UNIVERSITÉ DU QUÉBEC À MONTRÉAL

GENERELIZED COMPLEX STRUCTURES ON NILMANIFOLD

MÉMOIRE

PRÉSENTÉ

COMME EXIGENCE PARTIELLE

DE LA MAÎTRISE EN MATHÉMATIQUES

PAR

RAGHAD ALNOURI

AOÛT 2017

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DEDICATION

To Syrian refugee children who struggle to hold a pen.

To my parents, Rima and Moutaz. To my dear husband, Amjad and my dear son, Nizar. To my absent brother, Yaman.

CONTENTS

RÉSUMÉ

La géométrie complexe généralisée est une généralisation de la géométrie complexe obtenue en considérant des structures complexes sur le fibré généralisé $T \oplus T^*$, plutôt que sur le fibré tangent T. Cette géométrie nouvellement définie fournit un language unificateur pour la géométrie complexe et symplectique puisqu'elle contient chacune d'elles comme cas spécial. On étudie une classe d'exemples non-triviaux de structures complexes généralisées: les structures complexes généralisées invariantes à gauche sur des groupes de Lie nilpotents. On montre que toutes les 6-nilvariétés admettent des structures complexes généralisées. En suivant les travaux de (Cavalcanti et Gualtieri, 2004), on présente la classification des 6-nilvariétés selon le type de structures complexes généralisées invariantes à gauche qu'elles admettent. De plus, on montre que les structures complexes généralisées invariantes à gauche sont en correspondance biunivoque avec les structures hermitiennes sur le fibré cotangent d'un groupe de Lie (par rapport à la métrique standard naturelle). On utilise ensuite cette correspondance ainsi que les résultats de (de Andrés *et al.*, 2007) pour déduire que l'algèbre cotangente d'une algèbre de Lie six dimensionnelle quelconque admet une structure hermitienne.

MOTS-CLÉS: structure complexe généralisée, structure complexe, structure symplectique, structure hermitienne, nilvariétés.

ABSTRACT

Generalized complex geometry is a generalization of complex geometry obtained by searching for complex structures on the generalized bundle $T \oplus T^*$, rather than the tangent bundle T. The newly defined geometry provides a unifying language for complex and symplectic geometry because it contains each of them as a special case. We study a class of non-trivial examples of generalized complex structures: left-invariant generalized complex structures on nilpotent Lie groups. We show that all 6-nilmanifolds admit generalized complex structures. We present the classification of 6-nilmanifolds according to which type of left-invariant generalized complex structure they admit, following the work of (Cavalcanti et Gualtieri, 2004). In addition, we show that the left-invariant generalized complex structures are in one-to-one correspondence with the Hermitian structures on the cotangent bundle of a Lie group (with respect to the standard natural metric). We then use this correspondence together with the results of (de Andrés *et al.*, 2007) to derive that the cotangent algebra of any six dimensional Lie algebra admits a Hermitian structure.

keywords: generalized complex structure, complex structure, symplectic structure, Hermitian structure, nilmanifold.

INTRODUCTION

Unlike classical differential geometry where we define structures in the tangent and cotangent bundles, in *generalized geometry* we replace the tangent bundle with the direct sum of the tangent and the cotangent bundle, and we deal with differential 1-forms and vector fields on an equal basis. Generalized geometry was initiated by (Hitchin, 2003) and developed by (Gualtieri, 2004). It is basically defined in terms of two objects. First, an orthogonal structure coming from the natural split-signature inner product on the vector bundle $T \oplus T^*$. Second, a bracket of two generalized vector fields called the Courant bracket. This bracket provides various geometric structures with their integrability condition. The symmetry group of the defined geometry is a very large group and contains more than just diffeomorphisms. The extra symmetries are B-field transformations generated by closed 2-forms. Accordingly, any type of structure defined in this generalized geometry can be transferred either globally by diffeomorphisms or locally, by B-fields.

Generalized complex geometry is a straightforward generalization of complex geometry where we search for complex structures on the bigger generalized bundle $T \oplus T^*$. This generalization is characterized by the way it unifies complex and symplectic geometries, since both are extremal special cases of the generalized complex geometry.

There are different ways to produce exotic examples of generalized complex structures; in particular, examples on manifolds which admit no complex or symplectic structure. These ways include searching for them on symplectic fibrations and Lie algebras, or doing a surgery procedure. In this dissertation we are concerned mostly with left-invariant generalized complex structures on nilpotent Lie groups, which are equivalent to integrable linear generalized complex structures on their nilpotent Lie algebra. In the study carried out by (Cavalcanti et Gualtieri, 2004), the authors have found obstructions on nilmanifolds to admit an invariant generalized complex structure based on data encoded in the corresponding Lie algebra: the nilpotence step and the dimension of the spaces forming the descending central series. Based on that, they presented a classification of invariant generalized complex structures on 6-nilmanifolds; they all admit generalized complex structures including five examples that admit no symplectic or invariant complex structures.

Lie algebra examples are of particular interest due to the fact that the Courant bracket on a Lie algebra is equivalent to the Lie bracket on the cotangent algebra. Therefore, finding a generalized complex structure on a Lie algebra is equal to giving a complex structure on the cotangent algebra. In (de Andrés *et al.*, 2007), it was proved that left invariant generalized complex structures on a Lie group are in one-to-one correspondence with the invariant Hermitian structures on the cotangent Lie group. Using this correspondence, it follows that the cotangent algebra of any 6-dimensional Lie algebra admits a Hermitian structure.

We begin in the first chapter with the basic settings for the generalized geometry. This includes a highlight of geometric structures at the level of vector spaces, and a study of the linear algebraic aspects of $V \oplus V^*$. Then we define the generalized complex structure at the level of vector spaces.

In chapter 2, we transfer all the linear algebraic properties studied on $V \oplus V^*$ to a $2n$-manifold. We see that the generalized vector bundle $T \oplus T^*$, equipped with the same inner product and orientation as described on $V \oplus V^*$, has a natural structure group $\mathrm{SO}(2n, 2n)$. On generalized tangent space, we define a linear generalized complex structure. To find the holomorphic chart of the generalized complex manifold, we need an analogue of the Newlander-Nirenberg integrability theorem for the generalized case. For this reason, we define the Courant bracket on the sections of $T \oplus T^*$ and formulate a generalized integrability condition in terms of this bracket. Defining a generalized complex structure on a $2n$-manifold reduces the structure group of $T \oplus T^*$ from $\mathrm{SO}(2n, 2n)$ to the indefinite unitary group $\mathrm{U}(n, n)$. Following the presentation of (Gualtieri, 2004), we describe the algebraic consequences of having an almost generalized complex structure, as well as the topological obstruction to its existence. This obstruction is the same as the one for an almost complex structure or a non-degenerate 2-form to exist. Furthermore, we describe the action of a generalized complex structure on differential forms.

In chapter 3, we begin with a section on Lie algebras and Lie groups. Then, we study left invariant generalized complex structures on Lie groups and we relate them to Hermitian structures on cotangent Lie groups. In this chapter, we follow (de Andrés *et al.*, 2007), where they proved that the Courant bracket, when restricted to left invariant vector fields and left invariant 1-forms, is given by a special equation. Consequently, they established a correspondence between invariant Hermitian structures on T^*G and left invariant generalized complex structures on G. After that, we define what a nilmanifold is. We then state the results of (Cavalcanti et Gualtieri, 2004) about generalized complex structures on nilmanifolds in arbitrary dimensions, and the classification of left-invariant generalized complex structures on 6-dimensional nilmanifolds. Finally, We conclude with the fact that when a 6-dimensional Lie group G is nilpotent, the cotangent bundle T^*G has a Hermitian structure.

CHAPTER I

BASIC SETTINGS OF GENERALIZED GEOMETRY

Much of the geometry of manifolds can be described in terms of vector bundles, which associate a vector space to each point of the manifold in a continuous way. Therefore, we begin in the first section of this chapter with a highlight of the linear algebraic aspects of vector spaces before fibering them on a manifold. A study of the natural split-signature orthogonal structure on $V \oplus V^*$ follows in the second section, as we define a generalized complex structure on the sum of cotangent and tangent bundles. We then define the notion of generalized complex structure on a real vector space, where complex and symplectic structures can be seen as extremal cases of a more general structure.

1.1 Linear algebra of V

1.1.1 Complex structures and complexification

This section contains two basic operations: extension and restriction of the field of scalars of vector spaces.

Definition 1.1.1. A *complex structure* on a finite-dimensional real vector space V is a linear endomorphism J of V ($J \in End_{\mathbb{R}}(V)$) such that $J^2 = -\mathbb{I}$; $\mathbb{I} : V \to V$ being the identity.

Since $det(J^2) = det(-\mathbb{I}) = (-\mathbb{I})^{dimV} \neq 0$, J is an invertible endomorphism. The existence of a complex structure J on a real vector space V implies that V is of even dimension. Indeed, $[det(J)]^2 = det(J^2) = det(-\mathbb{I}) = (-1)^{dimV}$, then $dimV$ should be even.

A pair (V, J), where V is a real vector space and J is a complex structure on V, can be turned into a complex vector space V_c by defining scalar multiplication by complex numbers as follows:

$$\forall v \in V_c : (a + ib)v = av + bJ(v); a, b \in \mathbb{R}$$

.

Proposition 1.1.2. There exists a basis $\{e_1, ..., e_n\}$ for V_c such that

$$e_1, ..., e_n, J(e_1), ..., J(e_n)$$

is a basis of V. In particular

$$\dim_{\mathbb{C}} V_c = \frac{1}{2} \dim_{\mathbb{R}} V$$

.

Proof. V is a real vector space of even dimension $2n$. We construct the following flag in the finite dimensional space V: $L_0 = \{0\}$ and $L_1 = \text{span}\{e_1\}$. Since $J^2 = -1$, $e_1, J(e_1)$ are linearly independent, then we can set $L_2 = \text{span}\{e_1, J(e_1)\}$. Indeed, $J(e_1) \in L_2 \backslash L_1$ and $\text{span}\{e_1\} \subset \text{span}\{e_1, J(e_1)\}$. This flag can be extended up to the maximal flag in V, which has length $\dim V$. This can be done by inserting intermediate subspaces into the starting flag as long as it is possible to do so. By induction on the dimension $2n$, $e_1, J(e_1), ..., e_n, J(e_n)$ are linearly independent. Hence, the length of the maximal flag cannot exceed $2n$. We have

$$\forall v \in V : v = \sum_{i=1}^{n} a_i e_i + \sum_{i=1}^{n} b_i j(e_i); a_i, b_i \in \mathbb{R}$$

, So using the scalar multiplication of the complex vector space V_c, we obtain

$$\forall v \in V : v = \sum_{i=1}^{n} (a_i + ib_i)e_i$$

, so $\{e_1, ..., e_n\}$ is a basis of V_c. □

Conversely, given a complex vector space V of complex dimension n, there exists a linear endomorphism J of V, ($J \in End_{\mathbb{R}}(V)$) defined by: $Jv = iv; \forall v \in V$. This endomorphism J is linear over $\mathbb{R}$.

$$J(av_1 + bv_2) = i(av_1 + bv_2) = iav_1 + ibv_2 = aJ(v_1) + bJ(v_2)$$

. Moreover, J satisfies the condition $J^2 = -\mathbb{I}$.

$$Jv = iv \Rightarrow J^2 = i(iv) = -v; \forall v \in V$$

. If we consider V as a real vector space, then J is a complex structure of V, which we call the *canonical complex structure* with V_c being the corresponding complex space.

Let J be a complex structure for V, then $-J : V \to V$ is also a complex structure, and is said to be the *complex structure conjugate to J*.

Example 1.1.3. *(canonical complex structure of* $\mathbb{R}^{2n}$*)*
Consider the complex vector space of n-tuples of complex numbers $\mathbb{C}^n = \{z = (z^1, ..., z^n) | z^k \in \mathbb{C}$ for all $k\}$. Given $z \in \mathbb{C}^n$ and setting $z^k = x^k + iy^k; x^k, y^k \in \mathbb{R}$ for all $k = 1, ..., n$. We can identify $\mathbb{C}^n$ over $\mathbb{C}$ with $\mathbb{R}^{2n}$ over $\mathbb{R}$. In terms of the natural basis of $\mathbb{R}^{2n}$, the *canonical complex structure* J_0 is given by

$$J_0 = \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$$

.

Proposition 1.1.4. Let J and J' be two complex structures on real vector spaces V and V' respectively. If we consider V and V' as complex vector spaces in a natural manner, then a real linear map f of V into V' is complex linear if and only if $J' \circ f = f \circ J$.

Proof. Denote f_R for $f \in Hom_{\mathbb{R}}(V, V')$, f_c for $f \in Hom_{\mathbb{C}}(V, V')$.

We have f_c is complex linear $\Leftrightarrow f_c(iv) = if_R(v)$;$\forall v \in V \Leftrightarrow f_c(iv) = f_R(Jv) = J' f_R(v) = if_R(v) \Leftrightarrow (J' \circ f)(v) = (f \circ J)(v) \Leftrightarrow J' \circ f = f \circ J$. The commuting condition is also sufficient, because it automatically implies that f_c is complex linear. $\square$

Example 1.1.5. *(The real representation of* $GL(n, \mathbb{C})$*)*
$GL(n, \mathbb{C})$ can be identified with the subgroup of $GL(2n, \mathbb{R})$ consisting of matrices which commute with the canonical complex structure J_0, i.e. there is an injective map:

$$\begin{aligned} \rho : GL(n, \mathbb{C}) &\hookrightarrow GL(2n, \mathbb{R}) \\ A + iB &\longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} ; A + iB \in GL(n, \mathbb{C}),\, A, B \in M(n, \mathbb{R}), \end{aligned}$$

which called the *real representation* of $GL(n, \mathbb{C})$.

A complex structure is equivalent to a $GL(n, \mathbb{C})$-structure, since there is a natural one-to-one correspondence between the set of complex structures on $\mathbb{R}^{2n}$ and the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.

Proposition 1.1.6. Let J be a complex structure on a real vector space V. A real vector subspace V' of V is invariant by J if and only if V' is a complex subspace of V_c.

Proof. J is the multiplication by i on V_c . Let $U \subset V_c$, then U is J-invariant if and only if $(a+ib)u \in U; a+ib \in \mathbb{C}$ □

Another way to extend the real field of a vector space to a complex one is the complexification $V^{\mathbb{C}}$, in which we represent the complex vector space as the direct sum of two real vector spaces, the "real and imaginary parts" $V^{\mathbb{C}} = V \oplus V$.

Let V be a finite-dimensional real vector space, we can always find the complex structure $J \in End_{\mathbb{R}}(V \oplus V)$ defined as follows:

$$J : V \oplus V \longrightarrow V \oplus V$$

$$J(v_1, v_2) = (-v_2, v_1)$$

Indeed,

$$\forall v_1, v_2 \in V : J(v_1, v_2) = (-v_2, v_1)$$

$$\Rightarrow J^2(v_1, v_2) = J\left(J(v_1, v_2)\right) = J(-v_2, v_1) = (-v_1, -v_2) = -(v_1, v_2)$$

$$\Rightarrow J^2 = -\mathbb{I}.$$

Definition 1.1.7. A *complexification* $V^{\mathbb{C}}$ of a finite-dimensional real vector space V is the complex vector space $(V \oplus V)_c$ associated with the above complex structure J.

Identifying V with the subset of vectors of the form $(v, 0) \in V \oplus V$ and using the fact that $i(v, 0) = J(v, 0) = (0, v)$, we can write the following:

$$\forall (v_1, v_2) \in V^c : (v_1, v_2) = (v_1, 0) + (0, v_2) = (v_1, 0) + i(v_2, 0) = v_1 + iv_2$$

$$\Rightarrow V^{\mathbb{C}} = V \oplus iV.$$

Remark 1.1.8. Any basis of V over $\mathbb{R}$ is a basis of $V^{\mathbb{C}}$ over $\mathbb{C}$, so that:

$$dim_{\mathbb{R}} V = dim_{\mathbb{C}} V^{\mathbb{C}}.$$

Similarly, if $f : V \longrightarrow W$ is a linear mapping of real vector spaces, the *complexification of the mapping* f, is a mapping $f^{\mathbb{C}} : V^{\mathbb{C}} \longrightarrow W^{\mathbb{C}}$ defined by:

$$f^{\mathbb{C}}(v_1, v_2) = (f(v_1), f(v_2)).$$

Clearly, $f^{\mathbb{C}}$ is linear over $\mathbb{R}$ and commutes with J:

$$f^{\mathbb{C}}(J(v_1, v_2)) = f^{\mathbb{C}}(-v_2, v_1) = (-f(v_2), f(v_1)) = Jf^{\mathbb{C}}(v_1, v_2).$$

Therefore, it is complex linear. Moreover, it is easy to check the following: $\mathbb{I}^c = \mathbb{I}, (af + bg)^c = af^c + bg^c; a, b \in \mathbb{R}, (f \circ g)^c = f^c \circ g^c$.

Remark 1.1.9. There exists a canonical isomorphism between $V^{\mathbb{C}}$ and $V \otimes_{\mathbb{R}} \mathbb{C}$

$$\begin{aligned} V \otimes_{\mathbb{R}} \mathbb{C} &\cong V^{\mathbb{C}} \\ v \otimes z &\mapsto (\mathfrak{Re}(z)v, \mathfrak{Im}(z)v)\,. \end{aligned}$$

V is a real subspace of $V \otimes_{\mathbb{R}} \mathbb{C}$ in a natural manner,

$$\begin{aligned} i : V &\hookrightarrow V^{\mathbb{C}} \\ v &\mapsto v \otimes 1. \end{aligned}$$

More generally, $T^r_s(V)$ is also a real subspace of the space of tensor $T^r_s(V^{\mathbb{C}})$ in a natural manner.

$$\begin{aligned} i : T^r_s(V) &\hookrightarrow T^r_s(V^{\mathbb{C}}) \\ v_1 \otimes ... \otimes v_r \otimes v^1 \otimes ... \otimes v^s &\mapsto (v_1 \otimes 1) \otimes ... \otimes (v_r \otimes 1) \otimes (v^1 \otimes 1) \otimes \otimes (v^s \otimes 1) \end{aligned}$$

where $\dim_{\mathbb{R}} T^r_s(V) = (dim_{\mathbb{R}} V)^{r+s}$, $\dim_{\mathbb{R}} T^r_s(V^c) = (2dim_{\mathbb{R}} V)^{r+s}$.

Every complex vector space W has a real form $W_{\mathbb{R}}$ called *decomplexification*, this is obtained simply by ignoring the multiplication of vectors in W by all complex numbers, and retaining only the multiplication over $\mathbb{R}$. In this case $\dim_{\mathbb{R}} W_{\mathbb{R}} = 2\dim_{\mathbb{C}} W$.

Definition 1.1.10. Let W be an n-dimensional complex vector space. A *real structure* on W is an endomorphism of $W_{\mathbb{R}}$, σ, such that:

- $\sigma[(a+ib)w] = (a-ib)\sigma(w)$; $\forall w \in W$;
- $\sigma^2 = \mathbb{I}_W$.

Proposition 1.1.11. Any real structure σ on W defines an n-dimensional subspace $V_\sigma = \{w \in W_{\mathbb{R}} : \sigma(w) = w\}$ where $V_\sigma^{\mathbb{C}} \cong W$.

Proof. Let $e_1, e_2, ..., e_n$ be a basis of the complex vector space W. We set $E_i = \frac{e_i+\sigma(e_i)}{2}$, then $E_1, E_2, ..., E_n$ is still a basis. In fact,

$$\begin{aligned}
\forall w \in W : w &= \sum_{i=1}^{n}(a_i+ib_i)E_i \\
&= \sum_{i=1}^{n}(a_i+ib_i)\left(\frac{e_i+\sigma(e_i)}{2}\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}(a_i+ib_i)e_i + \sum_{i=1}^{n}\sigma[(a_i+ib_i)e_i)]\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}(a_i+ib_i)e_i + \sum_{i=1}^{n}(a_i-ib_i)\sigma(e_i)\right).
\end{aligned}$$

In particular, if $w \in V_\sigma : \sigma(e_i) = e_i$. i.e., $(a+ib) = a - ib = \overline{(a+ib)}$. Thus, the coefficient $a+ib$ is real and $V_{\mathbb{C}} = \text{span}_{\mathbb{R}}\{E_i\}$. ☐

Proposition 1.1.12. Given (V_c, σ), there is a unique $U \subset_{\mathbb{C}} V_c$ such that $U^{\mathbb{C}} = V_c$

Let V be a real $2n$-dimensional vector space with a complex structure $J \in End_{\mathbb{R}}(V)$. Then J can be uniquely extended to a complex linear endomorphism of V^c also denoted by $J \in Hom_{\mathbb{R}}(V^{\mathbb{C}})$ as follows:

$$J : V^{\mathbb{C}} \longrightarrow V^{\mathbb{C}}$$

$$J = J \otimes \mathbb{I}_{\mathbb{C}}.$$

The eigenvalues of the extended J are therefore $+i$ and $-i$. We set: $V^{1,0}(J) = \{z \in V^{\mathbb{C}}; Jz = iz\}$ which is the eigenspace of the eigenvalue $+i$, and its vectors are called *vectors of type* $(1,0)$. And $V^{0,1}(J) = \{z \in V^{\mathbb{C}}; Jz = -iz\}$ which is the eigenspace of the eigenvalue $-i$, and its vectors are called *vectors of type* $(0,1)$.

There is an induced complex structure on the dual vector space. Let V be a finite-dimensional real vector space, and V^* its dual. A complex structure J on V induces a *complex structure on* V^*, also denoted by J, as follows:

$$\langle Jv, v^* \rangle = \langle v, Jv^* \rangle; v \in V, v^* \in V^*.$$

where $V \times V^* \longrightarrow K$ is the canonical pairing. Indeed,

$$\forall v \in V, v^* \in V^*$$

$$J^2(v^*) = \langle v, J^2 v^* \rangle = \langle Jv, Jv^* \rangle = \langle J^2 v, v^* \rangle = \langle -v, v^* \rangle = -\langle v, v^* \rangle = -v^*.$$

$$; \forall v \in V, v^* \in V^*$$

Similarly, the *complexification* $(V^*)^{\mathbb{C}}$ is the dual vector space of $V^{\mathbb{C}}$. Consider the dual vector space V^* of the real vector space V consisting of all linear functions $v^* : V \to \mathbb{R}$. The value of $v^* \in V^*$ at $v \in V$ is denoted by either $v^*(v)$ or $\langle v, v^* \rangle$. $(V^*)^{\mathbb{C}}$ maybe identified with the complex vector space of all linear functions:

$$v^* + iw^* : V \to \mathbb{C};$$
$$v \mapsto (v^* + iw^*)(v); \forall v \in V, v^* + iw^* \in (V^*)^{\mathbb{C}}$$

$$(V^*)^{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C} \cong V^* \otimes_{\mathbb{R}} \mathbb{C}^* \cong (V \otimes_{\mathbb{R}} \mathbb{C})^* = (V^{\mathbb{C}})^*$$
$$\Rightarrow (V^*)^{\mathbb{C}} = (V^{\mathbb{C}})^* = (V^{1,0} \oplus V^{1,0})^* = V_{1,0} \oplus V_{0,1}.$$

The elements of type $(1,0)$ in $(V^*)^{\mathbb{C}}$ are those functions $v^* : V \to \mathbb{C}$ for which $Jv^*(v) = iv^*(v), \forall v \in V$. Similarly, the elements of type $(0,1)$ are those $\bar{v}^*$ for which $\bar{v}^*J(v) = -i\bar{v}^*(v); \forall v \in V$, then we can conclude that vectors in $(V^*)^{\mathbb{C}}$ admit a direct sum decomposition into complex conjugate subspaces,

$$(V^*)^{\mathbb{C}} = V_{0,1} \oplus V_{1,0};$$

$$V_{1,0} = \{x^* \in (V^*)^{\mathbb{C}}; \langle x, x^* \rangle = 0; \forall x \in V^{0,1}\};$$

$$V_{0,1} = \{x^* \in (V^*)^{\mathbb{C}}; \langle x, x^* \rangle = 0; \forall x \in V^{1,0}\}.$$

The tensor product space $T^r_s(V^{\mathbb{C}})$ can be represented as a direct sum,

$$T^r_s(V^{\mathbb{C}}) = \bigoplus \left(\bigotimes V^{1,0} \otimes \bigotimes V^{0,1} \otimes \bigotimes V_{1,0} \otimes \bigotimes V_{0,1} \right).$$

The exterior algebras $\Lambda V_{1,0}, \Lambda V_{0,1}$ considered as sub-algebras of $\Lambda(V^*)^{\mathbb{C}}$ in a natural manner:

$$\Lambda V_{1,0} \hookrightarrow \Lambda(V_{1,0} \oplus V_{0,1});$$

$$\Lambda V_{0,1} \hookrightarrow \Lambda(V_{1,0} \oplus V_{0,1}).$$

Proposition 1.1.13. The exterior algebra $\Lambda(V^*)^{\mathbb{C}}$ decomposes as follows,

$$\Lambda(V^*)^{\mathbb{C}} = \bigoplus_{r=0}^{n} (\bigoplus_{p+q=r} \Lambda^{p,q}(V^*)^{\mathbb{C}}).$$

Moreover, the complex conjugation in $(V^*)^{\mathbb{C}}$, extended to $\Lambda(V^*)^{\mathbb{C}}$ in a natural manner, gives a real linear isomorphism

$$\Lambda^{p,q}(V^*)^{\mathbb{C}} \longrightarrow \Lambda^{q,p}(V^*)^{\mathbb{C}}.$$

Let $\{e^1, ..., e^n\}$ be a basis for $V_{1,0}$, then $\{\bar{e}^1, ..., \bar{e}^n\}$ where $\bar{e}^k = \overline{e^k}$, is a basis of $V_{1,0}$ (using lemma 1.1). Moreover, the set of elements $\{e^{j_1} \wedge ... \wedge e^{j_p} \wedge e^{k_1} \wedge ... \wedge e^{k_q} | 1 \leq j_1 < ... < j_p, 1 \leq k_1 < ... < k_p \leq n\}$ forms a basis for $\Lambda^{p,q}(V^*)^{\mathbb{C}}$ over $\mathbb{C}$.

Remark 1.1.14. The complex structure induces an orientation on the vector spaces equipped with. The class of ordered bases,

$$\{[e_1, ..., e_n, J(e_1), ..., J(e_n)], [e'_1, ..., e'_n, J(e'_1), ..., J(e'_n)], ...\}$$

determines one equivalence class of the two possible orientations, which we called *the orientation induced by the complex structure J*.

1.1.2 Symplectic structures

In this section, we define symplectic vector spaces, linear symplectomorphisms and the symplectic linear group.

Definition 1.1.15. A *symplectic structure* ω on a finite dimesional vector space V is a non-degenerate skew-symmetric bilinear mapping $\omega : V \times V \to \mathbb{R}$.

The bilinear form ω induces a linear map $\tilde{\omega} : V \to V^*$ via $\tilde{\omega}(v) := (v, \bullet)$; $\forall v \in V$ and so $\omega(u, v) = (\tilde{\omega}(u), v)$. This induced linear map is an isomorphism if and only if the bilinear map is non-degenerate. Since ω is skew-symmetric, $\tilde{\omega}^* = -\tilde{\omega}$. This allows to view a symplectic structure on V as an isomorphism $\tilde{\omega} : V \to V^*$ satisfying $\tilde{\omega}^* = -\tilde{\omega}$, where $\tilde{\omega}^*$ is the linear dual of the mapping $\tilde{\omega}$. i.e, $\tilde{\omega}^* : (V^*)^* = V \to V^*$.

The existence of a symplectic structure ω on a real vector space V implies that V is of even dimension. Indeed, $\tilde{\omega}^* = -\tilde{\omega}$. Taking determinants, $\det(\tilde{\omega}^*) = \det(\tilde{\omega}) = (-1)^{\dim V}\det(\tilde{\omega})$, Thus $\dim V$ is even.

As in the more familiar Euclidean case, we can define a ω-orthogonal spaces of V. Let (V, ω) be a symplectic space, and $W \subset V$, its *symplectic orthogonal subspace* W^ω is defined by:

$$W^\omega = \{v \in V \mid \omega(v, \omega) = 0 \ ; \forall \omega \in W\}.$$

Note that one might have $W^\omega \subset W$, i.e, $W \cap W^\omega \neq \{0\}$. The symplectic complement has the following properties:

1. $\dim W + \dim W^\omega = \dim V$
2. $(W^\omega)^\omega = W$
3. $(W_1 \cap W_2)^\omega = W_1^\omega + W_2^\omega$
4. if $w_1 \subseteq w_2$, then $W_1{}^\omega \supseteq W_2{}^\omega$

Definition 1.1.16. : Let (V, ω) be a symplectic space. A subspace $W \subset V$ is called

- *symplectic* or *non-degenerate* if $\omega\mid_W$ is non-degenerate. i.e, if $(V, \omega\mid_W)$ is a symplectic space.
- *isotropic* if $\omega\mid_W = 0$. i.e, if $W^\omega \subseteq W$.
- *coisotropic* if $W^\omega \supseteq W$.
- *maximal isotropic* or *Lagrangian* if $W^\omega = W$.

From the properties of the symplectic complement we can see that a subspace $W \subset V$ is symplectic if and only if W^ω is symplectic. In addition, the dimension of any maximal isotropic subspace is half of the dimension of V, W is isotropic if and only if W^ω is coisotropic.

Using a skew-symmetric version of Gram-Schmidt, we can prove by induction on the dimension n that there is always a basis $\{e_1, ..., e_n, e'_1, ..., e'_n\}$, called *the symplectic basis* such that:

$$\omega(e_i, e'_j) = \delta_{i,j}, \quad \omega(e_i, e_j) = \omega(e'_i, e'_j) = 0.$$

Indeed, since ω is non-degenerate, there exist $e_1, e_1' \in V$ such that $\omega(e_1, e_1') = 1$. It follows that the subspace spanned by e_1, e_1' is symplectic. Then (W^ω, ω) is a symplectic space of dimension $2n-2$. By the induction hypothesis, there exists a symplectic basis $\{e_2, ..., e_n, e_2', ..., e_n'\}$ of W^ω. Thus, the vectors $e_1, ..., e_n, e_1', ..., e_n'$ form a basis of V.

This symplectic basis has a matrix of the form:

$$\begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$$

which is exactly the matrix J_0 that gives the canonical complex structure on $\mathbb{R}^{2n}$.

Example 1.1.17. (canonical symplectic structure of $\mathbb{R}^{2n}$)
Identify $\mathbb{R}^{2n}$ with $\mathbb{R}^n \times \mathbb{R}^n$, label the canonical basis by $p_1, ..., p_n, q_1, ..., q_n$. Let $p_1^*, ..., p_n^*, q_1^*, ..., q_n^*$ be the dual basis of V^*. Then we get the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ where:

$$\omega_0 = \sum_{j=1}^{n} p_j^* \wedge q_j^*.$$

Example 1.1.18. Let V be a real or complex vector space. Then $V \oplus V^*$ has a canonical symplectic structure given by the form:

$$\omega\left((v, \alpha), (u, \beta)\right) = \beta(v) - \alpha(u)\ ; \alpha, \beta \in V^*,\ \ u, v \in V.$$

If we identify $\mathbb{R}^{2n}$ with $\mathbb{R}^n \times (\mathbb{R}^n)^*$ then the previous example gives us the canonical symplectic form on $\mathbb{R}^{2n}$.

Definition 1.1.19. A *linear symplectomorphism* φ between two symplectic spaces (V_1, ω_1) and (V_2, ω_2) is a vector space isomorphism $\varphi : V_1 \to V_2$ which preserves the symplectic structure in the sense that $\varphi^*(\omega_2) = \omega_1$. i.e, $\omega_1(u.v) = \omega_2\left(\varphi(u), \varphi(v)\right),\ \forall u, v \in V_1$. If a linear symplectomorphism exists, the two spaces said to be *linearly symplectomorphic.*

Theorem 1.1.20. *(linear Darboux's theorem)*
Two symplectic spaces of the same dimension are linearly symplectomorphic.

The relation of being symplectomorphic is an equivalence relation on the set of all even-dimensional vector spaces. Furthermore, by linear Darboux's theorem, every $2n$-dimensional symplectic vector space is symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. Hence, non-negative even integers classify equivalence classes for the relation of being symplectomorphic.

Example 1.1.21. Let (V, ω) be a symplectic space, then an endomorphism $\Phi : V \to V$ is a linear symplectomorphism if and only if its graph $\Gamma_\Phi = \{(v, \phi(v)) \mid v \in V\}$ is maximal isotropic in $(V \oplus V, -\omega \oplus \omega)$ where $-\omega \oplus +\omega$ denotes the difference $-\pi_1^*\omega + \pi_2^*\omega$ of the pullbacks of ω along the two canonical projections π_1, π_2 : $V \oplus V \to V$.

Example 1.1.22. Let (V, ω) be a symplectic space, $\Phi : V \to V^*$ an endomorphism. The graph $\Gamma_\Phi = \{(v, \Phi(v)) \mid v \in V\}$ is maximal isotropic in $(V \oplus V^*, \omega)$ if and only if Φ is symmetric: $\Phi(v)(u) = \Phi(u)(v)$, $\forall u, v \in V$, and where ω is the symplectic structure described in example 1.1.18. This shows that maximal isotropic subspaces of V are in one to one correspondence with the quadratic forms of V^*.

Lemma 1.1.23. A bi-linear skew-symmetric form ω on V^{2n} is non-degenerate if and only if its n-th exterior power $\Lambda^n w$ is non-zero.

Proof. Assume first that w is degenerate. Let $v \neq 0$ such that $w(v, u) = 0$, $\forall u \in V$. Choose a basis $v_1, ..., v_{2n}$ for V such that $v_1 = v$, then $\Lambda^{2n}(v_1, ..., v_{2n}) = 0$. Conversely, suppose that w is non-degenerate, then since $\Lambda^n w_0$ is a volume form in $\mathbb{R}^{2n}$, it follows from linear Darboux's theorem that $\Lambda^n w \neq 0$. $\square$

The set of all isometries (linear symplectomorphisms) of a symplectic space forms a group. The set of matrices in $\mathrm{Gl}(2n, \mathbb{R})$, representing this group in a symplectic basis is called the *symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$. In the view of Darboux's theorem, all symplectic vector spaces of the same dimension are isomorphic, so it suffices to consider the case $V = \mathbb{R}^{2n}$ with the canonical symplectic form w_0. In this case $\mathrm{Sp}(2n) = \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{Sp}(\mathbb{R}^{2n}, w_0)$ is a real $2n \times 2n$ matrices A which satisfy the condition $A^T J_0 A = J_0$ so that $\det A = \mp 1$. By lemma 1.1.23, any symplectic space has a non-vanishing volume form, and so it is orientable and $\det A$ is always $+1$. In the complex case $\mathrm{Sp}(2n, \mathbb{C})$, we may identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, then the multiplication by J_0 in $\mathbb{R}^{2n}$ corresponds to multiplication by i in $\mathbb{C}^n$. With this identification the complex linear group $\mathrm{Gl}(n, \mathbb{C})$ is a subgroup of $\mathrm{Gl}(2n, \mathbb{R})$, and $\mathrm{U}(n)$ is a subgroup of $\mathrm{Sp}(2n)$. The following proposition describes the connection between symplectic and complex linear maps.

Proposition 1.1.24.

$$\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{Gl}(n, \mathbb{C}) \cap \mathrm{Sp}(2n) = \mathrm{Gl}(n, \mathbb{C}) \cap \mathrm{O}(2n)$$

Proof.

$$\begin{aligned} A \in \mathrm{Gl}(n, \mathbb{C}) &\Leftrightarrow AJ_0 = J_0 A \\ A \in \mathrm{Sp}(2n) &\Leftrightarrow A^T J_0 A = J_0 \\ A \in \mathrm{O}(2n) &\Leftrightarrow AA^T = \mathbb{I} \end{aligned}$$

Any two of these conditions implies the third. Also

$$A \in \mathrm{O}(2n) \cap \mathrm{Sp}(2n) \Leftrightarrow A = \begin{pmatrix} B & -C \\ C & B \end{pmatrix} \in \mathrm{Gl}(2n, \mathbb{R})$$

which commutes with J_0, and this is exactly the condition on $U = B + iC$ to be unitary. That is a linear map that preserves both orthogonal and symplectic structure on a vector space, will preserve the Hermetian structure induced by them on it. $\square$

1.1.3 Hermitian structures

Definition 1.1.25. A *Hermitian structure* on a complex vector space V is a Hermetian inner product which is an $\mathbb{R}$-bi-linear map, $H : V \times V \to \mathbb{C}$ satisfying the following properties:

- $H(v, w) = \overline{H(w, v)}$
- $H(v, \bar{v}) > 0;\ v \neq 0.$
- $H(iv, w) = -H(v, iw) = iH(v, w)$

A complex vector space V equipped with a positive-definite Hermitian structure called a *Unitary space.*

Hermitian structures in a unitary space V do not coinside with orthogonal structures in $V_{\mathbb{R}}$. From a real perspective, it make sense to split the Hermitian structure over the real numbers into its real and imaginary part, which are separately $\mathbb{R}$-linear:

$$h = \mathfrak{Re}(H),\ w = \mathfrak{Im}(H)$$

Each of these two parts is non-degenerate, h is symmetric whereas w is antisymmetric. Hence, h determines an orthogonal structure on $V_{\mathbb{R}}$, and w determines a symplectic one. Both of these structures are invariant under multiplication by i, that is, the canonical complex structure on $V_{\mathbb{R}}$:

$$h(iu, iv) = h(u, v),\ w(iu, iv) = w(u, v)$$

The two structures are related by the following relation:

$$h(u, v) = w(iu, v),\ w(u, v) = -w(iu, v)$$

Moreover, h is positive-definite if and only if w is.

Definition 1.1.26. A *Hermitian inner product* on a real vector space V with a complex structure J is an inner product h such that

$$\forall v_1, v_2 \in V : h(Jv_1, Jv_2) = h(v_1, v_2)$$

$\forall v \in V : h(Jv, v) = 0$ Indeed,

$$h(Jv, v) = h(J^2v, Jv) = h(-v, Jv) = -h(v, Jv) = -h(Jv, v)$$

Proposition 1.1.27. Let h be a Hermitian inner product on a $2n$-real dimension vector space V with a complex structure J. Then there exist elements $e_1, ..., e_n$ of V such that $\{e_1, ..., e_n, Je_1, ..., Je_n\}$ is an orthonormal basis for V with respect to the inner product h.

Proof. Let e_1 to be the unit vector,$\{e_1, Je_1\}$ would be an orthonormal and linearly independent set. Let W be the subspace spanned by e_1 and Je_1, and let $W^{\perp}$ be the orthogonal complement $V = W \oplus W^{\perp}$. Then $W^{\perp}$ is invariant by J, i.e., $\forall w \in W^{\perp}, Jw \in W^{\perp}$;

$$\forall v \in W, w \in W^{\perp} : \langle Jw, v\rangle = \langle J^2w, Jv\rangle = \langle -w, Jv\rangle = -\langle w, Jv\rangle \Rightarrow \langle Jw, v\rangle = 0$$

this shows that J is a complex structure on $W^{\perp}$. By induction on the dimension $2n$ of V, assume the proposition holds for $W^{\perp}$ of dimension $2(n-1)$, then it has an orthonormal basis $\{e_2, ..., e_n, Je_2, ...Je_n\}$. Since $W \perp W^{\perp}$ we can get the desired orthonormal basis. $\square$

Example 1.1.28. *(Canonical Hermitian inner product on* $\mathbb{R}^{2n}$)
If h_0 is the canonical inner product on $\mathbb{R}^{2n}$, i.e., the inner product with respect to which the natural basis of $\mathbb{R}^{2n}$ is orthonormal, then h_0 is a Hermitian inner product for $\mathbb{R}^{2n}$ with the canonical complex structure J_0 of $\mathbb{R}^{2n}$.

There is a natural one-to-one correspondence between the set of Hermitian inner products in $\mathbb{R}^{2n}$ with respect to the canonical complex structure J_0 and the homogeneous space $GL(n, \mathbb{C})/U(n)$.

The Hermitian inner product h in (V, J) can be extended uniquely to a complex symmetric bi-linear form, denoted also by h of $V^{\mathbb{C}}$ such that

1. $h(\bar{z}, \bar{w}) = \overline{h(z, w)}; \forall z, w \in V^{\mathbb{C}}$
2. $h(z, \bar{z}) > 0; \forall z \in V^{\mathbb{C}} - \{0\}$
3. $h(z, \bar{w}) = 0; \forall z \in V^{1,0}, w \in V^{0,1}$

Conversely, every complex symmetric bi-linear form h on $V^{\mathbb{C}}$ satisfying the previous properties is the natural extension of a Hermitian inner product of V.

To each Hermitian inner product h on V with respect to a complex structure J, we associate an element φ of $\Lambda^2 V^*$ as follows:

$$\varphi(x, y) = h(x, Jy); \forall x, y \in V$$

φ is skew-symmetric:

$$\varphi(y, x) = h(y, Jx) = h(Jx, y) = h(Jx, -J^2 y) = h(x, -Jy) = -\varphi(x, y)$$

φ is invariant by J:

$$\varphi(Jx, Jy) = h(Jx, J^2 y) = -h(y, Jx) = -\varphi(y, x) = \varphi(x, y)$$

Since $\Lambda^2 V^*$ can be considered as a subspace of $\Lambda^2 V^{*\mathbb{C}}$, φ may be considered as an element of $\Lambda^2 V^{*\mathbb{C}}$. In other words, φ may be uniquely extended to a skew-symmetric bi-linear form on $V^{\mathbb{C}}$, denoted also by φ.

If $\varphi \in \Lambda^2 V^*$ the skew-symmetric bi-linear form on $V^{\mathbb{C}}$ associated to a Hermitian inner product h of V, then $\varphi \in \Lambda^{1,1} V^{*\mathbb{C}}$.

1.2 Linear algebra of $V \oplus V^*$

1.2.1 Symmetries of $V \oplus V^*$

We can define three classes of objects on V as orthogonal symmetries of $V \oplus V^*$. These three classes are: endomorphisms (the usual symmetries of V), 2-forms, and bi-vectors. We examine how these different classes act on $V \oplus V^*$.

Let V be an n-dimensional real vector space. The space $V \oplus V^*$ has a natural symmetric inner product defined by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2} \left(\xi \left(Y \right) + \eta \left(X \right) \right), \quad X, Y \in V,\ \xi, \eta \in V^*$$

Let $e_1, e_2, ..., e_n$ be a basis of V, and let $e^1, e^2, ..., e^n$ be the dual basis of V^*. Then $e_1 + e^1, e_2 + e^2, ..., e_n + e^n, e_1 - e^1, e_2 - e^2, ..., e_n - e^n$ is a basis for $V \oplus V^*$ such that

$$\left\langle e_i + e^i, e_i + e^i \right\rangle = 1$$

$$\left\langle e_i - e^i, e_i - e^i \right\rangle = -1$$

$$\left\langle e_i \pm e^i, e_j \pm e^j \right\rangle = 0\ ; i \neq j$$

Thus $\langle .,. \rangle$ is non-degenerate with signature (n, n), called *split signature.* The symmetry group of the structure $(V \oplus V^*, \langle .,. \rangle)$ is therefore

$$\mathrm{O}\left(V \oplus V^*\right) = \left\{A \in \mathrm{GL}\left(V \oplus V^*\right) : \langle A., A. \rangle = \langle .,. \rangle \right\} \simeq O\left(n, n\right)$$

Since we have on $V \oplus V^*$ the natural orientation coming from the canonical symplectic structure

$$w\left(X + \alpha, Y + \beta\right) = \frac{1}{2} \left(\beta\left(X\right) - \alpha\left(Y\right) \right)$$

, we reduce the symmetry group to $\mathrm{SO}\left(n, n\right)$. The Lie algebra of $\mathrm{SO}(V \oplus V^*)$ is

$$\mathfrak{so}\left(V \oplus V^*\right) = \wedge^2 \left(V \oplus V^*\right) = \wedge^2 V \oplus \left(V \otimes V^*\right) \oplus \wedge^2 V^* \simeq \wedge^2 V \oplus \mathrm{End}\left(V\right) \oplus \wedge^2 V^*$$

and so a skew-adjoint transformation $T \in \mathfrak{so}(V \oplus V^*) = \{T \mid \langle Tx, y\rangle + \langle x, Ty\rangle = 0 \; x, y \in V \oplus V^*\}$, can be written in block-diagonal form as

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix},$$

where $A \in \mathrm{End}(V), B \in \wedge^2 V, \beta \in \wedge^2 V^*$. Each part of the previous decomposition acts on $V \oplus V^*$, and by exponentiation we obtain orthogonal symmetries of $V \oplus V^*$ in the identity component of $\mathrm{SO}(V \oplus V^*)$.

Let $A \in \mathrm{End}\,(V)$ corresponds to

$$T_A = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \in \mathfrak{so}\,(V \oplus V^*)$$

which acts on $V \oplus V^*$ as the linear transformation

$$e^{T_A} = \begin{pmatrix} e^A & 0 \\ 0 & \left((e^A)^*\right)^{-1} \end{pmatrix} \in \mathrm{SO}\,(V \oplus V^*)$$

Since any transformation of positive determinant is e^A for some $A \in \mathrm{End}\,(V)$, we can regard $\mathrm{GL}^+\,(V)$ as a subgroup of $\mathrm{SO}(V \oplus V^*)$, which we can extend to the full $\mathrm{GL}(V)$. Then the usual symmetries of V is part of larger group of symmetries.

The case of a 2-form $B \in \wedge^2 V^*$viewed as a map $B : V \to V^*$ with $B^* = -B$, which corresponds to

$$T_B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \in \mathfrak{so}(V \oplus V^*)$$

which acts on $V \oplus V^*$ as the linear transformation

$$e^B = e^{T_B} = \exp\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} + 0 = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

since $T_B^2 = 0$. i.e. e^{T_B} is the transformation sends

$$\begin{pmatrix} X \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} X \\ \xi + B(X) \end{pmatrix} = \begin{pmatrix} X \\ \xi + i_X B \end{pmatrix}$$

Thus B gives rise to an orthogonal transformation preserving projection onto V and act by shearing in the V^* direction, which called *B-field* transformations.

The case of a bivector $\beta \in \wedge^2 V$ viewed as a map $\beta : V^* \to V$, is the same as the 2-form, let β corresponds to

$$T_\beta = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

which acts on $V \oplus V^*$ as

$$e^\beta = e^{T_\beta} = \exp\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} X \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} X + i_\xi \beta \\ \xi \end{pmatrix}$$

which is an orthogonal transformation preserving projection onto V^* and act by shearing in the V direction, which called *β-field* transformations. The B-field action will be fundamental giving extra transformation in generalized geometry, which represent a breaking in the symmetry since the bi-vector $\beta \in \wedge^2 V$ plays a lesser role.

1.2.2 Maximal isotropic subspaces

We have seen in Example 1.1.18 that the pairing $V \times V^* \to \mathbb{R}$ defines a symplectic structure,

$$w(X + \alpha, Y + \beta) = \frac{1}{2}\left(\beta(X) - \alpha(Y)\right)$$

In addition, we defined a maximal isotropic subspace L to be an isotropic subspace of maximal dimension, i.e. $w(X, Y) = 0 \quad \forall X, Y \in L$.

Since both subspaces V and V^* are null under the pairing, they are examples of maximal isotropic subspaces. Another example is the graph Γ_B of the map $B : V \to V^*$, $\Gamma_B = e^B.V = \{X + \imath_X B \ : X \in V\}$. Also for $\beta : V^* \to V$, The space $e^{\beta}.V^* = \{\imath_\xi \beta + \xi \ : \ \xi \in V^*\}$ is maximal isotropic subspace. In general, $\forall A \in \mathrm{O(V \oplus V^*)}$: A(V) is a maximal isotropic subspace of $V \oplus V^*$. Maximal isotropic subspaces of $V \oplus V^*$ are also called *linear Dirac structures.* They provide an alternative splittings for $V \oplus V^*$; If L, L' are any maximal isotropics such that $L \cap L' = \varnothing$, then the inner product define an isomorphism $L' \simeq L^*$. Thus, $V \oplus V^* = L \oplus L'$. The space of maximal isotropics is disconnected into two components, and elements of these are said to have *odd* or *even* type (cf. Definition 1.2.4) depending on whether their component intersect V or not. The following two examples are important.

Example 1.2.1. Let $E \subset V$ be any subspace of dimension d, and consider the space $E \oplus \mathrm{Ann\,(E)} \subset \mathrm{V \oplus V^*}$ where Ann (E) is the annihilator of E in V which have the dimension $n - d$. Then $E \oplus \mathrm{Ann\,(E)}$ is maximal isotropic subspace.

Example 1.2.2. Let $E \subset V$ be any subspace, and let $\epsilon \in \wedge^2 E^*$ considering it as a map $E \to E^*$ via $X \to i_X \epsilon$. Consider the subspace analogous to the graph of ϵ ,

$$L\,(E, \epsilon) = \{X + \xi \in E \oplus V^* \ : \ \xi\,|_E = \epsilon\,(X)\}$$

Then for $X + \xi, Y + \eta \in L\,(E, \epsilon)$,

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}\left(\xi\,(Y) + \eta\,(X)\right) = \frac{1}{2}\left(\epsilon\,(Y, X) + \epsilon\,(X, Y)\right) = 0$$

and so $L\,(E, \epsilon)$ is a maximal isotropic subspace.

Remark 1.2.3. When $\epsilon = 0$, $L\,(E, 0) = E \oplus \mathrm{Ann\,(E)}$ which was the first example. In addition, $L\,(V, 0) = V$ and $L\,(\{0\}, 0) = V^*$.

If we define $E = \pi_V L$, where π_V is the projection $V \oplus V^* \to V$, then $L \cap V^* = \mathrm{Ann}(E)$ since L is maximal isotropic. Also $V^* = E^*/\mathrm{Ann}(E)$ and so if we define

$$\begin{aligned} \epsilon : E &\to E^* \\ e &\mapsto \pi_{V^*}\left(\pi_V^{-1}\,(e) \cap L\right) \in V^*/\mathrm{Ann}\,(E) \end{aligned}$$

then every maximal isotropic L is of the form $L(E,\epsilon)$.

Definition 1.2.4. The *type* of a maximal isotropic $L(E,\epsilon)$, is the codimension k of its projection into V,

$$k = \dim \mathrm{Ann}(E) = n - \dim \pi_V(L)$$

Since B-transformations preserve projections to V, it does not effect E and so it does not change the type of the maximal isotropic. However, β-transformations do change the type of the maximal isotropic, since it change projections to V and therefore may change the dimension of E.

1.2.3 Spinors of $V \oplus V^*$, pure spinors

A spin structure for the orthogonal structure on $V \oplus V^*$ is always exists, and it is isomorphic to the exterior forms $\Lambda^\bullet V^*$. We show that there is a correspondence between maximal isotropic subspaces and special types of spinors called pure spinors.

We have the natural Cartan's action of $V \oplus V^*$ on $S = \wedge^\bullet V^*$. If $X + \xi \in V \oplus V^*$ and $\rho \in \wedge^\bullet V^*$, let

$$(X+\xi)\cdot\rho = \imath_X\rho + \xi\wedge\rho$$

Then

$$\begin{aligned}(X+\xi)^2\cdot\rho &= \imath_X(\imath_X\rho + \xi\wedge\rho) + \xi\wedge(\imath_X\rho + \xi\wedge\rho)\\ &= (\imath_X\xi)\rho - \xi\wedge\imath_X\rho + \xi\wedge\imath_X\rho\\ &= \langle X+\xi, X+\xi\rangle\rho\end{aligned}$$

Thus, we have an action of $v \in V \oplus V^*$ with $v^2\rho = \langle v, v\rangle\rho$. This is the defining relation for the Clifford algebra $\mathrm{CL}(V \oplus V^*)$ using the natural pairing $\langle .,. \rangle$ on $V \oplus V^*$.

The spin group Spin $(V \oplus V^*) \subset$ CL $(V \oplus V^*)$ is defined by

$$\text{Spin}\,(V \oplus V^*) = \{v_1 \dots v_{2k} : v_i \cdot v_i = \pm 1; k \in \mathbb{N}\}$$

which is a double cover of the spacial orthogonal group SO (V $\oplus$ V*) via

$$\begin{aligned} \phi : \text{Spin}\,(V \oplus V^*) &\rightarrow \text{SO}\,(V \oplus V^*) \\ \phi\,(v)\,(X) &\mapsto v \cdot X \cdot v^{-1} \quad ; v \in \text{Spin}\,(V \oplus V^*)\,, X \in V \oplus V^* \end{aligned}$$

This map identifies the Lie algebras $\mathfrak{spin}(n,n) \cong \mathfrak{so}(n,n) \cong \Lambda^2 V \oplus \Lambda^2 V^* \oplus End(V)$:

$$\begin{aligned} \mathfrak{spin}(n,n) &\rightarrow \mathfrak{so}(n,n) \\ d\phi(v)(X) &\mapsto [v, X] = v \cdot X - X \cdot v \end{aligned}$$

Above, we identify $\mathfrak{spin}(n,n)$ with elements of CL $(V \oplus V^*)$ of the form $\sum_{i=1}^{2k} v_i v_j$; $v_j.v_j = \mathbb{I}_i$ Since the exterior algebra of V^* is the space of spinors, each element in $\mathfrak{spin}(n,n)$ acts naturally on $\Lambda^\bullet V^*$.

Example 1.2.5. Let $B = \sum b_{ij} e^i \wedge e^j \in \Lambda^2 V^*$ be a 2-form. It acts on $V \oplus V^*$ via $X + \xi \mapsto \imath_X B$ this defines an element of $\mathfrak{so}(n,n)$. The corresponding element in $\mathfrak{spin}(n,n)$ inducing the same action on $V \oplus V^*$ is given by $\sum b_{ij} e^j e^i \in$ CL $(V \oplus V^*)$. It follows that the spinorial action of B on a form ρ is given by $\sum b_{ij} e^j e^i \cdot \rho = -B \wedge \rho$.

In a similar way we can see the Lie algebra action of a bi-vector

Example 1.2.6. Let $\beta = \sum \beta^{ij} e_i \wedge e_j \in \Lambda^2 V \subset \mathfrak{so}(n,n)$, its action is given by $\beta \cdot (X + \xi) \mapsto \imath_\xi \beta$. And the corresponding element in $\mathfrak{spin}(n,n)$ with the same action is $\sum \beta^{ij} e_j e_i$. So the action on a form ρ is given by $\beta \cdot \rho = \imath_\beta \rho$.

Let σ be the anti-automorphism of CL (V $\oplus$ V*) defined on decomposables by

$$\sigma\,(v_1 \otimes \cdots \otimes v_k) = v_k \otimes \cdots \otimes v_1$$

Then we have the following bi-linear form on $\Lambda^\bullet V^* \subset \mathrm{CL}\,(V \oplus V^*)$

$$(\xi_1, \xi_2) = (\sigma(\xi_1) \wedge \xi_2)_{\mathrm{top}}$$

where top indicates taking the top degree component on the form. This bi-linear form called the *Mukai pairing.* The natural inner product on $V \oplus V^*$ extends by complexification to $(V \oplus V^*) \otimes \mathbb{C}$. Now given $\rho \in \Lambda^\bullet V^* \otimes \mathbb{C}$, we can consider its Clifford annihilator

$$L_\rho = \{v \in (V \oplus V^*) \otimes \mathbb{C} : v \cdot \rho = 0\}$$

Since for $v \in L_\rho$,

$$0 = v^2 \cdot \rho = \langle v, v \rangle \rho$$

thus L_ρ is always isotropic.

Definition 1.2.7. A form $\rho \in S = \Lambda^\bullet V^*$ is *pure* if L_ρ is maximal isotropic, i.e. if $\dim_{\mathbb{C}} L_\rho = \dim_{\mathbb{R}}\mathrm{V}$.

Given a maximal isotropic subspace $L \subset V \oplus V^*$, we can always find a pure form annihilating it and conversely, if two pure forms annihilate the same maximal isotropic, they are multiple of each other. Thus, maximal isotropics are in one to one correspondence with lines of pure forms. Algebraically, the requirement of a form to be pure implies that it is of the form $e^{B+iw}\Omega$ where B and w are bi-vectors and $\Omega = \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_k$ is a decomposable complex k-vector.

Example 1.2.8. Let $\rho = e^1 \wedge \cdots \wedge e^n$, then $L_\rho = V^*$ and ρ is a pure form. Take $1 \in \Lambda^0 V^*$, then $L_1 = V$, hence 1 is a pure spinor.

1.3 Linear generalized complex structures

Let V be a real vector space, and consider the direct sum $V \oplus V^*$,

Definition 1.3.1. A *generalized complex structure* on V is an endomorphism $\mathcal{J} \in \mathrm{End}\,(V \oplus V^*)$ which satisfies two conditions

- It is complex, i.e. $\mathcal{J}^2 = -\mathbb{I}$.
- It is symplectic, i.e $\mathcal{J}^* = -\mathcal{J}$.

If $\mathcal{J}^2 = -\mathbb{I}$ and $\mathcal{J}^* = -\mathcal{J}$, this is equivalent to $\mathcal{J}^*\mathcal{J} = \mathbb{I}$ which means that $\mathcal{J}$ is an orthogonal choice of complex structure. Equivalently, a *generalized complex structure* can be defined on V as a complex structure on $V \oplus V^*$ which is orthogonal in the natural inner product.

Since $\mathcal{J}^2 = -\mathbb{I}$, it splits the complexification $(V \oplus V^*) \otimes \mathbb{C}$ into a direct sum of $\pm i$-eigenspaces, L and $\bar{L}$. Further, since $\mathcal{J}$ is orthogonal,

$$\langle u, v\rangle = \langle \mathcal{J}u, \mathcal{J}v\rangle = \langle iu, iv\rangle = -\langle u, v\rangle \Rightarrow \langle u, v\rangle = 0\, \forall u, v \in L$$

and so L is maximal isotropic subspace with respect to the inner product. Conversely, specifying such an L as the i-eigenspace determines a unique generalized complex structure on V. Thus, a *generalized complex structure* on V of dimension n is equivalent to a maximal isotropic subspace $L \in (V \oplus V^*) \otimes \mathbb{C}$ such that $L \cap \bar{L} = \{0\}$.

Proposition 1.3.2. (Theorem 4.5 in (Gualtieri, 2004))
The vector space V admits a generalized complex structure if and only if it is of even dimension.

Proof. Since the inner product on $V \oplus V^*$ is indefinite, we can find $v \in V \oplus V^*$ such that $\langle v, v\rangle = 0$. Since $\mathcal{J}$ is orthogonal complex structure $\mathcal{J}v$ is orthogonal to v and $\langle \mathcal{J}v, \mathcal{J}v\rangle = 0$. Thus, the subspace $N = \mathrm{span}\{v, \mathcal{J}v\} \subset V \oplus V^*$ is an isotropic subspace. We can extend N by adding pairs of vectors v' orthogonal to N with $\mathcal{J}v'$ until N becomes maximal isotropic. Since the inner product has split signature, $\dim N = \dim V$ and so V must be of even dimension. $\square$

Remark 1.3.3. generalized complex structures reduces the structure from $\mathrm{O}(2n, 2n)$ to $\mathrm{U}(n, n) = \mathrm{O}(2n, 2n) \cap \mathrm{GL}(2n, \mathbb{C})$

An extra characterization of generalized complex structure on V can be obtained from the interpretation of forms as spinors. They can be determined by line of the complex differential forms. To any maximal isotropic we can associate a one dimensional space U_L it annihilates. Since the generalized complex structure is given by the maximal isotropic L, it can equivalently determined by the pure spinor line $U_L \subset \wedge^{\bullet} V^* \otimes \mathbb{C}$ generated by

$$\rho_L = e^{B+iw}\Omega$$

where $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$, and the integer k is the *type* of the maximal isotropic as in Definition 1.2.4.

Definition 1.3.4. We call the complex line U_L *the canonical line* of the generalized complex structure, and the integer k the *type* of the generalized complex structure.

Since L is the i-eigenspace of a complex structure, then $L \cap \bar{L} = \{0\}$. This intersection property can also be viewed from the pure spinors, using the Mukai bilinear form,

Proposition 1.3.5. (Chevalley, 1997)
Maximal isotropics L, L' satisfy $\dim L \cap L' = \{0\}$ if and only if their pure spinor representatives ρ, ρ' satisfy $(\rho, \rho') \neq 0$.

Therefore, determining a generalized complex structure on V^{2n} is equivalent to an additional constraint on the generator ρ_L:

$$(\rho, \bar{\rho}) = w^{2n-2k} \wedge \Omega \wedge \bar{\Omega} \neq 0 \quad ; \Omega = \theta_1 \wedge \cdots \wedge \theta_k$$

or in other words

- $\theta_1, \ldots, \theta_k, \bar{\theta}_1, \ldots, \bar{\theta}_k$ are linearly independent.
- w is non-degenerate when restricted to the real $(2n-2k)$-dim subspace $\triangle \subset V$ where $\triangle = \mathrm{Ker}\left(\Omega \wedge \bar{\Omega}\right)$.

Example 1.3.6. (symplectic type $k = 0$)

The generalized complex structure determined by a symplectic structure ω is

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

where the matrix is written in the splitting $V \oplus V^*$. In fact, $\mathcal{J}_\omega^2 = -1$ and $\mathcal{J}_\omega^* = -\mathcal{J}_\omega$. The i-eigenspace of $\mathcal{J}_\omega$ is the maximal isotropic $L = \{X - iw\,(X) \,:\, X \in V \otimes \mathbb{C}\}$ which is the Clifford annihilator of the spinor line generated by

$$\rho = e^{i\omega}$$

This generalized complex structure has type $k = 0$ (the codimension of the projection of L to $V \otimes \mathbb{C}$).

Example 1.3.7. (complex type $k = n$)

The generalized complex structure corresponding to a complex structure J is

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

where the matrix written in the splitting $V \oplus V^*$. In fact, $\mathcal{J}_J^2 = -1$ *and* $\mathcal{J}_J^* = -\mathcal{J}_J$. The i-eigenspace of $\mathcal{J}_J$ is the maximal isotropic $L = V^{0,1} \oplus V^{1,o*}$ where $V^{1,0}$ is the i-eigenspace of the complex structure J. The space L is the Clifford annihilator of the spinor line generated by

$$\rho_L = \Lambda^n \left(V^{1,0*}\right)$$

This generalized complex structure is of type $k = n$.

CHAPTER II

GENERALIZED COMPLEX MANIFOLDS

After studying the pointwise structure of the generalized tangent bundle in the first chapter, we transfer all the properties studied to a manifold. Let M be a smooth manifold of dimension $2n$, with the tangent bundle T, and consider the direct sum of the tangent and cotangent bundle $T \oplus T^*$. This generalized vector bundle is equipped with the same inner product and orientation described on $V \oplus V^*$, then $T \oplus T^*$ have a natural structure group $\mathrm{SO}(n, n)$. We consider an extension of the Lie bracket of two vector fields. This is the Courant bracket which was introduced in the literature first by (Courant, 1990), in the context of his work with Weinstein to define Dirac structure. In generalized geometry, as well as in Dirac geometry, Courant bracket defines the integrability condition for different geometric structures.

In the first section, we study Courant bracket on $T \oplus T^*$. Next, we describe the topological implication of having generalized complex structure on the manifold. After that, we state the Courant integrability condition.

2.1 The Courant bracket on $T \oplus T^*$

The Courant bracket is a skew-symmetric bracket defined on smooth sections of $T \oplus T^*$ by

$$[\![X+\xi, Y+\eta]\!] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d\left(\imath_X\eta - \imath_Y\xi\right)$$

where $X+\xi, Y+\eta \in C^\infty\left(T \oplus T^*\right)$.

Although the Courant bracket reduces on vector fields to the Lie bracket $[X,Y]$ and vanishes on the 1-forms, it is not a Lie bracket since it fails to satisfy the Jacobi identity. The Courant bracket structure emerges from the interpretation of forms as spinors, where they can be obtained as a derived bracket of the differential operator d acting on differential forms. This is exactly analogous to the fact that the Lie bracket can be seen as a derived bracket of the exterior derivative d using Cartan's formulas

$$\mathcal{L}_X = \imath_X d + d\imath_X = \left[d, \imath_X\right], \ \imath_{[X,Y]} = \left[\mathcal{L}_X, \imath_Y\right]$$

which yield

$$\imath_{[X,Y]} = \left[\left[d, \imath_X\right], \imath_Y\right]$$

Remark 2.1.1. We recall the definition of the *Lie derivative* of a differential form, suppose φ^t is the one-parameter (locally defined) group of diffeomorphisms defined by a vector field X. Then there is a naturally defined *Lie derivative*

$$\mathcal{L}_X\alpha = \frac{\partial}{\partial t}\varphi_t^*\alpha\,|_{t=0}$$

of a p-form α by X. It is again a p-form. Now, given a vector field X on a manifold M, there is a linear map $\imath_X : \Omega^p(M) \to \Omega^{p-1}(M)$ called *the interior product* such that: $\imath_X df = X(f)$ and $\imath_X\left(\alpha \wedge \beta\right) = \imath_X(\alpha) \wedge \beta + (-1)^p \alpha \wedge \imath_X(\beta)$ if $\alpha \in \Omega^p(M)$.

The Courant bracket on $T \oplus T^*$ is a natural extension of the Lie bracket of vector fields but acting via the Clifford action

$$[\![u, v]\!] \cdot \rho = [[d, u\cdot], v\cdot] \rho$$

for $u, v \in C^\infty (T \oplus T^*)$, $\rho \in \wedge^\bullet V^*$. The proof of the last identity is in the following lemma.

Lemma 2.1.2. For any differential form ρ and any sections $u, v \in C^\infty (T \oplus T^*)$ we have the following identity

$$[[d, u\cdot], v\cdot] \rho = [\![u, v]\!] \cdot \rho$$

Proof. Let $u = X + \xi, v = Y + \eta$ so that $\imath_X \rho = -\xi \wedge \rho$, $\imath_Y \rho = -\eta \wedge \rho$ then

$$\begin{aligned}
\imath_{[X,Y]}\rho &= [\mathcal{L}_X, \imath_Y] \rho \\
&= \mathcal{L}_X (-\eta \wedge \rho) - \imath_Y (d(-\xi \wedge \rho) + \imath_X d\rho) \\
&= -\mathcal{L}_X (\eta \wedge \rho) - \eta \wedge (d(-\xi \wedge \rho) + \imath_X d\rho) \\
&\qquad -\imath_Y (-d\xi \wedge \rho + \xi \wedge d\rho + \imath_X d\rho) \\
&= (-\mathcal{L}_X \eta + \imath_Y d\xi) \wedge \rho - (\imath_Y + \eta\wedge)(\imath_X + \xi\wedge) d\rho \\
&= (-\mathcal{L}_X \eta + \imath_Y d\xi) \wedge \rho + u \cdot v \cdot d\rho
\end{aligned}$$

showing that

$$[\![u, v]\!] \cdot \rho = (u \circ v) \cdot \rho = u \cdot v \cdot d\rho = [[d, u\cdot], v\cdot] \rho$$

□

Courant bracket is also invariant under the action of the usual symmetries of Lie bracket: diffeomorphisms of the manifold M. In addition, there are extra symmetries which are B-field transformations. We have seen that the B-field action is the orthogonal transformation $e^B (X + \xi) = X + \xi + \imath_X B$.

Proposition 2.1.3. If $B \in \wedge^2 T^*$ is a closed 2-form, then its action on sections of $T \oplus T^*$ commute with the Courant bracket.

Proof.

$$\begin{aligned}
\left[\!\left[e^B (X+\xi), e^B (Y+\eta) \right]\!\right] &= [\![X+\xi+\imath_X B, Y+\eta+\imath_Y B]\!] \\
&= [\![X+\xi, Y+\eta]\!] + [\![X, \imath_Y B]\!] + [\![\imath_X B, Y]\!] \\
&= [\![X+\xi, Y+\eta]\!] + \mathcal{L}_X \imath_Y B - \frac{1}{2} d\imath_X \imath_Y B \\
&\quad -\mathcal{L}_Y \imath_X B + \frac{1}{2} d\imath_Y \imath_X B \\
&= [\![X+\xi, Y+\eta]\!] + \mathcal{L}_X \imath_Y B - \imath_Y \mathcal{L}_X B + \imath_Y \imath_X dB \\
&= [\![X+\xi, Y+\eta]\!] + \imath_{[X,Y]} B + \imath_Y \imath_X dB = \\
&= e^B \left([\![X+\xi, Y+\eta]\!] \right) + \imath_Y \imath_X dB
\end{aligned}$$

and when B is closed, then $dB = 0 \Rightarrow \imath_Y \imath_X dB = 0\,; \forall X, Y$

and so

$$\left[\!\left[e^B (X+\xi), e^B (Y+\eta) \right]\!\right] = [\![X+\xi, Y+\eta]\!] + \imath_{[X,Y]} B = e^B \left([\![X+\xi, Y+\eta]\!] \right)$$

□

Thus, a closed 2-form B acts also preserving the Courant bracket, which means that we have an action of the semi-direct product of the closed 2-forms with diffeomorphisms

$$\mathrm{Sym}\left(T \oplus T^*, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]\right) = \Omega^2_{cl}(M) \rtimes \mathrm{Diff}\,(\mathrm{M})$$

where the first piece is a local point-wise transformation and the second is a global transformation.

The infinite dimensional Lie algebra of the extended group $\Omega^2_{cl}(M) \rtimes \mathrm{Diff}\,(\mathrm{M})$ consists of sections $X + B$ of $T \oplus \wedge^2 T^*$ where B is closed. Let $u = X + \xi$ be a section of $T \oplus T^*$ or a *generalized vector field.* If we take $B = -d\xi$ then $\tilde{u} = X - d\xi \in \mathrm{Lie}\,(\Omega^{\mathbf{2}}_{\mathbf{cl}}(\mathbf{M}) \rtimes \mathrm{Diff}\,(\mathrm{M}))$. We can regard the map $\tilde{u} \to u$ given by $X + \xi \to X - d\xi$ as the *Lie derivative* $\mathcal{L}_u$ in the direction of the generalized vector field $u \in C^\infty(T \oplus T^*)$. The Lie algebra action of $\tilde{u}$ on $v = Y + \eta$ is

$$\tilde{u} \cdot v = (X - d\xi) \cdot (Y + \eta) = \mathcal{L}_X (Y + \eta) - \imath_Y d\xi = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + d(\imath_Y \xi)$$

which is a non- skew symmetric version of the Courant bracket, and so by skew-symmetrization we can recover the Courant bracket,

$$\begin{aligned} \frac{1}{2}((\tilde{u} \cdot v) - (\tilde{v} \cdot u)) &= \frac{1}{2}((X - d\xi) \cdot (Y + \eta) - (Y - d\eta) \cdot (X + \xi)) \\ &= \frac{1}{2}([X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + d(\imath_Y \xi)) \\ &\quad + \frac{1}{2}([Y, X] - \mathcal{L}_Y \xi + \mathcal{L}_X \eta - d(\imath_X \eta)) \\ &= [\![X + \xi, Y + \eta]\!] = [\![u, v]\!] \end{aligned}$$

Nevertheless, although Courant bracket is derived this way from a Lie algebra action, it is not itself a bracket of any Lie algebra since the Jacobi identity fails as mentioned before.

2.2 Generalized almost complex structures and Courant integrability condition

We define a generalized almost complex structure as following,

Definition 2.2.1. A *generalized almost complex structure* is a differentiable bundle automophism $\mathcal{J} : T_pM \oplus T_p^*M \to T_pM \oplus T_p^*M$ which is a linear generalized complex structure on each fiber of the generalized bundle $T \oplus T^*$.

Toplogically, a generalized complex structure is a reduction to $U(n, n)$, which is homotopic to its maximal compact subgroup $U(n) \times U(n)$. Thus, $U(n, n)$ structure is reduced to $U(n) \times U(n)$. Geometrically, this can be interpreted as a choice of a positive definite sub-bundle $C_+ \subset T \oplus T^*$ which is complex with respect to $\mathcal{J}$. The orthogonal complement $C_- = C_+^\perp$ is negative-definite and also complex. Therefore we have an orthogonal decomposition

$$T \oplus T^* = C_+ \oplus C_-$$

Because C_+ and C_- are definite and T is null, the projection $\pi_T : C\pm \to T$ is an isomorphism, and we can transfer the complex structure on $C\pm$ to T. This give us two almost complex structures J_+, J_- on T. In addition, we have seen in Example 1.3.7 that if a manifold admits an almost complex structure, it actually admits a generalized almost complex structure of type n (we called it the complex type). Consequently, a generalized almost complex structure exists on a manifold if and only if an almost complex structure does, and we can deduce the following important corollary,

Corollary 2.2.2. (Theorem 4.15 in (Gualtieri, 2004))

The obstruction to the existence of a generalized almost complex structure is the same as that for an almost complex structure, which itself is the same as that for a nondegenerate 2-form (almost symplectic structure).

We proceed now to the integrability condition:

Definition 2.2.3. A generalized almost complex structure $\mathcal{J}$ is said to be *integrable* to a generalized complex structure when its $+i$-eigenbundle $L \subset (T \oplus T^*) \otimes \mathbb{C}$ is Courant involutive. i.e., Sections of the subbundle L defined by the $+i$-eigenspaces of $\mathcal{J}$ are closed under the Courant bracket.

Fact 2.2.1. *The obstruction for a generalized complex structure to be integrable is tensorial. In other words, if $[\![u, v]\!]$ is a section of L, so is $[\![u, fv]\!]$.*

Proof. The main reason is that L is isotropic with respect to the inner product. Indeed, If $\mathcal{J}u = iu$ then

$$i\langle u, u\rangle = \langle \mathcal{J}u, u\rangle = -\langle u, \mathcal{J}u\rangle = -i\langle u, u\rangle$$

Using the property of the Courant bracket:

$$[\![u, fv]\!] = f\,[\![u, v]\!] + (Xf)v - \langle u, v\rangle\, df$$

If u, v are sections of L then $\langle u, v\rangle = 0$. Then, $[\![u, fv]\!] = f\,[\![u, v]\!] + (Xf)v$.

Thus, if $u, v, [\![u, v]\!]$ are sections, so is $[\![u, fv]\!]$. □

We have seen that the generalized tangent bundle $T \oplus T^*$ have a natural structure group $\mathrm{SO}(n,n)$. We consider without proof the fact that this $\mathrm{SO}(n,n)$-bundle always admits $\mathrm{Spin}(n,n)$ structure. Then, we can view differential forms on a manifold M as spin representation of $T \oplus T^*$. Accordingly, we can characterize a generalized complex structure in terms of the so-called canonical bundle K. The subbundle L has rank n and is isotropic in a $2n$-dimensional space. It is also of maximal dimension (For a nondegenerate inner product n is the maximal). For a spinor ρ,

$$(v, v)\rho = v \cdot v{\cdot}\rho = 0$$

Thus, the space of $v \in W$ such that $v \cdot \rho = 0$ is isotropic. Because to any maximal isotropic subspace we can associate a one-dimensional space of pure spinors it annihilates, a generalized complex manifold has a complex line subbundle K of $\Lambda^\bullet T^* \otimes \mathbb{C}$, which called *the canonical bundle.* It consists of multiples of a pure spinor defining $\mathcal{J}$. The condition $L \cap \bar{L} = 0$ is equivalent to the Mukai pairing $\langle \rho, \bar{\rho}\rangle \neq 0$.

Example 2.2.4. Let M be a complex manifold. The subspace generated by $\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, \ldots, dz_1, dz_2, \ldots$ annihilates $dz_1 \wedge dz_2 \wedge \ldots \wedge dz_m$. This generates the usual canonical bundle of a complex manifold.

Example 2.2.5. The tangent space T annihilates 1 by Clifford multiplication. Thus, for a symplectic manifold, the transform of T by $-iw$ annihilates the form e^{iw} where the canonical bundle K is trivialized by this form.

The following theorem has been proved in (Hitchin, 2010), which give us the integrability condition in this context:

Theorem 2.2.6. *Let ρ be a form which is a pure spinor such that $\langle \rho, \bar{\rho} \rangle \neq 0$. Then ρ defines a generalized complex structure if and only if $d\rho = e \cdot \rho$ for some local section $e \in \Gamma$.*

Remark 2.2.7. The simplest use of this integrability is when there is a global closed form which is a pure spinor. Such manifolds are called g*eneralized Calabi-Yau* manifolds. They include Calabi-Yau manifolds such that the holomorphic n-form is ρ, and symplectic manifolds where $\rho = e^{iw}$.

In the following examples, we see that the integrability condition on generalized almost complex structures yields the classical conditions on symplectic and complex structures.

Remark 2.2.8. By type of the generalized complex structure on the manifold, we mean the codimension of $E = \pi_T(L)T \otimes \mathbb{C}$. This type might not be constant throughout the manifold.

Example 2.2.9. Let M be a complex manifold with an almost complex structure. This almost complex structure induces a generalized almost complex structure with i-eigenspace $T^{0,1}M \oplus T^{1,0}M$. If this generalized almost complex structure is integrable, then $T^{0,1}M$ has to be closed with respect to the Lie bracket. Thus, the almost complex structure is actually a complex structure. Conversely, any complex structure gives rise to an integrable generalized complex structure.

Example 2.2.10. Let M be a complex manifold. If it has a non-degenerate 2-form ω, then the induced generalized almost complex structure is integrable if for some $X + \xi$ we have $de^{i\omega} = (X + \xi) \cdot e^{i\omega}$. The degree 1 part gives that $\imath_X \omega + \xi = 0$ and the degree 3 part, that $d\omega = 0$ and hence M is a symplectic manifold.

Example 2.2.11. (Example 1.24 in (Cavalcanti, 2007))

Consider $\mathbb{C}^2$ with complex coordinates z_1, z_2. The differential form $\rho = z_1 + dz_1 \wedge dz_2$ is equal to $dz_1 \wedge dz_2$ along the locus $z_1 = 0$, while away from this locus it can be written as $\rho = z_1 \exp(\frac{dz_1 \wedge dz_2}{z_1})$. Since it also satisfies $d\rho = -\partial_2 \cdot \rho$, we see that it generates a canonical bundle K for a generalized complex structure which has type 2 along $z_1 = 0$ and type 0 elsewhere, showing that a generalized complex structure does not necessarily have constant type. In order to obtain a compact type-change locus we observe that this structure is invariant under translations in the z_2 direction, hence we can take a quotient by the standard $\mathbb{Z}^2$ action to obtain a generalized complex structure on the torus fibration $D^2 \times T^2$, where D^2 is the unit disc in the z_1-plane. Using polar coordinates, $z_1 = re^{2\pi\theta_1}$, the canonical bundle is generated, away from the central fibre, by

$$\begin{aligned} \exp(B + i\omega) &= \exp(d\ \log r + id\theta_1)(d\theta_2 + id\theta_3) \\ &= \exp\left(d \log r \wedge d\theta_2 - d\theta_1 \wedge d\theta_3 + i\left(d \log r \wedge d\theta_2 - d\theta_1 \wedge d\theta_3\right)\right) \end{aligned}$$

where θ_2 and θ_3 are coordinates for the 2-torus with unit periods. Away from $r = 0$, therefore, the structure is a B-field transform of a symplectic structure ω, where

$$\begin{aligned} B &= d \log r \wedge \theta_2 - d\theta_1 \wedge \theta_3 \\ \omega &= d \log r \wedge \theta_3 + d\theta_1 \wedge \theta_2. \end{aligned}$$

The type jumps from 0 to 2 along the central fibre $r = 0$, inducing a complex structure on the restricted tangent bundle, for which the tangent bundle to the fibre is a complex sub-bundle.

Remark 2.2.12. Since we know the Lie algebra action of $\mathfrak{spin}(n,n)$ on forms and $\mathcal{J} \in \mathfrak{spin}(n,n)$, we can compute its action on forms. For example, in the case of a generalized complex structure induced by a symplectic form ω, we have $\mathcal{J}$ is the sum of the 2-form ω, and a bivector $-w^{-1}$. Hence its Lie algebra action on a form ρ is $\mathcal{J}\rho = \imath_{(-\omega\wedge-\omega^{-1})}\rho$.

CHAPTER III

GENERALIZED COMPLEX STRUCTURES ON NILMANIFOLDS

In this chapter we introduce some interesting examples of generalized complex structures on Lie algebra, in which Courant bracket on the Lie algebra $\mathfrak{g}$ is a Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$. Hence, finding a generalized complex structure on $\mathfrak{g}$ is equal to the search for a complex structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ orthogonal with respect to the natural pairing endowed with. In (Cavalcanti et Gualtieri, 2004), they proved that each 6-dimensional nilpotent Lie algebra admits a generalized complex structure. A classification of which of those algebras admit complex or symplectic structures was done before in (Salamon, 2001; Goze et Khakimdjanov, 2010), and according to them, 5 nilpotent Lie algebras does not admit neither (left invariant) complex nor symplectic structures. Luckily, these 5 examples are the first exotic examples of generalized complex structures Gualtieri & Cavalcanti have produced. Moreover, (de Andrés *et al.*, 2007) have related left invariant generalized complex structures on Lie groups to Hermitian structures on cotangent Lie groups as we present in this chapter.

3.1 Lie algebras and Lie groups

Definitions 3.1.1. A real *Lie group* is a set G with two structures: G is a group and G is a manifold. The two structures are compatible in the following sense:

- multiplication map $G \times G \to G$ is a smooth map.
- inversion map $G \to G$ is a smooth map.

Lie groups are so frequently studied because they usually appear as symmetry groups of various geometric objects. Therefore, we study the action of Lie groups on manifolds and representations

Definitions 3.1.2. An *action* of a real Lie group G on a manifold M is an assignment to each $g \in G$ a diffeomorphism $\rho(g) \in \mathrm{Diff}(M)$ such that $\rho(1) = \mathbb{I}$, $\rho(gh) = \rho(g)\rho(h)$, and such that the map

$$\begin{aligned} G \times M &\to M \\ (g, m) &\mapsto \rho(g) \cdot m \end{aligned}$$

is a smooth map.

A *representation* of a real Lie group G is a vector space V (either real or complex) together with a group morphism $\rho : G \to \mathrm{End}(V)$. If V is finite-dimensional, we require that ρ be smooth (respectively, analytic), so it is a morphism of Lie groups. In other words, we assign to every $g \in G$ a linear map $\rho(g) : V \to V$ so that $\rho(g)\rho(h) = \rho(gh)$.

Important examples of group action are the following actions of G on itself:

1. Left action: $L_g : G \to G$, defined by $L_g(h) = gh$
2. Right action: $R_g : G \to G$, defined by $R_g(h) = hg^{-1}$.
3. Adjoint action: $Ad_g : G \to G$, defined by $Ad_g(h) = ghg^{-1}$.

where the left and right actions commute and $\mathrm{Ad}_g = L_g R_g$.

A vector field $X \in \mathrm{Vect}(G)$ is *left-invariant* if $g.X = X$ for every $g \in G$, and *right-invariant* if $X \cdot g = X$ for every $g \in G$. A vector field is called *bi-invariant* if it is both left and right-invariant.

Definition 3.1.3. A *Lie algebra* over a field $\mathbb{K}$ ($\mathbb{K}$=$\mathbb{R}$ or $\mathbb{C}$) is a vector space $\mathfrak{g}$ over $\mathbb{K}$ with a $\mathbb{K}$-bilinear map called Lie bracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is skew-symmetric: $[X, Y] = -[Y, X]$ and satisfies Jacobi identity

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

Let G be a real Lie group. Then $\mathfrak{g} = T_{\mathbb{I}}G$ has a canonical structure of a Lie algebra over $\mathbb{K}$ with the Lie bracket. We will denote this Lie algebra by $\mathrm{Lie}(G)$.

A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a *Lie subalgebra* if it is closed under the Lie bracket, i.e. for any $X, Y \in \mathfrak{h}$, $[X, Y] \in \mathfrak{h}$. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an *ideal* if for any $X \in \mathfrak{g}, Y \in \mathfrak{h}$, we have $[X, Y] \in \mathfrak{h}$.

For any real or complex finite-dimensional Lie algebra $\mathfrak{g}$, there is a unique (up to isomorphism) connected simply-connected Lie group G (respectively, real or complex) with $\mathrm{Lie}(G) = \mathfrak{g}$.

An interesting application of the correspondence between Lie groups and Lie algebras is the interplay between real and complex Lie algebras and groups.

3.1.1 Complex structure and complexification of a Lie algebra

Let $\mathfrak{g}$ be a complex Lie algebra, consider $\mathfrak{g}$ as a real vector space, the natural complex structure J defined by $Jv = iv$, satisfies

$$\forall v_1, v_2 \in \mathfrak{g} : [Jv_1, v_2] = J[v_1, v_2] = [v_1, Jv_2]$$

(This is equivalent to say the complex structure J commutes with the adjoint representation, $\forall v_1, v_2 \in \mathfrak{g} : J \circ ad_{v_1}(v_2) = ad_{v_1}(v_2) \circ J$, where $ad_{v_1}(v_2) = [v_1, v_2]$). Conversely, suppose $\mathfrak{g}$ is a real Lie algebra with a complex structure J on the underlying vector space, such that J commutes with the adjoint representation. Then we can get a complex Lie algebra by defining the scalar multiplication as follows:

$$(a + ib)v = av + bJv; a, b \in \mathbb{R}$$

and the Lie bracket would be complex bi-linear. The complex structure satisfying the above additional condition is called sometimes *ad-invariant* complex structure.

The complexification of $\mathfrak{g}$ is the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ such that $[v_1 \otimes a, v_2 \otimes b] = [v_1, v_2] \otimes ab; v_1, v_2 \in \mathfrak{g}, a, b \in \mathbb{C}$. This complexification yields a decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ into the $\mp i$ eigenspaces of the complex linear extension of J. The projection $\mathfrak{g} \to \mathfrak{g}^{1,0}$ is a canonical isomorphism of complex vector spaces. The exterior algebra of the dual vector space $\mathfrak{g}^*$decomposes as:

$$\Lambda^k \mathfrak{g}^* = \bigoplus_{p+q=k} \Lambda^p \mathfrak{g}^{*1,0} \otimes \Lambda^q \mathfrak{g}^{*0,1} = \bigoplus_{p+q=k} \Lambda^{p,q} \mathfrak{g}^*$$

and $\overline{\Lambda^{p,q}\mathfrak{g}^*} = \Lambda^{q,p}\mathfrak{g}^*$.

Definition 3.1.4. A *left invariant complex structure* on a real Lie group G is a complex structure on the underlying manifold such that left multiplication by elements of the group are holomorphic. Equivalently, there exists an endomorphism $J \in \mathrm{End}(\mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$, such that: $J^2 = -\mathbb{I}$ and J is integrable.

Definition 3.1.5. An almost complex structure J on a real Lie algebra $\mathfrak{g}$ is said to be *integrable* if the Nijenhuis tensor vanishes. i.e.,

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0, \forall X, Y \in \mathfrak{g}$$

The pair $(\mathfrak{g}, J)$ is called a *Lie algebra with complex structure.* Equivalently, J is integrable if and only if $\mathfrak{g}^{1,0}$ (and hence $\mathfrak{g}^{0,1}$) satisfying the decomposition in Example 3.1.1, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Remark 3.1.6. Let G be a real Lie group with Lie algebra $\mathfrak{g}$. Giving a left-invariant almost complex structure on G is equivalent to giving an almost complex structure J on $\mathfrak{g}$. In addition J is integrable if and only if it is integrable as an almost complex structure on G. It then induces a complex structure on G by the Newlander-Nirenberg theorem and G becomes a complex manifold and the elements of G act holomorphically by left multiplication. Whereas, G is not a complex Lie group in general.

Corollary 3.1.7. We can specify a complex structure on a given Lie algebra either by giving an endomorphism J on the basis of $\mathfrak{g}$ such that $J^2 = -\mathbb{I}$, or by giving a complex subalgebra $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{h} \cap \overline{\mathfrak{h}} = 0$ and $\mathfrak{h} \oplus \overline{\mathfrak{h}} = \mathfrak{g}^{\mathbb{C}}$.

3.1.2 Hermitian structures on Lie algebras

Let $\mathfrak{g}$ be a real Lie algebra with an ad-invariant[1] complex structure J on the underlying vector space. Let h be a Hermitian inner product on the underlying vector space with respect to J. We call the pair (J, h) a *Hermitian structure on* $\mathfrak{g}$. If h denotes also the extension to a complex symmetric bi-linear form on the underlying vector space of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$, then both $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are isotropic with respect to the extension h on $\mathfrak{g}^{\mathbb{C}}$ (since J is orthogonal). Moreover, these subalgebras are maximal isotropic since h is non-degenerate. Conversely, if B is a complex symmetric bilinear form on $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{q}$ is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$, then the pair $(\mathfrak{q}, B)$ gives rise to a Hermitian structure (J, h) on $\mathfrak{g}$ if $\mathfrak{q}$ is a maximal B-isotropic and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{q} \oplus \bar{\mathfrak{q}}$.

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[1]satisfying $[Jv_1, v_2] = J[v_1, v_2] = [v_1, Jv_2]\ \forall v_1, v_2 \in \mathfrak{g}$.

3.2 Hermitian structures on cotangent Lie groups

The cotangent bundle T^*G of a Lie group G with Lie algebra $\mathfrak{g}$ has a canonical Lie group structure induced by the coadjoint action of G on $\mathfrak{g}^*$. It has also a canonical ad-invariant metric h defined by

$$h\left((X,\alpha),(Y,\beta)\right) = \beta\left(X\right) + \alpha(Y),\ \ X, Y \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^*$$

Hermitian structures on T^*G with respect to h where left translations are holomorphic isometries, are complex structures on $\mathfrak{g} \oplus \mathfrak{g}^*$ orthogonal with respect to the pairing on it. These complex structures are integrable, i.e. $N_J = 0$ such that $J : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^*$. Note that N_J is defined with respect to the Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ to be defined in a following construction.

The action of G on itself $Lg : G \to G$ can be lifted to an action of G on TG given by $dLg : TG \to TG$. Then,

Definitions 3.2.1. A left invariant complex structure is an equivariant endomorphism of TG with respect to the lifted action of G given by left multiplication.

A *left invariant symplectic structure* on G is an equivariant isomorphism $w : TG \to T^*G$ where the action of G on T^*G is $L^*g^{-1} : T^*G \to T * G$.

A *left invariant Hermitian structure* on G is a pair (J, h) of a left invariant complex structure J together with a left invariant Hermitian metric h.

In this section we study a special type of left invariant Hermitian structures, those that defined on the cotangent bundle of Lie groups with respect to the canonical ad-invariant metric.

Construction 3.2.2. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{v}$ is a $\mathfrak{g}$-module. We have a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{v})$. The semidirect product $\mathfrak{g} \ltimes \rho\mathfrak{v}$ has a Lie bracket defined by

$$[(X,u),(Y,v)] = [X,Y] + \rho(X)v - \rho(Y)u, \ X,Y \in \mathfrak{g}, u,v \in \mathfrak{v}$$

In particular, we take $\mathfrak{v} = \mathfrak{g}^*$and $\rho = \mathrm{ad}^*$, the coadjoint representation

$$\begin{aligned} \mathrm{ad}^* : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}^*) \\ \mathrm{ad}^*(X)(\alpha) &\mapsto -\alpha \circ \mathrm{ad}(X), X \in \mathfrak{g}, \alpha \in \mathfrak{g}^* \end{aligned}$$

Let $\mathfrak{g} \ltimes \mathrm{ad}^*$ be denoted by $(T^*\mathfrak{g}, \mathrm{ad}^*)$, called *the cotangent algebra*, such that the Lie bracket is defined by

$$[(X,\alpha),(Y,\beta)] = [X,Y] - \beta \circ \mathrm{ad}(X) + \alpha \circ \mathrm{ad}(Y), \ X,Y \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^*$$

and the canonical non-degenerate symmetric ad-invariant bi-linear form h defined in the beginning of the section. The sub-algebra $\mathfrak{g}$ and the ideal $\mathfrak{g}^*$ are maximal isotropic in $(T^*\mathfrak{g}, h)$.

Left invariant Hermitian structures on cotangent Lie group T^*G are given by endomorphisms $J : T^*\mathfrak{g} \to T^*\mathfrak{g}$. With respect to the decomposition $\mathfrak{g} \oplus \mathfrak{g}^*$, J can be written in the following matrix

$$\begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$$

such that

1. J is integrable.

2. $J_4 = -J_1^*$, $J_2 = J_2^*$, $J_3 = J_3^*$.

3. $J_1^2 + J_2 J_3 = -\mathbb{I}$, $J_1 J_2 = -(J_1 J_2)^*$, $J_3 J_1 = (J_3 J_1)^*$,

Example 3.2.3. (Example 2.2 in (de Andrés *et al.*, 2007))

Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra with a complex structure J. Define $\mathfrak{J}_J$ on $T^*\mathfrak{g}$ by:

$$\mathfrak{J}_J(X,\alpha) = (J(X), J^*(\alpha)),\ X \in \mathfrak{g}, \alpha \in \mathfrak{g}^*,\ J^*(\alpha) = \alpha \circ J$$

In fact, $\mathfrak{J}_J$ is orthogonal with respect to the canonical pairing h on $T^*\mathfrak{g}$. In addition, the integrability of J make $\mathfrak{J}_J$ a complex structure on the cotangent algebra $(T^*\mathfrak{g}, \mathrm{ad}^*)$. Thus, $(\mathfrak{J}_J, h)$ is a Hermitian structure on it.

Example 3.2.4. (Example 2.3 in (de Andrés *et al.*, 2007))

Let $w : \mathfrak{g} \to \mathfrak{g}^*$ be a linear isomorphism. Let $\mathfrak{J}_w(X,\alpha) = (-w^{-1}(\alpha), w(X))$. In fact, $\mathfrak{J}_w$ is orthogonal with respect to the canonical pairing h if and only if w is skew-symmetric. In other words, w is symplectic on $\mathfrak{g}$. The integrability of $\mathfrak{J}_w$ is equivalent to

$$w([X,Y]) = w(X) \circ \mathrm{ad}(Y) - w(Y) \circ \mathrm{ad}(X)$$

Then, $\mathfrak{J}_w$ define a Hermitian structure on the cotangent algebra.

3.3 Left invariant generalized complex structures on Lie groups

According to the last section, Hermitian structures on T^*G with respect to h are complex structures on $\mathfrak{g} \oplus \mathfrak{g}^*$ orthogonal with respect to the pairing on it. We can relate these structures to generalized complex structures by observing that $\mathfrak{g} \oplus \mathfrak{g}^*$ is the fiber at the identity component of the bundle $TG \oplus T^*G$ over G. If we can extend J above to $TG \oplus T^*G$ by lifting left multiplication in G, the Courant bracket when restricted to left invariant vector fields and left invariant 1-forms is given by

$$[\![(X,\xi),(Y,\eta)]\!] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - d(\imath_X\eta - \imath_Y\xi)$$

On the Lie group G, consider the left action of on the generalized tangent bundle $TG \oplus T^*G$ induced by left multiplication of G on itself,

$$\begin{aligned} \lambda : G \times (TG \oplus T^*G) &\rightarrow G \oplus T^*G \\ (g, (X, \alpha)) &\mapsto \left((dL_g)_f X, \left(L^*_{g^{-1}}\right)_{gf} \alpha \right), \; X \in T^*_f G, g, f \in G \end{aligned}$$

where $(L^*_{g^{-1}})_{gf}\alpha(Y) = \alpha((dL_{g^{-1}})_{gf}Y)$, $\forall Y \in T_{gf}G$.

Definition 3.3.1. A generalized complex structure $\mathcal{J}$ on G is said to be *left invariant* if $\mathcal{J} : TG \oplus T^*G \rightarrow TG \oplus T^*G$ is equivariant with respect to the induced left action of G on$TG \oplus T^*G$. i.e., for any $g \in G$, the following diagram is commutative:

$$\begin{tikzcd} TgG \oplus T^*gG \arrow[r, "\mathcal{J}g"'] \arrow[d, "\lambda g^{-1}"] & TgG \oplus T^*gG \arrow[d, "\lambda g^{-1}"'] \\ \mathfrak{g} \oplus \mathfrak{g}^* \arrow[r, "\mathcal{J}e"] & \mathfrak{g} \oplus \mathfrak{g}^* \end{tikzcd}$$

Proposition 3.3.2. (Proposition 3.1 in (de Andrés *et al.*, 2007))

There is a one-to-one correspondence between left invariant generalized complex structures on G and Hermitian structures (J, h) on T^*G, where h is the canonical bi-invariant metric on T^*G.

Proof. By identifying the space of left invariant sections of $TG \oplus T^*G$ with $\mathfrak{g} \oplus \mathfrak{g}^*$, the restriction of Courant bracket to $\mathfrak{g} \oplus \mathfrak{g}^*$ is precisely the Lie bracket on the cotangent algebra $(T^*\mathfrak{g}, \mathrm{ad}^*)$. Concequently, Courant integrability condition of left invariant generalized complex structure $\mathcal{J}$ on G is equivalent to the integrability of Je on the cotangent algebra $(T^*\mathfrak{g}, \mathrm{ad}^*)$. Moreover, if $\mathcal{J}$ is a left invariant generalized complex structure on G, (J_e, h) is a Hermitian structure on $T^*\mathfrak{g}$, since $\mathcal{J}$ is orthogonal with respect to h if and only if J_e is invariant by h. (this is because

$\lambda g, g \in G$ isometries of $h\cdot$ on the generalized tangent bundle). Conversely, given a Hermitian structure (J, h) on $(T^*\mathfrak{g}, \mathrm{ad}^*)$, we can it extend it to a left invariant generalized complex structure $\mathcal{J}$ on G such that $\mathcal{J}e = J$. □

Remark 3.3.3. As shown in the examples: 3.2.3 and 3.2.4, If G has left invariant complex or symplectic structure, then each of them induces a left invariant generalized complex structure on G. Now, using the previous proposition, a Hermitian structure on the cotangent algebra with respect to h is a generalized complex structure on $\mathfrak{g}$ and denoted by $(\mathcal{J}, h)$.

3.4 Nilmanifolds

we begin by defining the object of our study, nilmanifolds.

Definition 3.4.1. A *nilmanifold* is a compact homogeneous space of the form N/Γ, where N is a simply connected nilpotent Lie group and Γ is a lattice in N (i.e. a discrete co-compact subgroup).

Definition 3.4.2. *A nilmanifolds with left-invariant complex structure* M_J is given by a triple $(\mathfrak{g}, J, \Gamma \subset G)$ where $\mathfrak{g}$ is a real nilpotent Lie algebra associated to a simply connected nilpotent Lie group G, J is an integrable complex structure on $\mathfrak{g}$ and Γ is a lattice (discrete cocompact subgroup). We will use the same letter $M_J = M$ for the compact complex manifold $G\backslash\Gamma$ endowed with the left-invariant complex structure induced by J.

We now address the question of the compatibility of the lattice $\Gamma \subset G$ with the other two structures $\mathfrak{g}$ and J. Most of the results originate from the work of (Malcev, 1951).

Definition 3.4.3. A *rational structure* of a nilpotent Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_{\mathbb{Q}}$ defined over the rationals such that $\mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R} = \mathfrak{g}$.

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be rational with respect to a given rational structure $\mathfrak{g}_{\mathbb{Q}}$ if $\mathfrak{g}_{\mathbb{Q}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$ is a rational structure for $\mathfrak{h}$. By a lattice in the Lie algebra $\mathfrak{g}$ we mean a lattice in the underlying vector space which is closed under bracket and we say that $\Gamma \subset G$ is induced by a lattice in $\mathfrak{g}$ if $\log \Gamma$ is a lattice in $\mathfrak{g}$.

Theorem 3.4.4. *(Malcev)*

There exists a lattice in a nilpotent simply connected Lie group G if and only if the corresponding Lie algebra admits a rational structure.

Example 3.4.5. Let $\mathfrak{h}_1$ be the $2(1)+1$-dimensional Heisenberg algebra, the Lie algebra with basis $\{X, Y, Z\}$ whose pairwise brackets are equal to zero except for $[X, Y] = Z$. $\mathfrak{h}_1$ has a rational structure determined by $\mathfrak{h}_{1,\mathbb{Q}} = \mathbb{Q}\text{−span}\{X, Y, Z\}$

The Hiesenberg group:

$$\mathrm{Heis}^3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} ; \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{R} \right\}$$

is a nilpotent Lie group (closed subgroup of $\mathrm{GL}(3, \mathbb{R})$). The corresponding lattice in $Heis^3$ is:

$$\Gamma = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in \mathbb{Z} \right\}$$

and Heis^3/Γ is a nilmanifold.

Consider a Lie algebra $\mathfrak{g}$ spanned by a basis $e_1, ..., e_n$. Then the Lie bracket is uniquely determined by structure constants a_{ij}^k such that $[ei, ej] = \sum_{k=1}^{n} a_{ij}^k e^k$ satisfying $a_{ij}^k = -a_{ij}^k$ (encoding the Jacobi identity). Let $\langle e^1, ..., e^n \rangle$ be the dual basis, i.e. $e^i(e_j) = \delta_{ij}$. Then for any $\alpha \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$ we define

$$\begin{aligned} d : \mathfrak{g}^* &\rightarrow \wedge^2\mathfrak{g}^* \\ d\alpha(X,Y) &:= -\alpha([X,Y]) \end{aligned}$$

and get a dual description of the Lie bracket by $de^k = -\sum a^k_{ij} e^{ij}$ where we abbreviate $e^i \wedge e^j = e^{ij}$. The map d induces a map on the exterior algebra $\Lambda^* g^*$ and $d^2 = 0$ is equivalent to the Jacobi identity.

Remark 3.4.6. We will use a notation like $\mathfrak{h}_2 = (0,0,0,0,12,34)$ by which we mean the following: Let $e_1, ..., e_6$ be the Malcev basis for the Lie algebra and $e^1, ..., e^6$ be the dual basis. Then, the defining relations for $\mathfrak{h}_2$ are given by $de^1 = de^2 = de^3 = de^4 = 0$, $de^5 = e^{12} = e^1 \wedge e^2$, $de^6 = e^{34}$. In other words the bracket relations are generated by $[e_1, e_2] = -e_5$ and $[e_3, e_4] = -e_6$. (The name $\mathfrak{h}_2$ is according to the classification of Salamon (Salamon, 2001)).

The descending central series of a Lie algebra $\mathfrak{g}$ is the chain of ideals defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]\ \forall i \geq 1$. Recall that, $\mathfrak{g}$ is *s-step nilpotent* if $\mathfrak{g}^s = 0$ and $\mathfrak{g}^{s-1} \neq 0$. We call the finite number s the *nilpotency index,* nil$(\mathfrak{g})$ (or the nilpotency index of the $\mathfrak{g}$-nilmanifold).

The nilpotency condition can be interpreted in terms of differential forms by defining subspaces $\{V_i\} \subset \mathfrak{g}^*$ inductively as follows:

$$V_0 = \{0\}, V_i = \left\{\alpha \in \mathfrak{g}^* : d\alpha \in \wedge^2 V_{i-1}\right\}; i \geq 1.$$

Each space V_i is the annihilator of $\mathfrak{g}_i$. We Choose a basis of V_1, then we extend it to a basis of $V_k; \forall k$. We obtain a Malcev basis $\{e^1, ..., e^n\}$ of $\mathfrak{g}^*$ such that $de^i \in \langle e^1, ..., e^{i-1}\rangle\ \forall i$.

We define the *nilpotent degree* of a p-form α, nil(α), to be the smallest i such that $\alpha \in \wedge^p V_i$. In addition, if α is a 1-form of nilpotent degree i then nil$(d\alpha) = i - 1$.

3.5 Generalized complex strustures on nilmanifolds

In this section, we present the two propositions proved in (Cavalcanti et Gualtieri, 2004): first, any left-invariant generalized complex structure on a nilmanifold is generalized Calabi-Yau, i.e. the canonical bundle[2] K_L has a closed trivialization. Next, the type[3] of a left-invariant generalized complex structure has an upper bound that depend only on the nilpotent structure.

Let M be a $2n$-nilmanifold, left-invariant generalized complex structures have constant type k throughout M, and its canonical bundle K_L is trivial. Thus, we can choose a global trivialization $\rho = e^{B+iw}\Omega$, where B and w are real left-invariant 2-forms and Ω is a globally decomposable complex k-form, $\Omega = \theta_1 \wedge ... \wedge \theta_k$. Let $X + \xi \in \Gamma(T \oplus T^*)$ be a left-invariant section such that the integrability condition is satisfied, $d\rho = (X + \xi) \cdot \rho$. If we order $\{\theta_1, ..., \theta_k\}$ according to nilpotent degree, then it is possible to choose them such that $\mathrm{nil}(i) \leq \mathrm{nil}(j)$, $i < j$, and such that $\{j : \mathrm{nil}(j) > i\}$ is linearly independent modulo V_i for each i.

Lemma 3.5.1. (Lemma 3.2. in (Cavalcanti et Gualtieri, 2004))

Let V be a subspace of a vector space W. Let $\alpha \in \wedge^p V$, and suppose $\{1, ..., m\} \subset W$ are linearly independent modulo V. Then $\alpha \wedge \theta_1 \wedge ... \wedge \theta_m = 0$ if and only if $\alpha = 0$.

Proof. Let $\pi : W \to W/V$ be the projection, and choose a splitting $W \cong V \oplus W/V$ such that $\alpha \wedge \theta_1 \wedge ... \wedge \theta_m$ has a component in $\wedge^p V \otimes \wedge^m (W/V)$ equal to $\alpha \wedge \pi(\theta_1) \wedge ... \wedge \pi(\theta_m) = 0$ which vanishes if and only if $\alpha = 0$. □

Proposition 3.5.2. (Theorem 3.1. in (Cavalcanti et Gualtieri, 2004))

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[2]cf. Definition 1.3.4.

[3]cf. Definition 1.3.4 and Remark 2.2.8.

Any left-invariant generalized complex structure on a nilmanifold must be generalized Calabi-Yau. That is, any left-invariant global trivialization of the canonical bundle must be a closed differential form. In particular, any left-invariant complex structure has holomorphically trivial canonical bundle.

Proof. (sketch)

Let $\rho = e^{B+iw}\Omega$ be the left-invariant trivializationof the canonical bundle such that its k- decomposables are ordered according to nilpotent degree as mentioned above. We use the integrability condition to deduce that the form is closed.

$$\begin{aligned} d\rho &= (X+\xi)\cdot\rho \\ d\left(e^{B+iw}\right) &= (X+\xi)\cdot\left(e^{B+iw}\right) \\ d(B+iw)\wedge\Omega + d\Omega &= (\imath_X(B+iw))\wedge\Omega + \imath_X\Omega + \xi\wedge\Omega \end{aligned}$$

Which can be devided into two parts: the $(k+1)$-degree part, $d\Omega = (\imath_X(B+iw))\wedge \Omega + \xi\wedge\Omega$. By wedging with θ_i and applying lemma 3.5.1, we conclude that

$$d\theta_i \wedge \theta_1 \wedge \dots \wedge \theta_j = 0, \;\; j < i, \forall i$$

hence, $d\Omega = 0$. For the $(k+3)$-degree part,

$$\begin{aligned} d(B+iw)\wedge\Omega &= 0 \\ e^{B+iw} d\Omega &= 0 \\ d\rho &= 0 \end{aligned}$$

□

Remark 3.5.3. Integrability condition imposes constrains on ρ as well as the non-degeneracy condition does on θ_i appearing in the decomposition of Ω . In other words, if $\theta_1, ..., \theta_j \in V_i$, then nondegeneracy implies that $\dim(V_i) \geq 2j$. For a fixed nilpotent algebra, this places an upper bound on the number of j which can be chosen from each V_i. Another strong constrain is imposed on the 1-forms θ_i. If $\dim V_{j+1}/V_j = 1$ occurs in a nilpotent Lie algebra, then either some i has nilpotent degree j, or no i has nilpotent degree $j+1$. To conclude, the size of the nilpotent steps $\dim V_{j+1}/V_j$ constrains the possible types of left-invariant generalized complex structures. This was proven in (Cavalcanti et Gualtieri, 2004) in the following proposition,

Proposition 3.5.4. (Theorem 3.2. in (Cavalcanti et Gualtieri, 2004))

Let M^{2n} be a nilmanifold with associated Lie algebra $\mathfrak{g}$. Suppose there exists a $j > 0$ such that, $\forall i \geq j$, $\dim(V_{i+1}/V_i) = 1$. Then M cannot admit left-invariant generalized complex structures of type k for $k \geq 2n - \mathrm{nil}(\mathfrak{g}) + j$. In particular, if M has maximal nilpotency index (i.e. $\dim V_1 = 2, \dim V_i/V_{i-1} = 1 \forall i > 1$), then it does not admit generalized complex structures of type k for$k \geq 2$.

3.6 Generalized complex structures on 6-nilmanifolds

The classification of 6-dimensional nilmanifolds has been done in (Goze et Khakimdjanov, 2010; Salamon, 2001), for those which admit left-invariant complex (type 3 GC)[4] and symplectic (type 0 GC)[5] structures. In (Cavalcanti et Gualtieri, 2004), they completed the study for generalized complex structures of types 1 and 2 as follows.

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[4]cf. Example 1.3.7

[5]cf. Example 1.3.6

3.6.1 Generalized complex structures of type 2

We have seen that a left-invariant structure of type 2 is given by a closed form $\rho = \exp(B+iw)\theta_1\theta_2$ such that $w \wedge \theta_1\theta_2\bar{\theta}_1\bar{\theta}_2 \neq 0$. Using Proposition 3.5.4, we deduce that any 6-nilmanifold with maximal nilpotence step cannot admit a structure of this type.

Example 3.6.1. (Lemma 4.1 in (Cavalcanti et Gualtieri, 2004))

A 6-nilmanifold that has nilpotent Lie algebra given by $(0, 0, 0, 12, 14, -)$, and has nilpotency index 4, does not admit left-invariant generalized complex structures of type 2.

Proof. Suppose the nilmanifold admits a structure of type 2. The nilpotency index is 4, then $\dim V_{i+1}/V_i = 1$, $i \geq 1$. Using $d\rho = 0$, we see that $\theta_1 = z_1e_1 + z_2e_2 + z_3e_3$ and $\mathrm{nil}(2) \geq 2$. Thus, $\theta_2 = w_1e_1 + w_2e_2 + w_3e_3 + w_4e_4$. The conditions $d(\theta_1\theta_2) = 0$ and $\theta_1\theta_2\overline{\theta_1\theta_2} \neq 0$ together gives $z_3 = 0$. Moreover, the annihilator of $\theta_1\theta_2\overline{\theta_1\theta_2}$ is generated by $\{e^5, e^6\}$. So the nondegeneracy condition $w_2 \wedge \Omega \wedge \Omega \neq 0$ implies that

$$B + iw = (k_1e_1 + ... + k_4e_4 + k_5e_5)e_6 + \alpha$$

where $k_5 \neq 0$ and $\alpha \in \wedge^2 \langle e_1, .., e_5 \rangle$. Using the structure constants, we see that d must have a nonzero multiple of e_6, and so it is not closed. □

In a similar way, it is shown in (Cavalcanti et Gualtieri, 2004) that:

Example 3.6.2. Nilmanifolds associated to the algebras defined by $(0, 0, 0, 12, 14, 13-24)$, $(0, 0, 0, 12, 14, 23+24)$ do not admit left-invariant generalized complex structures of type 2.

Example 3.6.3. Nilmanifolds associated to the algebras defined by $(0, 0, 12, 13, 23, 14)$, $(0, 0, 12, 13, 23, 14 - 25)$ do not admit left-invariant generalized complex structures of type 2.

Based on that, they deduced the following:

Corollary 3.6.4. (Theorem 4.1. in (Cavalcanti et Gualtieri, 2004))

The only 6-dimensional nilmanifolds not admitting left-invariant generalized complex structures of type 2 are those with maximal nilpotency index and those excluded by the previous three examples.

3.6.2 Generalized complex structures of type 1

A left-invariant structure of type 1 is given by a closed form $\rho = \exp(B + iw)\theta_1$ such that $w^2 \wedge \theta_1\bar{\theta}_1 \neq 0$. In other words, w is a symplectic form on the 4-dimensional leaves of the foliation determined by $\theta_1 \wedge \overline{\theta_1}$.

Corollary 3.6.5. (Theorem 4.2. in (Cavalcanti et Gualtieri, 2004))

The only 6-nilmanifolds which do not admit left-invariant generalized complex structures of type 1 are those associated to the algebras defined by
$(0, 0, 12, 13, 23, 14)$ and $(0, 0, 12, 13, 23, 14 - 25)$.

A complete table of classification's result can be found in (Cavalcanti et Gualtieri, 2004), in which they listed explicit examples of all types of left-invariant generalized complex structures whenever they exist.

According to the correspondence described in Section 3.3, and using the fact that every six dimensional nilpotent Lie group admits a left invariant generalized complex structure, we can deduce the following,

Corollary 3.6.6. The cotangent algebra $(T^*\mathfrak{g}, \mathrm{ad}^*)$ of any six dimensional nilpotent Lie algebra $\mathfrak{g}$ admits a Hermitian structure (J, h), where h is the canonical ad-invariant metric on $T^*\mathfrak{g}$.

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