able to construct this ideal heptagon. This brings them to consider in parallel the notion of exact representation of a rational number.

It may be an opportunity for asking students to do the long division of 360 by 7 by hand. The teacher may then explain the period and insists that the exact value of 360/7 needs, to be written exactly into a decimal form, an infinity of digits or else, the representation with a bar over the period to account for this infinity. “Under what form can we propose this exact value to the software ? Is it but possible ? And what about entering 360/7 into the box, as a value for α ?” Even with “360/7” entered in the input box, GeoGebra displays 51,42857142857143. The teacher may then confirm that the numbers handled by the software are approximations. He/she moves on to say that the regular polygon with 7 sides does exist (in theory), and that the exact measure (in degrees) of its central angle does not admit a finite decimal expansion, but can nevertheless be exactly represented by the notations $\frac{360}{7}$ or $51,\overline{428571}$. He/she concludes by stating that there exists a regular polygon with n sides for each integer n greater than 2 and that for each one, the central angle measures $\frac{360}{n}$ degrees.

CONCLUSION

So in a meaningful context, the students become aware of the dualities *ideal object – visual representation*, *exact measurements – approximations* in geometry, and link these to the representations of rational numbers. They know that some fractions don't have a finite decimal expansion and they now understand that one must not make use of the equality sign between such a fraction written in the form $\frac{p}{q}$ and any notation referring to a finite decimal expansion.

The activity has been the object of a pre-experimentation outside school with five children (aged 11 to 13), and our hypotheses have then been largely confirmed. More systematic classroom experimentations, in Quebec and Mexico, are to come. The activity is part of a larger research programme about the geometer-physicist paradigm (Tanguay et Geeraerts, 2012, 2013), in which more issues pertaining to measure and measurements will be explored. This programme is also in its early stages.

NOTES

1. The French word for *field* used by Kuzniak is *domaine*. It is closely related to the word and concept *cadre* considered by Douady (1986). Here, *field* should be understood according to a meaning referring to school mathematics, rather than to advanced mathematics.
2. By *synthetic geometry*, we mean geometry without coordinates, as opposed to *analytic geometry*.
3. They are indeed distinct registers in Duval's sense (1993), since for example one cannot add or multiply 5,6 with $4\sqrt{2}$ or with 360/7 without changing the representation of one of the two numbers.
4. Hence from the teacher and a teaching perspective, providing a slider for n is a form of *instrumentalization* of GeoGebra, in Rabardel's sense (e.g. Vérillon & Rabardel, 1995).
5. The zoom is easy to use with GeoGebra : use the mouse wheel ! To avoid the figure being pushed off the zone we want to zoom in – in this instance the neighbourhood of point A – one must just insert the mouse cursor in this neighbourhood.

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