UNIVERSITÉ DU QUÉBEC À MONTRÉAL

ON A CONJECTURE RELATING LEFT-ORDERABLE FUNDAMENTAL GROUPS, L-SPACES, AND CO-ORIENTED TAUT FOLIATIONS

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UNIVERSITÉ DU QUÉBEC À MONTRÉAL

SUR UNE CONJECTURE RELIANT LES GROUPES FONDAMENTAUX ORDONNABLES À GAUCHE, LES ESPACES L, ET LES FOLIATIONS TENDUES CO-ORIENTÉES

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RÉSUMÉ

Ce mémoire tentera de décrire la motivation d'une conjecture récemment formulée, que nous appellerons la *conjecture des espaces L (L-space conjecture)*, établissant un lien entre trois différentes méthodes d'étudier les trois-variétés. Énoncée de manière approximative, ladite conjecture prétend qu'il y a équivalence entre le fait d'imposer une certaine condition sur le groupe fondamental d'une trois-variété (le fait d'être *ordonnable à gauche*), la taille de son homologie de Heegaard-Floer, et l'existence de certains types de feuilletages. Après avoir introduit quelques notions élémentaires de la théorie des trois-variétés, nous donnerons un traitement approfondi du fait d'être ordonnable à gauche ainsi qu'un survol de l'homologie de Heegaard-Floer et de la théorie des foliations. Les deux derniers sujets seront exposés de manière plus sommaire, et aux seules fins de correctement énoncer la conjecture. En cours de route, nous étudierons également les variétés de Seifert en détail puisqu'elles fournissent une large famille de variétés pour lesquelles la conjecture a été démontrée. Nous présenterons ensuite une partie de la démonstration de la conjecture des espaces L dans le cas des variétés de Seifert, due à Boyer, Rolfsen, Wiest, Gordon, et Watson, suivie d'une application de ce résultat aux revêtements ramifiés des noeuds toriques dûe à Gordon et à Lidman. On terminera en énonçant la conjecture des espaces L, puis en donnant quelques résultats en faveur de sa validité ainsi que d'autres conjectures et problèmes y étant reliés.

mots-clé: topologie des trois-variétés, variété de Seifert, groupe fondamental ordonnable à gauche, foliation tendue co-orientée, espace L.

ABSTRACT

This memoir will attempt to describe the motivation for a recent conjecture, which we will call the *L-space conjecture*, relating three different methods of studying three-manifolds. Vaguely stated, the conjecture claims equivalence between imposing a certain condition (called *left-orderability*) on the fundamental group of a closed, connected, orientable three-manifold, the size of its Heegaard-Floer homology, and the existence of certain types of foliations it carries. After introducing some basic notions in three-manifold theory, we will give a thorough treatment of left-orderability as well as an overview of Heegaard-Floer homology and of foliation theory. The latter two subjects will be exposited more superficially, and only as far as is necessary to state the conjecture properly. Along the way, we will also study Seifert spaces in some detail, as they constitute a large family of manifolds for which the conjecture has been proven. We will then present a portion of the proof of the L-space conjecture for Seifert spaces due to Boyer, Rolfsen, Wiest, Gordon, and Watson, followed by an application of this result to the branched cyclic covers of torus knots due to Gordon and Lidman. We will end by stating the L-space conjecture, additional evidence in its favour, and other related conjectures and problems.

keywords: three-manifold topology, Seifert space, left-orderable fundamental group, co-oriented taut foliation, L-space.

CHAPTER 1

BASIC FACTS ABOUT MANIFOLDS

1.1. The fundamental group in dimension three

It is well known that the homeomorphism type of a closed, connected, orientable surface Σ is entirely determined by the rank of its first homology group, which is free abelian on $2g$ generators for some $g \geq 0$. This is also the minimal number of generators of its fundamental group, which can be presented as

$$\pi_1(\Sigma) = \langle l_1, ..., l_g, m_1, ..., m_g \mid [l_1, m_1][l_2, m_2]...[l_g, m_g] \rangle.$$

One can ask if the betti numbers of a closed, connected, orientable n-dimensional manifold determine its homeomorphism type in general. This question was posed and solved by Poincaré in his *Cinquième complément à l'Analysis Situs* in which he constructed what came to be known as the Poincaré homology sphere, a closed, connected, orientable three-dimensional manifold whose fundamental group is nontrivial, but perfect. By Poincaré duality, this space has the same betti numbers as S^3 without being homeomorphic to it. It is at the end of this paper that Poincaré formulated the following conjecture, which would remain unresolved for almost a century:

CONJECTURE (Poincaré). *If a closed, connected, three-dimensional manifold has trivial fundamental group, it is homeomorphic to S^3.*

Before discussing this conjecture, we will ask ourselves a more naive question, inspired by the classification of surfaces mentioned earlier:

Does the fundamental group distinguish between closed, connected, orientable n-dimensional manifolds?

This fails almost immediately: in dimension four, one need only consider the complex projective plane $\mathbb{CP}^2$, which is simply-connected but has nontrivial second betti number, and as such cannot be homeomorphic to S^4. But what about dimension three? There are certainly reasons to believe that the fundamental group captures a lot of topological information in this setting: for example, the homology groups of a closed, connected, orientable, three-dimensional manifold M are entirely determined by $\pi_1(M)$.

As it turns out, we can construct a family of three-dimensional manifolds which are not determined by their fundamental group, called lens spaces. The surprising fact is that the *only* ambiguities present in the fundamental group, with regards to classifying closed, connected, and orientable three-manifolds, come from this family of spaces and from questions of orientation. This will be made more precise later on; for the moment, we give a description of these spaces.

Let p be a nonzero integer. Writing S^3 as $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, we define a free action of $\mathbb{Z}_p$ on S^3 by taking some integer q coprime to p and setting

$$\overline{1} \cdot (z_1, z_2) = (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2).$$

The lens space $L(p, q)$ is defined as the quotient of S^3 by this $\mathbb{Z}_p$ action. Note that $L(p, q)$ is then universally covered by S^3 with $\mathbb{Z}_p$ acting by deck transformations, so that $\pi_1(L(p, q)) \cong \mathbb{Z}_p$. For the fundamental group to distinguish between these spaces, it would be necessary for the above construction to be independent of q. This is not the case: in fact, the complete classification of lens spaces is known up to homotopy and up to homeomorphism:

Theorem 1. *(Classification of three-dimensional lens spaces) The lens spaces $L(p, q)$ and $L(p, q')$ are*

- *Homotopy equivalent if and only if qq' is congruent to $\pm n^2$ modulo p for some integer n;*
- *Homeomorphic if and only if q' is congruent to $\pm q^{\pm 1}$ modulo p.*

Proof. For the homotopy classification, see theorem 10 of **[52]**. For the homeomorphism classification, see theorem 2.5 of **[22]**. □

Although the fundamental group fails to distinguish between lens spaces, this is the only family of "sufficiently simple" three-manifolds for which this sort of ambiguity occurs. In order to make this statement more precise, we introduce some key vocabulary and foundational results in the theory of three-manifolds.

A three-dimensional manifold M is called *irreducible* if any smooth submanifold of M homeomorphic to S^2 bounds a subset that is homeomorphic to the closed ball in $\mathbb{R}^3$. A three-dimensional manifold M is called *prime* if it is not S^3 and if whenever M can be expressed as a connected sum $N_1 \# N_2$, one of the manifolds N_1 or N_2 must be S^3. In general, an irreducible three-manifold which is not S^3 is always prime. The converse is true for all but two prime three-manifolds: the two S^2-bundles over S^1 are the only prime three-manifolds that are not irreducible (for a proof, see lemma 3.13 of [**24**]). The importance of these concepts comes from the following result:

Theorem 2. *(Prime decomposition theorem) Let M be a closed, connected, orientable three-manifold. Then there exists a finite set $\{P_1, \dots P_n\}$ of prime manifolds such that*

$$M \cong P_1 \# \dots \# P_n.$$

Furthermore, the components in this decomposition are uniquely determined up to homeomorphism, so that this decomposition is unique up to permutation of the summands.

Proof. See theorem 1.5 of [**22**], as well as [**32**]. □

Because the fundamental group of a three-manifold behaves well with respect to the connected sum operation (the latter induces a free product on the level of the former), this result allows us to restrict our focus to prime manifolds. In fact, we may further restrict to irreducible manifolds because we know precisely which prime three-manifolds are not irreducible.

As it turns out, this notion of irreducibility (coupled with connectedness, closedness and orientability) is restrictive enough to allow the fundamental group to function as a near-complete invariant. More precisely, the following holds: if M is a closed, connected, orientable, irreducible three-manifold, then it is uniquely determined by its fundamental group, unless it is a lens space. We shall see why this is the case in the next section; for the moment, we introduce one more definition.

Definition 3. Let Σ be a compact surface properly embedded in some three-manifold M. If there exists a disk D in M such that $D \cap \Sigma = \partial D$, and such that ∂D does not bound a disk in Σ, D is called a *compression disk* for Σ and Σ is said to be *compressible*. If there exists no such disk, and Σ is not homeomorphic to S^2 nor to D^2, Σ is said to be *incompressible*. A closed, orientable three-manifold is said to be *Haken* if it contains an orientable, incompressible surface.

1.2. The Geometrization conjecture

In the context of three-manifold theory, a *geometry* is defined to be a simply connected manifold Y equipped with a Riemannian metric for which it is complete and homogeneous. A manifold M is said to have *geometric structure modelled on* Y if M is homeomorphic to Y/Γ where Γ is a subgroup of $Isom(Y)$ acting freely on Y. In 1982, William Thurston formulated the geometrization conjecture, describing a decomposition of three-manifolds into pieces admitting a geometric structure based on one of the eight three-dimensional geometries. We will not describe these notions in detail; we refer the reader to **[45]** for an exposition of the notions involved in Thurston's conjecture. Because we will later refer to them by name, we list the eight geometries here:

- Spherical;
- Euclidean;
- Hyperbolic;
- $S^2 \times \mathbb{R}$;
- $\mathbb{H}^2 \times \mathbb{R}$;
- $\widetilde{SL_2(\mathbb{R})}$;
- Nil;
- Solv.

The geometrization conjecture has since been proved; see **[1]** for a survey of the implications of this result. We will content ourselves with giving a short rundown of the power conferred to the fundamental group in dimension three by the geometrization theorem:

THEOREM 4. *(Geometrization) Let M be a closed, orientable, irreducible three-manifold. Then there exists a (possibly empty) finite set $\{T_1, \dots T_n\}$ of pairwise disjoint incompressible tori in M such that the components of $M - (T_1 \cup \dots \cup T_n)$ each admit a finite volume geometric structure based on one of the eight three-dimensional geometries.*

If the set $\{T_1, \dots T_n\}$ is nonempty, then M is Haken by definition. This case is settled by the following result:

THEOREM 5. **[51]** *Let M and N be closed, orientable, irreducible three-manifolds with isomorphic fundamental groups. If M is Haken, then M and N are homeomorphic.*

If the set $\{T_1, \dots T_n\}$ is empty, then M itself admits a geometric structure and has finite volume. Thus the problem reduces to determining whether the fundamental group distinguishes between the manifolds admitting such geometric structures. The case of spherical manifolds is dealt with by the Elliptization theorem:

THEOREM 6. *(Elliptization) A closed, orientable, irreducible three-manifold has finite fundamental group if and only if it is spherical.*

Within the class of closed, oriented, irreducible spherical manifolds, it is known that the fundamental group is a complete invariant except when it comes to lens spaces (see the comment on page 113 of **[39]**). Furthermore, this theorem implies the Poincare conjecture, for it is known that a spherical manifold admits S^3 as its universal cover. If its fundamental group is trivial, it must therefore be homeomorphic to S^3. We are thus left with manifolds with infinite fundamental group. If a closed, oriented, irreducible three-manifold is neither hyperbolic nor Solv, but belongs to one of the other six geometries, it is in fact Seifert-fibred (we will define this notion in chapter 5). This case is covered by the following rigidity theorem:

THEOREM 7. **[46]** *Let M be a closed, orientable, irreducible Seifert fibred space with infinite fundamental group, and let N be a closed, orientable, irreducible 3-manifold such that $\pi_1(M)$ is isomorphic to $\pi_1(N)$. Then M is homeomorphic to N.*

We are left with closed, orientable, irreducible Solv and hyperbolic manifolds. The former have been studied directly, and classified, and it turns out that the fundamental group is a complete invariant (see page 244 of **[2]**). For the hyperbolic case, we have the Mostow rigidity theorem:

THEOREM 8. **[35]** *Let M and N be closed, orientable, irreducible three-manifolds with isomorphic fundamental groups. If M is hyperbolic, then M and N are homeomorphic.*

This concludes the (quasi) classification of closed, oriented, irreducible three manifolds via their fundamental group, and the prime decomposition theorem allows us to summarize these results as follows:

THEOREM 9. *Let M and M' be closed connected, oriented, three-dimensional manifolds with isomorphic fundamental groups. There exist integers $p_1, \dots p_k$, $q_1, \dots q_k$, $q'_1, \dots q'_k$, as well as orientable prime manifolds $N_1, \dots N_m$, such that*

- *None of the N_j are lens spaces;*
- *q_i and q'_i are coprime to p_i for every i in $\{1, \dots, k\}$;*
- *$M \cong N_1\# \dots \#N_m\#L(p_1, q_1)\# \dots \#L(p_k, q_k)$;*
- *$M' \cong N_1\# \dots \#N_m\#L(p_1, q'_1)\# \dots \#L(p_k, q'_k)$.*

It is worth noting that even when $q_i = q'_i$ for all i, M and M' are not necessarily homeomorphic, because the N_j or the $L(p_i, q_i)$ may have opposite orientations, and the connected sum operation depends on this choice of orientation. Thus closed, oriented three-manifolds are uniquely determined by their fundamental group up to lens spaces and orientation of their prime factors.

Now that we are thoroughly convinced that the fundamental group is a powerful invariant of closed three-manifolds, we can impose certain restrictions on it and try to understand the resulting restrictions on three-manifolds.

1.3. Miscellaneous facts

We gather here some general facts about three-manifolds which will be useful later.

LEMMA 10. *Let M be a compact, connected three-manifold such that either*

- *M is closed and non-orientable; or*
- *∂M is nonempty but contains no S^2 or P^2 components.*

Then $b_1(M) > 0$.

PROOF. Let W denote the manifold obtained by gluing two copies of M along their common boundary (if M is closed, this is just two disjoint copies of M). Then W is a closed 3-manifold, and as such has Euler characteristic 0 by Poincaré duality (if M is non-orientable, we need to consider coefficients in $\mathbb{Z}_2$ instead; see corollary 3.37 of [**23**]).

Appyling the Mayer-Vietoris sequence to W decomposed as two copies of M and a collar neighbourhood of ∂M, we get

$$0 = \chi(W) = 2\chi(M) - \chi(\partial M),$$

so that $2\chi(M) = \chi(\partial M)$.

On the other hand, if M is non-orientable or has nonempty boundary, we have $b_3(M) = 0$. By hypothesis, each component of ∂M has non-positive Euler characteristic. Thus

$$0 \geq \frac{1}{2}\partial M = \chi(M) = 1 - b_1(M) + b_2(M),$$

so that $b_1(M) \geq 1 + b_2(M) \geq 1$. □

DEFINITION 11. A surface S in a three-manifold M is called *two-sided* if there is an embedding

$$h : S \times [-1, 1] \to M$$

such that $h(x, 0) = x$ for all $x \in S$ and such that $h(S \times [-1, 1]) \cap \partial M = h(\partial S \times [-1, 1])$. A three-manifold M is called *P^2-irreducible* if it is irreducible and contains no two-sided P^2.

LEMMA 12. *Let M be compact and P^2-irreducible with $b_1(M) = 0$. Then M is a 3-ball or is closed and orientable.*

PROOF. Suppose M is not a 3-ball. We proceed by contradiction: suppose ∂M is nonempty. Because M is P^2-irreducible and is not a 3-ball, its boundary cannot contain any S^2 or P^2 components. Indeed, if ∂M contained a 2-sphere, said 2-sphere would bound a 3-ball B. But B would be both closed in M as it is compact, and open in M by invariance of domain. Thus M would be a 3-ball, which is contrary to our hypotheses. Thus by lemma 10, we have $b_1(M) > 0$. Therefore M must be closed. If it were non-orientable, lemma 10 would again give a contradiction. □

Theorem 13. *Let M be an irreducible three-manifold. Then M contains a two-sided P^2 if and only if it is non-orientable and its fundamental group contains an element of order two.*

Proof. See theorem 8.2 of [**17**]. □

A *knot* is a smooth embedding $K : S^1 \to S^3$. Its *complement* is defined to be the set $S^3 - K(S^1)$.

Given a knot K, let ν be an open tubular neighbourhood of $K(S^1)$ and set $X = S^3 - \nu$. Note that because ν deformation retracts onto $K(S^1)$, the homology of X is the same as the homology of the complement of K. Let $\tilde{X}$ be the covering of X associated to the kernel of the Hurewicz homomorphism

$$h : \pi_1(X) \to H_1(X).$$

By Alexander duality, we have $H_1(X) \cong \mathbb{Z}$. For this reason, we shall call $\tilde{X}$ the *infinite cyclic covering* of X. The *n-fold cyclic cover* of X will be the covering associated to the kernel of the map

$$\rho \circ h : \pi_1(X) \to H_1(X) \to \mathbb{Z}_n,$$

where ρ denotes the canonical projection of $\mathbb{Z}$ onto $\mathbb{Z}_n$ (we may identify $H_1(X)$ with $\mathbb{Z}$ by fixing an orientation for K).

Let K be a knot in S^3, let ν be a closed tubular neighbourhood of $K(S^1)$, and fix a meridian m on $\partial\nu$. Fix an integer n and let X_n denote the n-fold cyclic cover of $S^3 - \mathrm{int}(\nu)$. The preimage T of $\partial\nu$ in X_n via the projection map is a torus and the preimage of m is a loop $\tilde{m}$ on T. Let $\mathbb{T}$ be a solid torus, and fix a meridian c on $\partial\mathbb{T}$. The space obtained by gluing $\mathbb{T}$ to X_n along their respective boundaries $\partial\mathbb{T}$ and T such that c is identified with $\tilde{m}$ is denoted $\Sigma_n K$. The covering map on X_n can be extended to a map $\Sigma_n K \to S^3$ which is an n-fold covering map outside of the preimage of K. The space $\Sigma_n K$ along with this projection is called the *n-fold branched cyclic cover of S^3 ramified over K,* or the *n-fold branched cyclic cover of K* for short.

We will be interested in a specific family of knots, the *torus knots.* They can be defined as follows: let p and q be coprime integers, and let T be an unknotted solid torus in $\mathbb{R}^3 \subset S^3$. Fixing a meridian m and a 0-framed longitude l on ∂T, a *(p, q) torus knot,* denoted $T_{p,q}$, is defined to be a simple closed curve representing the homology class $q[m] + p[l]$. We now state some facts, without proof, about torus knots and the fundamental groups of their branched cyclic covers.

Let n, p, q be integers ≥ 2 , and set $\Gamma(n, p, q) = \langle x, y, z | x^n = y^p = z^q = xyz \rangle$ and $\Delta(n, p, q) = \langle x, y, z | x^n = y^p = z^q = xyz = 1 \rangle$

Proposition 14. *$\Gamma(n, p, q)$ is finite if and only if $\Delta(n, p, q)$ is finite, that is, if and only if $\frac{1}{n} + \frac{1}{p} + \frac{1}{q} > 1$.*

Proof. See page 68 of [**12**]. □

Proposition 15. *$[\Gamma(n, p, q), \Gamma(n, p, q)]$ has finite index in $\Gamma(n, p, q)$ if and only if $\frac{1}{n} + \frac{1}{p} + \frac{1}{q} \neq 1$. In particular, if $\frac{1}{n} + \frac{1}{p} + \frac{1}{q} \neq 1$, then $[\Gamma(n, p, q), \Gamma(n, p, q)]$ is finite if and only if $\Gamma(n, p, q)$ is finite.*

Proof. This follows by computing the order of $Ab(\Gamma(n, p, q))$. Rewriting $\Gamma(n, p, q)$ as

$$\Gamma(n, p, q) = \langle x, y, z | xyzx^{-n} = xyzy^{-p} = xyzz^{-q} = 1 \rangle$$

expresses $Ab(\Gamma(n, p, q))$ as the free abelian group on three generators quotiented by the relations given by the matrix

$$A = \begin{bmatrix} 1-n & 1 & 1 \\ 1 & 1-p & 1 \\ 1 & 1 & 1-q \end{bmatrix}$$

and whose order is infinite if $\det A$ is zero, and equal to $|\det A|$ otherwise. We have

$$\det A = np + pq + pn - pqn = npq(\frac{1}{n} + \frac{1}{p} + \frac{1}{q} - 1),$$

proving our claim. □

Proposition 16. *If $\frac{1}{n} + \frac{1}{p} + \frac{1}{q} = 1$, then $[\Gamma(n, p, q), \Gamma(n, p, q)]$ is infinite.*

Proof. Suppose $\frac{1}{n} + \frac{1}{p} + \frac{1}{q} = 1$. Because $\Delta(n, p, q)$ arises as a quotient of $\Gamma(n, p, q)$, it suffices to show that $[\Delta(n, p, q), \Delta(n, p, q)]$ is infinite. Note that by proposition 14, $\Delta(n, p, q)$ is infinite. If it were the case that $[\Delta(n, p, q), \Delta(n, p, q)]$ is finite, then the quotient $\Delta(n, p, q)/[\Delta(n, p, q), \Delta(n, p, q)]$ would have to be infinite. However up to permutation, the only possible values for the triplet (n, p, q) are $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$. Abelianizing the presentation

$$\Delta(n, p, q) = \langle x, y, z | x^n = y^p = z^q = xyz = 1 \rangle$$

given above for these choices of triplets yields

- $Ab(\Delta(2,3,6)) = \mathbb{Z}_6$;
- $Ab(\Delta(2,4,4)) = \mathbb{Z}_2 \oplus \mathbb{Z}_4$;
- $Ab(\Delta(3,3,3)) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Thus $[\Delta(n,p,q), \Delta(n,p,q)]$ is infinite, so that $[\Gamma(n,p,q), \Gamma(n,p,q)]$ is infinite as well. □

Definition 17. The (n,p,q) Brieskorn manifold is the compact 3-manifold given as a subset of $\mathbb{C}^3$ by

$$M(n,p,q) := \{z_1^n + z_2^p + z_3^q = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$$

Theorem 18. $\pi_1(M(n,p,q)) = [\Gamma(n,p,q), \Gamma(n,p,q)]$

Proof. This is the main result of **[33]**. □

Theorem 19. *$M(n,p,q)$ is homeomorphic to $\Sigma_n(T_{p,q})$.*

Proof. See lemma 1.1 of **[33]**. □

Corollary 20. *The spaces $\Sigma_n(T_{p,q})$ have finite fundamental group in exactly the following cases:*

(1) $\{n,p,q\} = \{2,2,n\}$ with $n \geq 2$;
(2) $\{n,p,q\} = \{2,3,3\}$;
(3) $\{n,p,q\} = \{2,3,4\}$;
(4) $\{n,p,q\} = \{2,3,5\}$.

Proof. This follows from the results listed above, combined with the observation that the only unordered triples $\{n,p,q\}$ such that $\frac{1}{n} + \frac{1}{p} + \frac{1}{q} > 1$ are those given in the corollary statement. □

CHAPTER 2

LEFT ORDERABILITY

2.1. Algebraic facts

Definition 21. A group is called *left-orderable* if it is nontrivial and admits a strict total ordering $<$ which is invariant by multiplication on the left, that is, such that for every $f, g, h \in G$,

$$f < g \implies hf < hg.$$

We shall call such an ordering a *left-ordering* on G. Note that $g > 1$ if and only if $g^{-1} < 1$.

Corollary 22. *Suppose G is left-orderable. Then G is nontrivial and torsion-free. In particular, G is infinite.*

Proof. That G is nontrivial follows immediately from the definition. To see that G is torsion-free, we proceed by contradiction, and suppose that G contains an element g different from the identity such that $g^n = 1$ for some $n \in \mathbb{N}$. By taking g^{-1} if necessary, we can assume without loss of generality that $1 < g$. Successively multiplying both sides of this inequality by g yields the relation

$$1 < g < \ldots < g^{n-1} < g^n = 1,$$

which is absurd. □

A left-ordering on a group is entirely determined by its *positive cone*

$$P = \{g \in G \mid 1 < g\}$$

. Elements of P will be called *positive.* Given a subset E of a group G, we will write

$$E^{-1} = \{g^{-1} \mid g \in E\}.$$

That is, E^{-1} is the set of inverses of elements of E. The proof of the following proposition is direct:

Proposition 23. *Let G be a group equipped with a left-ordering. Its positive cone P satisfies the following conditions:*

(1) *P is a subsemigroup of G;*
(2) *$G = \{1\} \sqcup P \sqcup P^{-1}$.*

Conversely, any group G containing a subset P satisfying the two above conditions can be equipped with a left-ordering by setting

$$g < h \Longleftrightarrow g^{-1}h \in P.$$

In light of the above proposition, given a group G, we will call any subset of G satisfying conditions 1 and 2 a *positive cone* by abuse of language. Note that a positive cone P of G satisfies the following properties:

(1) P is a subsemigroup of G not containing 1;
(2) If $fg \in P$, then $f \in P$ or $g \in P$,

but these properties do not characterize positive cones of G. We will call a subset of G satisfying the two above properties a *subpositive cone* of G. Note that a subpositive cone is a positive cone if it satisfies the following third property:

(3) For every $g \in G - \{1\}$, either g or g^{-1} belongs to G.

We will need the following result on subpositive cones in the following:

Lemma 24. *If P is a subpositive cone of G, then $Q = (P^{-1})^c$ is a subsemigroup of G.*

Proof. Suppose f and g belong to Q. Then f^{-1} and g^{-1} do not belong to P, so that $g^{-1}f^{-1}$ does not belong to P either as P is a subpositive cone. Thus fg belongs to Q. □

Given a finite family $E_1, \dots, E_n$ of subsets of a group G, we let $S(E_1, \dots, E_n)$ denote the semigroup generated by the E_i, that is, the set of nonempty products of elements of the E_i. In the event where some of the E_i are singletons, we will drop the brackets from the semigroup notation; i.e. we will write $S(g, h)$ instead of $S(\{g\}, \{h\})$. Given a finite subset $\{g_1, \dots, g_n\}$ of G and an arbitrary subset X of G, we may form 2^n potentially distinct semigroups as follows: for each set of

$\epsilon_i = \pm 1$, $1 \leq i \leq n$, form the semigroup $S(X, g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$. The intersection of all 2^n semigroups of this form will be denoted $I(X, g_1, \ldots, g_n)$.

Following **[38]**, we will shortly prove a useful criterion for determining when a group is left-orderable, but for the moment we need a few lemmas concerning positive cones and subpositive cones.

LEMMA 25. *Let G be a group satisfying the following property, which we will denote (†): for any finite subset $\{g_1, \ldots, g_n\}$ of G, we have $I(1, g_1, \ldots, g_n) = 1$. Then for every $g \in G - \{1\}$, there exists a subpositive cone C_g containing g.*

PROOF. Fix a $g \in G - \{1\}$, and consider the family $\mathcal{F}$ of subsets X of G such that

(1) X is a subsemigroup of G containing 1;
(2) for any finite subset $\{g_1, \ldots, g_n\}$ of G, we have $g \notin I(X, g_1, \ldots, g_n)$.

This family is nonempty: by hypothesis, it contains $\{1\}$. Let C be an ascending chain in $\mathcal{F}$, and consider the set $M = \bigcup_{X \in C} X$, which is clearly an upper bound for C with regards to set inclusion. We claim that M belongs to $\mathcal{F}$. To see this, first note that a union of nested subsemigroups of G containing 1 is also a subsemigroup of G containing 1, so that condition 1 is verified for M.

To see that M also satisfies condition 2, let $\{g_1, \ldots, g_n\}$ be a finite subset of G, and suppose for the sake of contradiction that g belongs to $I(M, g_1, \ldots, g_n)$. This yields 2^n ways of expressing g in terms of elements of M and of the $g_i^{\pm 1}$. Only a finite number of elements of M are required to form these 2^n words; therefore there is some element X' of C containing all of said elements. But this means that g belongs to $I(X', g_1, \ldots, g_n)$, contradicting the fact that X' is in $\mathcal{F}$. Thus g does not belong to $I(M, g_1, \ldots, g_n)$, so that M is in $\mathcal{F}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element X_g. We claim that for any $f \in G$, X_g contains either f or f^{-1}.

Indeed, if this claim were false for some element f of $G - \{1\}$ (note that X_g contains 1 by construction), then the semigroups $S(X_g, f)$ and $S(X_g, f^{-1})$ would both properly contain X_g. By maximality of X_g, condition 2 must fail for both of these semigroups, so that there exist finite subsets $\{h_1, \ldots, h_k\}$ and $\{h_1', \ldots, h_l'\}$ of G such that

- $g \in I(S(X_g, f), h_1, \ldots, h_k)$;
- $g \in I(S(X_g, f^{-1}), h_1', \ldots, h_l')$.

Therefore given any set of $\epsilon_i = \pm 1$, $1 \le i \le n$, we may write g

- as a product of f and of the $h_i^{\epsilon_i}$, and
- as a product of f^{-1} and of the $h_i'^{\epsilon_i}$.

In other words, $I(X_g, f, h_1, \ldots, h_k, h_1', \ldots, h_l')$ contains g, contradicting condition 2 for X_g. Thus X_g must contain f or f^{-1} as claimed.

Note that X_g cannot contain g: this follows by taking $\{g_1, \ldots, g_n\} = \emptyset$ in condition 2. We claim that $C_g = G - X_g$ is the desired subpositive cone: first, we have already seen that C_g contains g and does not contain 1. To see that C_g is a subsemigroup of G, let f and h belong to C_g. Then f^{-1} and h^{-1} both belong to X_g, so that if fh also belongs to X_g, we must have $f \in X_g$ and $h \in X_g$, which is absurd. To verify the final property of subpositive cones, suppose $fh \in C_g$. If both f and h belong to X_g, then fh i also belongs to X_g, which is absurd. Thus either f or h must belong to C_g.

Note that C_g is only subpositive in general, as X_g may contain both f and f^{-1} for some $f \in G$. □

Lemma 26. *Let G be a group satisfying property* (†). *Then G is left-orderable.*

Proof. We consider the family $\mathcal{F}$ of subpositive cones X of G. Given an ascending chain C in $\mathcal{F}$, the union M of all its elements is again a subpositive cone of G: indeed, M cannot contain 1 as none of the elements of C does; and if $fg \in M$, then fg belongs to some element X' of C, so that either f or g belongs to X' (and therefore to M). Thus $\mathcal{F}$ has a maximal element P by Zorn's lemma. We claim that P is in fact a positive cone of G. To see this, we must show that for any $g \in G - \{1\}$, we have either $g \in P$ or $g^{-1} \in P$.

We proceed by contradiction: suppose there exists a $g \in G - \{1\}$ such that neither g nor g^{-1} is contained in P. By lemma 25, we may find a subpositive cone P_g containing g. Note that the set $P_g \cap (P^{-1})^c$ is a subsemigroup by lemma 24. We claim that the set $C = P \cup (P_g \cap (P^{-1})^c)$ is a subpositive cone properly containing P.

First, C clearly contains P and does not contain 1. To see that C is a subsemigroup of G, we consider elements $f \in P$ and $h \in P_g \cap (P^{-1})^c$ (all other cases being direct). Suppose $fh \notin P$. Then $h^{-1}f^{-1} \notin P^{-1}$, so that $h^{-1}f^{-1} \in (P^{-1})^c$. But this means $f^{-1} \in (P^{-1})^c$, as $(P^{-1})^c$ is a subsemigroup of G. This in turn implies that $f^{-1} \notin P^{-1}$, which would mean $f \notin P$, which is absurd.

To verify that C satisfies the final property of subpositive cones, consider elements f and h of G such that $fh \in C$. Suppose f is not in C. Then f belongs neither to P nor to $P_g \cap (P^{-1})^c$. If fh belongs to P, we are done as P is a subpositive cone. We may therefore suppose that $fh \notin P$. Summarizing the situation, we have

(1) $f^{-1} \in (P^{-1})^c$ as $f \notin P$;
(2) $fh \in P_g \cap (P^{-1})^c$ as $fh \in C$ and $fh \notin P$;
(3) $h^{-1}f^{-1} \in (P^{-1})^c$ as $fh \notin P$.

Combining (1) and (2), we see that $h \in (P^{-1})^c$. It remains to see that $h \in P_g$. Because P_g is a subpositive cone, by (2) it suffices to show that $f \notin P_g$. This is necessarily the case if $f \in (P^{-1})^c$, because we supposed $f \notin C$. On the other hand, if $f \notin (P^{-1})^c$, then $f^{-1} \in P$, so that we directly have $h \in C$ by (2).

We have thus shown that C is a subpositive cone containing P. Furthermore, as P does not contain g^{-1}, C must contain g, so that C properly contains P (as P does not contain g either), contradicting the maximality of P.

We have thus proved that P is a positive cone for G, so that G admits a left-ordering by the second part of proposition 23. □

Proposition 27. *(Finite subset criterion for left-orderability) A group G is left-orderable if and only if for every finite subset $\{g_1, \ldots, g_n\}$ of G which does not contain the identity element of G, there exists a set of $\epsilon_i = \pm 1$, $i = 1, \ldots, n$, such that $S(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$ does not contain the identity element of G.*

Proof. Suppose G is a left-orderable group, and let $\{g_1, \ldots, g_n\}$ be a finite subset of G which does not contain the identity. For each g_i, take ϵ_i such that $g_i^{\epsilon_i} > 1$. Then every element of $S(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$ is strictly positive.

Conversely, suppose that G is a group satisfying the finite subset criterion. We claim that G satisfies property (†). To see this, we proceed by contradiction and suppose we have a finite subset $\{g_1, \ldots, g_n\}$ of G such that the intersection $I(g_1, \ldots, g_n)$ contains a nontrivial element g. We may assume without loss of generality that $\{g_1, \ldots, g_n\}$ does not contain the identity element of G. For each choice of $\epsilon_i = \pm 1$, $1 \leq i \leq n$, we therefore have the following:

- g is a word in the $g_i^{\epsilon_i}$;
- g is a word in the $g_i^{-\epsilon_i}$.

Thus for each choice of $\epsilon_i = \pm 1$, both g and g^{-1} belong to $S(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$, so that each of the 2^{n+1} semigroups $S(g^{\pm 1}, g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$ contains the identity element, contradicting the fact that G satisfies the finite subset criterion of the hypothesis. We conclude that G must satisfy property (†), so that by lemma 26, G is left-orderable. □

THEOREM 28. *(Burns-Hale) A group G is left-orderable if and only if for every nontrivial finitely generated subgroup H of G, there exists a left-orderable group L and an epimorphism $H \to L$.*

PROOF. Suppose G is left-orderable. Then every nontrivial sugroup H of G is also left-orderable, and the identity map from H to itself suffices.

Conversely, suppose G is a group satisfying the condition in the theorem's statement. We will prove that G satisfies the finite subset criterion. We proceed by induction on the cardinality of a given finite subset $\{g_1, \ldots, g_n\}$ of G not containing the identity.

For $n = 1$, suppose that $S(g_1)$ contains the identity. Then g_1 must have finite order; but this contradicts the hypothesis that the finitely generated group $\langle g_1 \rangle$ surjects onto a left-orderable (hence infinite) group.

We now suppose that the finite subset criterion is satisfied for all subsets of G of cardinality less than or equal to n and not containing the identity, and we consider a subset $\{g_1, \ldots, g_{n+1}\}$ of G, also not containing the identity. By hypothesis, there exists a left-orderable group L and a nontrivial homomorphism

$$\phi : \langle g_1, \ldots, g_{n+1} \rangle \to L.$$

We may reindex the g_i so that $\{g_1, \ldots, g_{n+1}\}$ is partitioned into the sets

- $A = \{g_1, \ldots, g_k\}$, all of whose elements have nontrivial image under ϕ; and
- $B = \{g_{k+1}, \ldots, g_{n+1}\}$, all of whose elements are mapped to 1 by ϕ.

Note that because ϕ is nontrivial, A is nonempty, so that B has cardinality less than or equal to n. By our induction hypothesis, we may take a set of $\epsilon_i = \pm 1$, $i = k+1, \dots, n+1$, such that $S(g_{k+1}^{\epsilon_{k+1}}, \dots, g_{n+1}^{\epsilon_{n+1}})$ does not contain the identity element of G. Furthermore, because no element of A is mapped to the identity element of the left-orderable group L, we may take a set of $\epsilon_i = \pm 1$, $i = 1, \dots, k$, such that $h(g_i)^{\epsilon_i} > 1$ for each such i.

We claim that $S(g_1^{\epsilon_1}, \dots, g_{n+1}^{\epsilon_{n+1}})$ does not contain the identity element 1 of G. For if it did, we could express 1 as a word ω in the $g_i{}^{\epsilon_i}$. Such a word would have to contain a power of $g_j{}^{\epsilon_j}$ for some g_j in the set A, as $S(g_{k+1}^{\epsilon_{k+1}}, \dots, g_{n+1}^{\epsilon_{n+1}})$ does not contain 1. Writing e for the identity element in L, we would therefore have

$$e = \phi(1) = \phi(\omega) = \tilde{\omega},$$

with $\tilde{\omega} \in S(\phi(g_1)^{\epsilon_1}, \dots, \phi(g_k)^{\epsilon_k})$. However, by our choice of the ϵ_i, each such word is strictly greater than e in L, contradicting the fact that $\tilde{\omega} = e$. □

2.2. Order-preserving homeomorphisms of the real line

Left-orderability of a group is strongly related to the existence of actions by orientation-preserving homeomorphisms of $\mathbb{R}$. We begin with the following:

Proposition 29. *$Homeo_+(\mathbb{R})$ is left-orderable.*

Proof. Take a sequence (x_n) of points which is dense in $\mathbb{R}$. For any pair $f, g \in Homeo_+(\mathbb{R})$, we then set $f \leq g$ if $f = g$ or if the first n such that $f(x_n) \neq g(x_n)$ is such that $f(x_n) < g(x_n)$. This defines a left-invariant total ordering on $Homeo_+(\mathbb{R})$ since f preserves orientation if and only if f preserves the natural order on $\mathbb{R}$. Indeed, if $x \in \mathbb{R}$, a generator for $H_1(\mathbb{R}, \mathbb{R} - \{x\})$ can be represented by a monotone map $\sigma : [0, 1] \to \mathbb{R}$ with $x \in \sigma((0, 1))$. If f is increasing, $f \circ \sigma$ is increasing if σ is increasing, and $f \circ \sigma$ is decreasing if σ is decreasing. Thus f preserves orientation as σ and $f \circ \sigma$ represent local homology classes that extend to the same fundamental class. Conversely, if f is decreasing, $f \circ \sigma$ is decreasing if σ is increasing, and $f \circ \sigma$ is increasing if σ is decreasing, so that f reverses orientation. □

The following theorem is well-known but detailed, self-contained proofs are hard to find in the literature. The following is an expanded version of the proof given in [**19**], supplemented with material provided by Steven Boyer.

THEOREM 30. *Let G be a countable, left orderable group. Then G is isomorphic to a subgroup of $Homeo_+(\mathbb{R})$.*

PROOF. **Step 1: Constructing an increasing map from G into $\mathbb{R}$.** Suppose G is a countable left-orderable group. We will construct an injective homomorphism from G into $Homeo_+(\mathbb{R})$. We write G as $\{g_i | i \in \mathbb{N}\}$, and construct a map $\phi : G \to \mathbb{R}$ inductively as follows: we first set $\phi(g_0) = 0$, then, assuming that $\phi(g_0), \dots, \phi(g_n)$ have been defined, we set

- $\phi(g_{n+1}) = \max\{\phi(g_0), \dots, \phi(g_n)\} + 1$ if $g_{n+1} > g_i$ for all $i \in \{0, \dots, n\}$;
- $\phi(g_{n+1}) = \min\{\phi(g_0), \dots, \phi(g_n)\} - 1$ if $g_{n+1} < g_i$ for all $i \in \{0, \dots, n\}$.

If neither of the above conditions is satisfied, there exists a unique pair integers $k, l \leq n$ such that

- $g_k < g_{n+1} < g_l$;
- there is no g_i with $i \in \{0, \dots, n\}$ such that $g_k < g_i < g_l$.

We then set
$$\phi(g_{n+1}) = \frac{\phi(g_k) + \phi(g_l)}{2}.$$
It is clear from our construction that ϕ is strictly increasing. Writing $X = \phi(G)$, we obtain an action $g \mapsto \phi_g$ of G on X by setting $\phi_g(\phi(h)) = \phi(gh)$. It is this action that we will seek to extend to all of $\mathbb{R}$. In order to do this, we first need to study the space X.

Step 2: Preliminary properties of X.

LEMMA 31. *X is unbounded.*

PROOF. Letting $E_n = \{g_0, \dots, g_n\}$, we take a strictly positive (resp. negative) element g in $G - E_n$. Then $g \max(E_n)$ is strictly larger than all elements of E_n. Let k be the index of $g \max(E_n)$ in our enumeration of G, i.e. write $g \max(E_n) = g_k$. We show by induction that there is necessarily some integer m such that $n \leq m < k$ and $g_{m+1} > g_i$ for all elements g_i of E_m. Indeed, if $g_{n+1} > g_i$ for all elements g_i of E_n, we are done. If not, then g_k is strictly larger than all elements of E_{n+1}. Now, given some integer l such that $n \leq l < k$ and such that g_k is strictly larger than all elements of E_l, either $g_{l+1} > g_i$ for all elements g_i of E_l, in which case we are done, or g_k is strictly larger than all elements of E_{l+1}. If this process does not terminate before we reach $k-1$, it will terminate at $k-1$ as we will have shown that g_k is strictly larger than all elements of E_{k-1}, so that $k-1$ will yield the desired integer m. In any case, $\phi(g_m) \geq \max\{\phi(h) : h \in E_n\} + 1$ by construction. By repeating the process with E_m, we see that we can find arbitrarily large elements of X. A similar argument with $g \min(E_n)$ shows that X is unbounded below. □

Lemma 32. *The closure $\overline{X}$ of X satisfies the condition that if (a, b) is a connected component of $\mathbb{R} - \overline{X}$, then $a \in X$ and $b \in X$.*

Proof. We proceed by contradiction: suppose for instance that $a \notin X$, and take $\delta < b - a$, so that for all $x \in (a - \delta, a]$, $y \in [b, b + \delta)$, we have

$$a < \frac{x+y}{2} < b.$$

As $a \in \overline{X} - X$ and $b \in \overline{X}$, we have a pair (m, n) such that $\phi(g_m) \in (a - \delta, a)$ and $\phi(g_n) \in [b, b + \delta)$.

Let $M = \max(m, n)$, and take $0 < \epsilon < \delta$ such that all $\phi(g_k)$ with $k \leq M$ are excluded from $(a - \epsilon, a + \epsilon)$. Note that this is possible because we supposed that $a \notin X$. Let i be the first index such that $\phi(g_i)$ is in $(a - \epsilon, a + \epsilon)$, so that in fact all $\phi(g_k)$ with $k < i$ are excluded from $(a - \epsilon, a + \epsilon)$. Then $\phi(g_i) < \phi(g_n)$ and $\phi(g_i) > \phi(g_m)$, with $m, n < i$, so that $\phi(g_i)$ cannot have been obtained in the first nor in the second manner listed in our construction. Thus we necessarily have

$$\phi(g_i) = \frac{\phi(g_k) + \phi(g_l)}{2}$$

for some (unique) pair (k, l) of integers with $k, l < i$, and such that there is no element g_j with $j < i$ such that $g_k < g_j < g_l$.

We may suppose without loss of generality that $\phi(g_k) < \phi(g_l)$. Note that we then have

$$a - \epsilon < \phi(g_k) < \phi(g_i) < \phi(g_l),$$

so that $\phi(g_l) \in [b, \phi(g_n)]$: indeed, $\phi(g_l)$ is excluded from $(a - \epsilon, a + \epsilon)$ because $l < i$, and is excluded from (a, b) because it is in X, so that $\phi(g_l) \geq b$ on the one hand. On the other hand, if we had $\phi(g_l) > \phi(g_n)$, we would have $g_l > g_n > g_k$; but this is impossible due to the fact that $n < i$.

To deal with $\phi(g_k)$, we first note that if it were the case that $\phi(g_k) \geq b$, we would have $\phi(g_i) \geq b$ as well, contrary to the assumption that $\phi(g_i) \in (a - \epsilon, a + \epsilon)$. A similar argument to the one in the previous paragraph then shows that $\phi(g_k) \in [\phi(g_m), a - \epsilon]$. Summarizing, we have

- $\phi(g_l) \in [b, \phi(g_n)] \subseteq [b, b + \delta)$ and
- $\phi(g_k) \in [\phi(g_m), a - \epsilon] \subseteq (a - \delta, a)$.

By our choice of δ, we therefore have $a < \frac{\phi(g_k)+\phi(g_l)}{2} < b$, so that $\phi(g_i) \in (a, b)$. But this contradicts the fact that (a, b) is in $\mathbb{R} - \overline{X}$. Thus $a \in X$ and similarly, $b \in X$. □

The bijection fixed above between G and $\mathbb{N}$ is no longer necessary and as such, we will be dropping the indices from the elements of G. The following terminology will be useful as we continue our analysis of X:

Definition 33. A *gap in G* is a pair (f, h) of elements of G such that $f < h$ and such that the set $\{\alpha \in G : f < \alpha < h\}$ is empty.

A *gap in X* is a partition $G = L \cup U$ of G such that $f < h$ for all $f \in L$ and $h \in U$, and such that $\inf \phi(U) - \sup \phi(L) > 0$.

Lemma 34. *For every gap $G = L \cup U$ in X, there is a corresponding gap (σ, ι) in G such that $f \leq \sigma$ for all $\phi(f) \in L$ and $\iota \leq h$ for all $\phi(h) \in U$. We call σ a* supremum for $\phi^{-1}(L)$ *and ι an* infimum for $\phi^{-1}(U)$.

Proof. To see this, set $\alpha = \sup \phi(L)$ and $\beta = \inf \phi(U)$. Then (α, β) is a connected component of $\mathbb{R} - \overline{X}$, so that α and β belong to X. We therefore have elements σ and ι of G such that $\phi(\sigma) = \alpha$ and $\phi(\iota) = \beta$. As ϕ is order-preserving, it is clear that σ and ι behave as desired because α and β do. □

We now show that G-translates of gaps in X are still gaps in X. More precisely:

Lemma 35. *If $G = L \cup U$ is a gap in X, and $g \in G$, then $G = gL \cup gU$ is also a gap in X.*

Proof. To see this, let (σ, ι) be the gap in G corresponding to the gap $G = L \cup U$ in X. The inequality

$$f \leq \sigma < \iota \leq h,$$

valid for all $f \in L$ and all $h \in U$, becomes

$$gf \leq g\sigma < g\iota \leq gh$$

for all $f \in L$ and all $h \in U$, so that $\tilde{f} < \tilde{h}$ for all $\tilde{f} \in gL$ and all $\tilde{h} \in gU$. Furthermore, we have

$$\inf gU - \sup gL \geq \phi(g\iota) - \phi(g\sigma) > 0,$$

so that $G = gL \cup gU$ is indeed a gap in X. □

We now turn to the business of extending our G-action from X to $\mathbb{R}$.

Step 3: Continuously extending the action from X to $\overline{X}$. We extend each map ϕ_g to a map $\overline{\phi_g} : \overline{X} \to \overline{X}$ as follows: given $x \in \overline{X} - X$, set $L_x = \{h \in G : \phi(h) < x\}$ and $U_x = \{h \in G : \phi(h) > x\}$. We claim that $G = L_x \cup U_x$ is not a gap in X. Indeed, if it were, we would have a gap (σ, ι) in G with σ a supremum for L and ι an infimum for U. This would entail

$$\phi(f) \leq \phi(\sigma) < x < \phi(\iota) \leq \phi(h)$$

for all $f \in L_x$ and all $h \in U_x$, contradicting the fact that $x \in \overline{X}$. Thus $G = L_x \cup U_x$ is not a gap in X, and in particular, we have

$$\bigcap_{f \in L_x, h \in U_x,} [\phi(f), \phi(h)] = \{x\}.$$

Indeed, x belongs to every interval of the form $[\phi(f), \phi(h)]$ with $f \in L_x, h \in U_x$, so that x belongs to their intersection. To see that no other point belongs to the set $\bigcap_{f \in L_x, h \in U_x,} [\phi(f), \phi(h)]$, we take a sequence (f_n) with $f_n \in L_x$ for all $n \in \mathbb{N}$ such that $(\phi(f_n))$ increases to x, and a sequence (h_n) with $h_n \in U_x$ for all $n \in \mathbb{N}$ such that $(\phi(h_n))$ decreases to x. By the nested interval theorem, we have $\bigcap_n [\phi(f_n), \phi(h_n)] = \{x\}$, so that $\bigcap_{f \in L_x, h \in U_x,} [\phi(f), \phi(h)] \subseteq \{x\}$.

Thus for any $g \in G$, the partition $G = gL_x \cup gU_x$ is not a gap in X. We therefore set $\overline{\phi_g}(x) = y$, where y is the unique real number such that

$$\bigcap_{f \in L_x, h \in U_x,} [\phi(gf), \phi(gh)] = \{y\}.$$

Note that the above equality shows that $y \in \overline{X}$, so that we have extended ϕ_g to $\overline{\phi_g} : \overline{X} \to \overline{X}$. By the above construction, $\overline{\phi_g}$ is invertible with inverse $\overline{\phi_{g^{-1}}}$. It remains to see that $\overline{\phi_g}$ is continuous on $\overline{X}$.

To see this, fix an element g in G, and let x be an element of $\overline{X}$. In order to prove that $\overline{\phi_g}$ is continuous at x, it suffices to verify that $\lim_{n\to\infty} \overline{\phi_g}(x_n) = \overline{\phi_g}(x)$ for all sequences (x_n) converging to x in $\overline{X}$ and such that

- $x_n \in X$ for all $n \in \mathbb{N}$;
- the sequence (x_n) is strictly monotone;

i.e., it is not necessary to take (x_n) in $\overline{X}^{\mathbb{N}}$ (these facts follow from elementary results on sequences in metric spaces). Taking a sequence (x_n) as described above, we may write each x_n as $\phi(h_n)$ for some $h_n \in G$. Note that the sequence (h_n) must then also be strictly monotone. We will deal with the case where (h_n) is strictly increasing; the strictly decreasing case is analogous.

We first consider the situation where x is in X. Writing $x = \phi(h)$ for some $h \in G$, we claim that

$$\bigcup_{n\in\mathbb{N}} (-\infty, h_n) = (-\infty, h),$$

where the notation $(-\infty, \alpha)$ for $\alpha \in G$ represents the set $\{\beta \in G : \beta < \alpha\}$. Indeed, if this were not the case, there would be some h' such that $h_n \leq h' < h$ for all $n \in \mathbb{N}$. Applying ϕ yields the inequality

$$x_n \leq \phi(h') < x$$

for all $n \in \mathbb{N}$, contradicting the fact that (x_n) converges to x.

We can therefore write

$$(-\infty, gh) = g(-\infty, h) = \bigcup_{n\in\mathbb{N}} g(-\infty, h_n) = \bigcup_{n\in\mathbb{N}} (-\infty, gh_n),$$

so that for every $h' \in G$ such that $h' < gh$, there exists an $n \in \mathbb{N}$ such that

$$h' < gh_n < gh.$$

If the sequence $(\overline{\phi_g}(h_n))$ did not converge to $\overline{\phi_g(h)}$, the partition $G = L \cup U$ given by $L = (-\infty, gh)$ and $U = G - L$ would be a gap in G, which would mean that $g^{-1}(L) \cup g^{-1}(U)$ is also a gap in G. But as $g^{-1}(L)$ is simply $(-\infty, h)$, this would prevent the sequence (x_n) from converging to x, which is contrary to our hypotheses. Thus if $x \in X$, $\overline{\phi_g}$ is continuous at x.

We now consider $x \in \overline{X} - X$. As above, let $(x_n) \in X^{\mathbb{N}}$ be a strictly increasing sequence converging to x in $\overline{X}$, which we write as $(\phi(h_n))$ for some stricly increasing sequence $(h_n) \in G^{\mathbb{N}}$. As in Step 3, consider the sets $L_x = \{h \in G : \phi(h) < x\}$ and $U_x = \{h \in G : \phi(h) > x\}$. Note that we have $h_n \in L_x$ for all $n \in \mathbb{N}$.

Because (x_n) converges to x, we see by applying ϕ^{-1} that for every $\alpha \in L_x$, there exists an $n \in \mathbb{N}$ such that $\alpha < h_m$ for all $m \geq n$. Thus for every $\beta \in gL_x$, there exists an $n \in \mathbb{N}$ such that $\beta < gh_m$ for all $m \geq n$. Thus $\sup \phi(gL_x) \leq \lim_{n\to\infty} \overline{\phi_g}(h_n) \leq \inf \phi(gU_x)$, so that if $\overline{\phi_g}(h_n)$ did not converge to $\overline{\phi_g}(x)$, the partition $G = gL_x \cup gU_x$ would be a gap in G. This in turn would imply that $L_x \cup U_x$ is also a gap in G, contradicting the fact that (x_n) converges to x.

For each $g \in G$, the map ϕ_g can therefore be extended continuously to $\overline{\phi_g} : \overline{X} \to \overline{X}$ as claimed. Note that the relation $\overline{\phi_g} \circ \overline{\phi_h} = \overline{\phi_{gh}}$ still holds on $\overline{X}$ by continuity, so that we have obtained an action of G on $\overline{X}$ by homeomorphisms. To lighten the notation, we will henceforth write ϕ_g instead of $\overline{\phi_g}$ for these extended maps.

Step 4: Continuously extending the action from $\overline{X}$ to $\mathbb{R}$. For every $g \in G$, we extend $\phi_g : \overline{X} \to \overline{X}$ to $\rho_g : \mathbb{R} \to \mathbb{R}$ as follows: given a connected component (a, b) of $\mathbb{R} - \overline{X}$, set

$$\rho_g((1-t)a + tb) = (1-t)\phi_g(a) + t\phi_g(b)$$

for all $t \in (0, 1)$. As defined, it is clear that each ρ_g is a strictly increasing bijection of $\mathbb{R}$ onto itself, and that the map $g \mapsto \rho_g$ is injective. It remains to show that each ρ_g is continuous and that the map $g \mapsto \rho_g$ is a homomorphism.

Fix $g \in G$ and $x \in \mathbb{R}$. We will show that ρ_g is continuous at x. If $x \in \mathbb{R} - \overline{X}$, then ρ_g is continuous at x because it is simply an affine function in a neighbourhood of x. Suppose then that $x \in \overline{X}$ and let $\epsilon > 0$. By continuity of ϕ_g at x on $\overline{X}$, there is a $\delta > 0$ such that

$$|\rho_g(x) - \rho_g(y)| = |\phi_g(x) - \phi_g(y)| < \frac{\epsilon}{2}$$

for all $y \in \overline{X} \cap (x - \delta, x + \delta)$. We now have four cases to consider:

Case 1. $\overline{X} \cap (x-\delta, x+\delta) = \{x\}$. Then ρ_g restricted to $(x-\delta, x+\delta)$ is affine on the intervals $(x-\delta, x]$ and $[x, x+\delta)$, hence is continuous at x.

Case 2. $\overline{X} \cap (x-\delta, x+\delta)$ has nonempty intersections with $(x-\delta, x)$ and $(x, x+\delta)$. Let u and v be elements of $\overline{X} \cap (x-\delta, x)$ and $\overline{X} \cap (x, x+\delta)$ respectively. Setting $\delta' = \min\{x-u, v-x\}$, we see that by construction, ρ_g is bounded above and below on $(x-\delta', x+\delta')$ by values taken by ϕ_g on $\overline{X} \cap (x-\delta', x+\delta')$. Thus $|\rho_g(x) - \rho_g(y)| \leq \frac{\epsilon}{2} < \epsilon$ for all $x, y \in (x-\delta', x+\delta')$.

Case 3. $\overline{X} \cap (x-\delta, x+\delta)$ has nonempty intersection with $(x-\delta, x)$ and empty intersection with $(x, x+\delta)$. Let u be an element of $(x-\delta, x) \cap \overline{X}$. Setting $\delta' = x-u$, we see as in case 2 that $|\rho_g(x)-\rho_g(y)| \leq \frac{\epsilon}{2} < \epsilon$ for all $x, y \in (x-\delta', x]$. Furthermore, ρ_g is affine on $(x, x+\delta')$, so that we can restrict to some $0 < \delta'' < \delta'$ such that $|\rho_g(x) - \rho_g(y)| < \epsilon$ for all $x, y \in (x-\delta'', x+\delta'')$.

Case 4. $\overline{X} \cap (x-\delta, x+\delta)$ has empty intersection with $(x-\delta, x)$ and nonempty intersection with $(x, x+\delta)$. This case is analogous to case 3.

We have thus shown that ρ_g is continuous on $\mathbb{R}$. To see that the map $g \mapsto \rho_g$ is a homomorphism, we must verify that given elements $g, h \in G$, we have $\rho_g \circ \rho_h = \rho_{gh}$ on $\mathbb{R}-\overline{X}$. Take $x \in \mathbb{R}-\overline{X}$ and let (a, b) be the connected component containing x. Then $(\phi_g(a), \phi_g(b))$ and $(\phi_{gh}(a), \phi_{gh}(b))$ are also connected components of $\mathbb{R} - \overline{X}$. Writing x as $(1-t)a + tb$ for some $t \in (0, 1)$, we have

$$\rho_{gh}((1-t)a + tb) = (1-t)\phi_{gh}(a) + t\phi_{gh}(b) = (1-t)\phi_g(\phi_h(a)) + t\phi_g(\phi_h(b)).$$

Rewriting the right-hand term as

$$(1-t)\phi_g(\phi_h(a)) + t\phi_g(\phi_h(b)) = \rho_g((1-t)\phi_h(a) + t\phi_h(b)) = \rho_g(\rho_h((1-t)a + tb)),$$

we see that we indeed have $\rho_g \circ \rho_h = \rho_{gh}$. We have thus constructed a faithful action of G on $\mathbb{R}$ via strictly increasing (and therefore orientation-preserving) homeomorphisms, as desired. □

Combining the above theorem with proposition 29, we obtain the following:

Corollary 36. *If G is a non-trivial countable group, then G is left orderable if and only if G is isomorphic to a non-trivial subgroup of $Homeo_+(\mathbb{R})$.*

We prove some lemmas which will also be useful later:

Lemma 37. *Let f be a nontrivial homomorphism from a group G into $Homeo_+(\mathbb{R})$. Then there exists another such homomorphism which induces an action on $\mathbb{R}$ without any global fixed points.*

PROOF. Given a nontrivial homomorphism $\phi : G \to Homeo_+(\mathbb{R})$, the set F of global fixed points of ϕ is closed in $\mathbb{R}$: indeed, if (x_n) is a sequence in F converging to a point x, then for any fixed γ in G, we have

$$\phi(\gamma)(x_n) = x_n$$

for all n. By continuity of $\phi(\gamma)$, we obtain $\phi(\gamma)(x) = x$ by passing to the limit, so that x is in F.

Because ϕ is nontrivial, $\mathbb{R} - F$ is nonempty. Then any component C of $\mathbb{R} - F$ is homeomorphic to $\mathbb{R}$ and is invariant under the action of ϕ. Restricting the action of ϕ to C, we obtain an action without any global fixed points. □

LEMMA 38. *Any fixed point free element of* $Homeo_+(\mathbb{R})$ *is conjugate to translation by* ± 1.

PROOF. Let ϕ be a fixed point free element of $Homeo_+(\mathbb{R})$. Then the sign of $x \mapsto \phi(x) - x$ is constant, and by passing to $x \to -\phi(-x)$ if necessary, we may assume that $x < \phi(x)$ for all $x \in \mathbb{R}$. Then the set

$$\{\phi^n(0) | n \in \mathbb{Z}\}$$

is unbounded: indeed, if it was bounded above (for instance) with least upper bound M, we would have $M = \lim_{n\to\infty} \phi^n(0)$, so that

$$\phi(M) = \phi(\lim_{n\to\infty} \phi^n(0)) = \lim_{n\to\infty} \phi^{n+1}(0) = M,$$

contradicting the fact that ϕ is fixed-point free.

We can therefore separate $\mathbb{R}$ into a union of intervals of the form

$$I_n = [\phi^n(0), \phi^{n+1}(0)],$$

with ϕ mapping each I_n homeomorphically onto I_{n+1}. We define an element of $Homeo_+(\mathbb{R})$ piecewise on the intervals $[n, n+1]$ by setting

$$\psi(x) = \phi^n(x - n) \text{ for } x \in [n, n+1].$$

This gives, for $x \in [n, n+1]$,

$$\psi^{-1} \circ \phi \circ \psi(x) = \phi^{n+1}(x - n) = \psi^{-1} \circ \phi^{n+1}(x + 1 - (n + 1)) = x + 1,$$

for all $n \in \mathbb{Z}$ (and hence for all $x \in \mathbb{R}$). □

2.3. Left-orderability of three-manifold groups

THEOREM 39. *(Scott core theorem) Let M be a connected 3-manifold with finitely-presented fundamental group. Then there exists a compact, connected three-dimensional submanifold N of M such that the inclusion of N into M induces an isomorphism on the level of fundamental groups.*

PROOF. This is the main result of **[44]**. □

The following theorem gives a necessary and sufficient condition for the fundamental group of a 3-manifold to be irreducible for a large class of such manifolds. Its proof is taken from **[9]**.

THEOREM 40. *Let M be compact, connected, and P^2-irreducible. Then M has a left-orderable fundamental group if and only if there exists an epimorphism from $\pi_1(M)$ onto a left-orderable group L.*

PROOF. If $\pi_1(M)$ is left-orderable, then the identity map provides the required epimorphism. Suppose now that there exists a left-orderable group L and an epimorphism $\phi : \pi_1(M) \to L$. By theorem 28, it suffices to show that given any non-trivial finitely-generated subgroup H of $\pi_1(M)$, we can produce a left-orderable group L' as well as an epimorphism from H onto L'. Let H be such a subgroup.

If H is of finite index in $\pi_1(M)$, $\phi(H)$ is also of finite index in L by surjectivity of ϕ. Indeed, ϕ induces a surjective map $\tilde{\phi} : \pi_1(M)/H \to L/\phi(H)$, so that every equivalence class in $L/\phi(H)$ can be written as the image of one of the finitely many equivalence classes in $\pi_1(M)/H$. In particular, $\phi(H)$ is a nontrivial subgroup of L (as L is left-orderable, hence infinite), so that the restricted map $\phi|_H : H \to \phi(H)$ yields the desired epimomorphism.

Suppose now that H has infinite index in $\pi_1(M)$, and let $p : \tilde{M} \to M$ denote the covering space associated to H. Note that because the fibres of p are closed, discrete, and infinite, $\tilde{M}$ is necessarily noncompact. By theorem 39, there exists a compact, connected submanifold N of $\tilde{M}$ such that the inclusion map induces an isomorphism between $\pi_1(N)$ and $\pi_1(\tilde{M})$. We claim that N necessarily has a nonempty boundary. To see this, we suppose that $\partial N = \emptyset$. By invariance of domain, the inclusion map $i : N \to \tilde{M}$ is an open map and N is an open subset of $\tilde{M}$. By compactness, N is also closed in $\tilde{M}$, which implies that $N = \tilde{M}$ by connectedness, contradicting the non-compactness of $\tilde{M}$. Thus $\partial N \neq \emptyset$, as claimed.

We may further assume that ∂N contains no 2-spheres. To see this, first note that the fact that M is P^2-irreducible implies that $\tilde{M}$ is also P^2-irreducible (see lemma 10.4 of **[24]** as well as the discussion preceding theorem 4 of **[25]**) . Thus any 2-sphere S contained in ∂N must bound a 3-ball B in $\tilde{M}$. Note that we have $S \subseteq B \cap N$. We claim that we also have $B \cap N \subseteq S$. Indeed, $\tilde{M} - S$ has two components: $\mathring{B}$ and its complement U. As $N - S$ is connected and is contained in $\tilde{M} - S$, it must wholly lie in $\mathring{B}$ or in U. If it were the case that $N - S \subseteq \mathring{B}$, we would have $N \subseteq B \subseteq \tilde{M}$. But this would force the map induced by inclusion to factor through the trivial group, which contradicts the fact that $\pi_1(N)$ and $\pi_1(\tilde{M})$ are isomorphic (note that $\pi_1(\tilde{M})$ cannot be the trivial group as $p_{\#}(\pi_1(\tilde{M})) = H$). Thus $N - S \subseteq U$, so that $B \cap N \subseteq S$ (recall that U is the complement of $\mathring{B}$ in $\tilde{M} - S$). Thus we have shown that $S = B \cap N$, so that we may attach B to N without affecting the fact that the inclusion induces an isomorphism on the level of fundamental groups.

Thus N is compact, connected and ∂N is nonempty but contains no S^2 or P^2 components (the latter is due to the fact that N is contained in $\tilde{M}$, which is P^2-irreducible). By lemma 10, we therefore have $b_1(N) > 0$, so that $H_1(N)$ surjects onto $\mathbb{Z}$. Abelianizing the fundamental group then yields the desired epimorphism:

$$H \cong \pi_1(\tilde{M}) \cong \pi_1(N) \to H_1(N) \to \mathbb{Z}.$$

□

This result immediately implies the following, which gives us an interpretation of left-orderability as an obstruction to the existence of certain maps:

Corollary 41. *Let M be compact, connected, and P^2-irreducible. If there exists a non-zero degree map from M onto a manifold N such that $\pi_1(N)$ is left-orderable, then $\pi_1(M)$ is also left-orderable.*

We also have the following:

Corollary 42. *Let M be a compact, connected, P^2-irreducible three-manifold with $b_1(M) > 0$. Then $\pi_1(M)$ is left-orderable.*

PROOF. We can map $\pi_1(M)$ onto $H_1(M)$ by the abelianization map, and project $H_1(M)$ onto $\mathbb{Z}$ because $H_1(M)$ is infinite. Theorem 40 then gives the desired result. □

PROPOSITION 43. *Let M be an integer homology sphere, and suppose that there exists a homomorphism $\pi_1(M) \to Homeo_+(S^1)$ with non-abelian image. Then $\pi_1(M)$ is left-orderable.*

PROOF. See proposition 1.2 of [**9**]. □

CHAPTER 3

HEEGAARD FLOER HOMOLOGY

We give a short rundown of Heegaard Floer homology, which is needed to define the notion of an L-space. This section will not contain any proofs as we will not need any of the technical machinery of Heegaard Floer homology. We refer the interested reader to [**49**].

3.1. Construction of $\widehat{HF}$

For any integer k, let B^k denote the k-dimensional ball. We call the decomposition $h_i^n = B^i \times B^{n-i}$ of B^n an *n-dimensional i-handle*. If N and M are oriented n-dimensional manifolds, we will say that N is obtained from M by *attaching an i-handle* if there exists an orientation-preserving diffeomorphism

$$\varphi : \partial B^i \times B^{n-i} \to \partial M$$

such that $N = M \cup_\varphi h_i^n$. We shall only be interested in the case $n = 3$, and as such, we will refer to h_i^3 simply as an i-handle and will write h_i instead of h_i^3.

A fundamental example of a manifold constructed by gluing handles is that of a *genus g handlebody*, which is defined to be any manifold obtained by gluing g 1-handles onto a 3-ball. The boundary of such a manifold is a closed, orientable surface of genus g, justifying the terminology.

Given two genus g handlebodies H and H', we can identify their boundaries via an orientation-reversing diffeomorphism ψ and consider the space $M = H \cup_\psi H'$, which turns out to be a closed, oriented, three-dimensional manifold. Given such a configuration, we call the data of H, H' and ψ a *Heegaard splitting* for M. Heegaard splittings will generally be written (H, H', Σ_g), where Σ_g denotes the common boundary of H and H' in M, as it will rarely be necessary for our purposes to describe ψ explicitely. Two Heegaard splittings (H, H', Σ_g) and $(\tilde{H}, \tilde{H}', \tilde{\Sigma}_g)$ will be said to be equivalent if there is an automorphism of M yielding a homeomorphism between the triples (H, H', Σ_g) and $(\tilde{H}, \tilde{H}', \tilde{\Sigma}_g)$.

Given a genus g surface Σ_g, let $\gamma_1, \dots \gamma_g$ be closed disjoint curves on Σ_g such that the set of homology classes $\{[\gamma_i] : 1 \leq i \leq g\}$ is linearly independent in $H_1(\Sigma_g; \mathbb{Z})$. We call such a set of curves a set of *attaching circles* on Σ_g. We can construct a handlebody $H(\Sigma_g, \gamma)$ from this data as follows: We first take annular neighbourhoods A_i of each γ_i. We then consider the manifold $\Sigma_g \times [0, 1]$, which has boundary components $\Sigma_g \times \{0\}$ and $\Sigma_g \times \{1\}$, and attach 2-handles to the $A_i \times \{1\}$, obtaining a manifold with boundary components homeomorphic to S^2 and to Σ_g. By attaching a 3-handle to the S^2 boundary component, we obtain a genus g handlebody. Note that we could also have constructed a handlebody $H'(\Sigma_g, \gamma)$ via the same process, using the closed interval $[-1, 0]$ instead of $[0, 1]$.

Let Σ_g be an oriented genus g surface, and let $\alpha_1, \dots, \alpha_g$ (henceforth referred to as α-curves) and $\beta_1, \dots, \beta_g$ (henceforth referred to as β-curves) be two sets of attaching circles on Σ_g. Using the notations from the previous paragraph, we set $H_\alpha = H(\Sigma_g, \alpha)$ and $H_\beta = H'(\Sigma_g, \beta)$. Orienting H_α and H_β such that $\partial H_\alpha = \Sigma_g$, and $\partial H_\beta = -\Sigma_g$, we obtain a Heegaard splitting $(H_\alpha, H_\beta, \Sigma_g)$ for $Y = H_\alpha \cup_\psi H_\beta$, where the gluing map ψ is the identity along $\Sigma_g \times \{0\}$. We shall call the data of Σ_g and the α- and β-curves a *Heegaard diagram* for Y and denote it $(\Sigma_g, \alpha_i, \beta_i)$. We will consider two Heegaard diagrams equivalent if their underlying Heegaard splittings are equivalent. A Heegaard diagram can be transformed via *Heegaard moves* (which we will not describe in detail here: for more on Heegaard moves, see [**49**], page 204), yielding another Heegaard diagram. The salient fact concerning Heegaard moves is the following:

Proposition 44. *Let M be a closed, oriented 3-manifold. Then any two Heegaard diagrams for M can be made equivalent via a sequence of Heegaard moves.*

Proof. See [**48**] for a full treatment of Heegaard splittings. □

Given a Heegaard diagram $(\Sigma_g, \alpha_i, \beta_i)$, we construct the space $\mathrm{Sym}^g(\Sigma_g) = \Sigma_g^g / S_g$, where S_g denotes the symmetric group on g elements, acting on Σ_g^g by permuting the factors in the product. Note that this is simply the set of unordered g-tuples of elements of Σ_g. It can be shown that this is a smooth manifold, and that a choice of complex structure on Σ_g induces a complex structure on $\mathrm{Sym}^g(\Sigma_g)$. We have the following:

Proposition 45. $\pi_1(\mathrm{Sym}^g(\Sigma_g)) \cong H_1(\mathrm{Sym}^g(\Sigma_g)) \cong H_1(\Sigma_g)$

PROOF. See proposition 4.2 of [**49**]. □

Inside $\mathrm{Sym}^g(\Sigma_g)$, the α curves and the β curves induce a pair of smoothly embedded tori,

$$T_\alpha = (\Pi_i \alpha_i)/S_g \text{ and } T_\beta = (\Pi_i \beta_i)/S_g.$$

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Let e_1 and e_2 be the arcs on $\partial\mathbb{D}$ with $\Re(z) \geq 0$ and $\Re(z) \leq 0$ respectively. Given points $x, y \in T_\alpha \cap T_\beta$, a *Whitney disk connecting x and y* is a continuous map $f : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$ such that $f(-i) = x$, $f(i) = y$, $f(e_1) \subset T_\alpha$, and $f(e_2) \subset T_\beta$. The set of homotopy classes of Whitney disks connecting x and y will be denoted $\pi_2(x, y)$.

As mentioned earlier, $\mathrm{Sym}^g(\Sigma_g)$ can be equipped with a complex structure. Viewing $\mathbb{D}$ as a complex manifold, it then makes sense to talk about holomorphic Whitney disks. Given a fixed homotopy class $\phi \in \pi_2(x, y)$, we can consider the moduli space $\mathcal{M}(\phi)$ of its holomorphic representatives. To this space is associated a numerical invariant called its *Maslov index* $\mu(\phi)$. We refer the interested reader to page 215 of [**49**] for more information on the meaning of this object.

THEOREM 46. *The space $\mathcal{M}(\phi)$ admits an $\mathbb{R}$-action. Set $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$ (the set of orbits for this $\mathbb{R}$-action). Given a complex structure on Σ_g, the induced complex structure on* $\mathrm{Sym}^g(\Sigma_g)$ *can be perturbed so that for all $x, y \in T_\alpha \cap T_\beta$ and for all $\phi \in \pi_2(x, y)$, we have that $\widehat{\mathcal{M}}(\phi)$ consists of finitely many points whenever $\mu(\phi) = 1$.*

See theorem 8.2 of [**49**].

We then define a map $c : \pi_2(x, y) \to \mathbb{Z}_2$ by setting $c(\phi)$ equal to

- 0 if $\mu(\phi) \neq 1$;
- the parity of the number of points in $\widehat{\mathcal{M}}(\phi)$ if $\mu(\phi) = 1$.

The last ingredient needed to define $\widehat{HF}$ is a choice of basepoint. Given a Heegaard diagram $(\Sigma_g, \alpha_i, \beta_i)$, let z be a point in the complement of the α- and β-curves. The data of $(\Sigma_g, \alpha_i, \beta_i)$ and z will be called a *pointed Heegaard diagram* and will be denoted $(\Sigma_g, \alpha_i, \beta_i, z)$. It turns out that a generic pointed Heegaard diagram does not yield a well-defined homology theory, but this problem can be overcome by imposing a technical condition on our Heegaard diagrams (thereby deeming them *admissible*). We will not expand on this here; we refer the interested reader to section 5 of **[49]**, and content ourselves stating the following vaguely-worded proposition:

Proposition 47. *Every pointed Heegaard diagram of M can be made admissible.*

Given a pointed Heegaard diagram $(\Sigma_g, \alpha_i, \beta_i, z)$, and viewing $Sym^g(\Sigma_g)$ as the set of unordered g-tuples of points of Σ_g, we let V_w denote the set of g-tuples of points of Σ_g among which z appears at least once. This set can be thought of as $\{z\} \times Sym^{g-1}(\Sigma_g)$, and is clearly disjoint from T_α and from T_β by our choice of z. Thus given $x, y \in T_\alpha \cap T_\beta$ and a homotopy class $\phi \in \pi_2(x, y)$, we may set

$$n_z(\phi) = \#\phi^{-1}(\{z\} \times Sym^{g-1}(\Sigma_g)),$$

where # denotes the algebraic intersection number.

We now define $\widehat{HF}$ as follows: given a pointed Heegaard diagram $(\Sigma_g, \alpha_i, \beta_i, z)$, we first define

$$\widehat{CF}(\Sigma_g, \alpha_i, \beta_i, z) = \bigoplus_{x \in T_\alpha \cap T_\beta} \mathbb{Z}_2 \cdot x,$$

which we make into a chain complex by setting

$$\partial x = \sum_{x \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} c(\phi) \cdot y.$$

This boundary map is well-defined by the following lemma (lemma 8.4 of **[49]**):

Lemma 48. *Given an admissible Heegaard diagram $(\Sigma_g, \alpha_i, \beta_i, z)$, there are only finitely many $\phi \in \pi_2(x, y)$ such that $c(\phi) \neq 0$ and $n_z(\phi) = 0$.*

It is proved in **[41]** that $\partial^2 = 0$. We then define $\widehat{HF}(\Sigma_g, \alpha_i, \beta_i, z)$ to be the homology of the above chain complex. This is a topological invariant of the manifold represented by $(\Sigma_g, \alpha_i, \beta_i, z)$ (see theorem 8.10 of **[41]**):

Theorem 49. *Let $(\Sigma_g, \alpha_i, \beta_i, z)$ and $(\Sigma'_g, \alpha'_i, \beta'_i, z')$ be two pointed Heegaard diagrams representing the same manifold M. Then $\widehat{HF}(\Sigma_g, \alpha_i, \beta_i, z)$ is isomorphic to $\widehat{HF}(\Sigma'_g, \alpha'_i, \beta'_i, z')$.*

We are therefore justified in introducing the notation $\widehat{HF}(M)$.

3.2. L-spaces

Given an unpointed Heegaard diagram $(\Sigma_g, \alpha_i, \beta_i)$ yielding a manifold M, we have the following:

Proposition 50. $\frac{H_1(Sym^g(\Sigma_g))}{H_1(T_\alpha)\oplus H_1(T_\beta)} \cong H_1(M)$

Let $x, y \in T_\alpha \cap T_\beta$, and choose paths $a : [0,1] \to T_\alpha$, $b : [0,1] \to T_\beta$. Then $a - b$ is a collection of arcs in $\mathrm{Sym}^g(\Sigma_g)$, well-defined up to multiples of the classes $[\alpha_i]$ and $[\beta_i]$ in $H_1(\Sigma_g) \cong \pi_1(Sym^g(\Sigma_g))$. By the above proposition, $a - b$ can be viewed as an element of $H_1(M)$ which we denote $\epsilon(x, y)$. We define the map $\epsilon : \mathrm{Sym}^g(\Sigma_g) \to H_1(M)$ by the assignment $a - b \mapsto \epsilon(x, y)$. This class is an obstruction to the existence of Whitney disks:

Proposition 51. *Let $x, y \in T_\alpha \cap T_\beta$. Then there exists a Whitney disk connecting x and y if and only if $\epsilon(x, y) = 0$.*

Proof. See proposition 4.6 of [**49**]. □

To every element $x \in T_\alpha \cap T_\beta$, we may assign a certain type of bundle called a Spin^c-structure. For a detailed account of how this is done, we refer the reader to section 9 of [**49**]. The space of Spin^c-structures on M will be denoted $\mathrm{Spin}^c(M)$, and the Spin^c-structure assigned to x will be denoted $\mathfrak{s}_z(x)$. Spin^c-structures are related to the homology class $\epsilon(x, y)$ as follows:

Theorem 52. *Let $(\Sigma_g, \alpha_i, \beta_i, z)$ be a pointed Heegaard diagram and take $x, y \in T_\alpha \cap T_\beta$. Then*

$$\mathfrak{s}_z(x) - \mathfrak{s}_z(y) = PD[\epsilon(x, y)].$$

The above theorem combined with the previous proposition gives us that $\mathfrak{s}_z(x) = \mathfrak{s}_z(y)$ if and only if there exists a Whitney disk connecting x to y. We can therefore express $\widehat{CF}(\Sigma_g, \alpha_i, \beta_i, z)$ as follows:

$$\bigoplus_{x \in T_\alpha \cap T_\beta} \mathbb{Z}_2 \cdot x = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M)} \bigoplus_{\mathfrak{s}_z(x) = \mathfrak{s}} \mathbb{Z}_2 \cdot x,$$

and by the above remark, for every fixed $\mathfrak{s} \in \mathrm{Spin}^c(M)$, ∂ restricts to the subspace $\widehat{CF}(\Sigma_g, \alpha_i, \beta_i, z, \mathfrak{s}) := \bigoplus_{\mathfrak{s}_z(x)=\mathfrak{s}} \mathbb{Z}_2 \cdot x$, yielding subcomplexes of $\widehat{CF}(\Sigma_g, \alpha_i, \beta_i, z)$ indexed by $\mathrm{Spin}^c(M)$. It can be shown that the homology of $\widehat{CF}(\Sigma_g, \alpha_i, \beta_i, z, \mathfrak{s})$

depends only on M and on $\mathfrak{s}$, so that we obtain a decomposition on the level of homology: we may therefore write

$$\widehat{HF}(M) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M)} \widehat{HF}(M, \mathfrak{s}),$$

where $\widehat{HF}(M, \mathfrak{s})$ is the homology of the complex $\widehat{CF}(\Sigma_g, \alpha_i, \beta_i, z, \mathfrak{s})$ for some pointed Heegaard diagram $(\Sigma_g, \alpha_i, \beta_i, z)$ yielding M.

PROPOSITION 53. *The Euler characteristic of $\widehat{HF}(M, \mathfrak{s})$ is*

- *1 if $b_1(M) = 0$,*
- *0 otherwise.*

PROOF. See proposition 5.1 of **[40]**, as well as section 4.1 of **[41]** for a description of the grading used to take the Euler characteristic of $\widehat{HF}(M, \mathfrak{s})$. □

By the decomposition of $\widehat{HF}(M)$ described above, and by the fact that $\mathrm{Spin}^c(M)$ can be put in bijective correspondence with $H^2(M) = H_1(M)$, we therefore have

$$\chi(\widehat{HF}(M)) = |H_1(M)|.$$

In particular, we have

$$|H_1(M; \mathbb{Z})| \leq rk(\widehat{HF}(M)).$$

DEFINITION 54. A $\mathbb{Q}$-homology sphere is an *L-space* if the above inequality is an equality.

We record here the following vacuous proposition:

PROPOSITION 55. *Let M be a closed, orientable three-manifold with $b_1(M) > 0$. Then M is not an L-space.*

CHAPTER 4

ORBIFOLDS

We shall need the basic vocabulary of orbifolds in order to discuss Seifert fibred spaces. For convenience, we work only in dimension two.

4.1. Basic definitions

A *2 dimensional orbifold* is a paracompact Hausdorff space X equipped with a maximal orbifold atlas, where an *orbifold atlas* on X is a set $\{(U_i, \Gamma_i, \tilde{U}_i, \phi_i)\}_{i \in I}$ such that:

- $\{U_i\}_{i \in I}$ is a covering of X by open sets, closed under finite intersections;
- each $\tilde{U}_i$ is an open set of $\mathbb{R}^2$ equipped with an action of the finite group Γ_i;
- each ϕ_i is a homeomorphism from U_i onto the quotient space $\tilde{U}_i/\Gamma_i$.

Furthermore, for every inclusion $U_i \subset U_j$, there is a group monomorphism $\tau_{ij} : \Gamma_i \to \Gamma_j$ as well as an embedding $\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$, such that for all g in Γ_i, we have

$$\tilde{\phi}_{ij}(gx) = \tau_{ij}(g)\tilde{\phi}_{ij}(x),$$

and such that the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\phi}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\tilde{\phi}_{ij}/\Gamma_i"] & \tilde{U}_j/\Gamma_i \arrow[d, "\tau_{ij}"] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\phi_i"] \arrow[r, "\text{inclusion}"] & U_j \arrow[u, "\phi_j"']
\end{tikzcd}$$

Note that when each Γ_i is the trivial group, we recover the definition of a (smooth) manifold.

Using the exponential map, the groups Γ_i can be viewed as finite subgroups of O_2, of which there are only three types. We summarize this in the following proposition (see section 13.3 of **[50]** for details) :

Proposition 56. *Given a 2 dimensional orbifold X, the quotient spaces $\tilde{U}_i/\Gamma_i$ take one of the following three forms:*

- $\mathbb{R}^2/\mathbb{Z}_2$, *where $\mathbb{Z}_2$ acts by a reflection;*
- $\mathbb{R}^2/\mathbb{Z}_n$, *where $\mathbb{Z}_n$ acts by a rotation;*
- $\mathbb{R}^2/D_n$, *where D_n is the dihedral group if order $2n$.*

In the first case, the points of $\tilde{U}_i$ left fixed by the action of $\mathbb{Z}_2$ correspond to a curve (called a *reflector curve*) in the orbifold and are called *reflector points.* In the second case, the action is free away from a single point, and it is called a *cone point of order n.* In the third case, two reflector curves meet in a point called a *corner reflector of order n.* We have been vague in describing this third case as it will not arise in our applications, and as such, is of little concern here.

Note that because we are in dimension two, every orbifold is homeomorphic to a manifold (possibly with boundary). Thus we can speak of the *underlying surface* of an orbifold. The cone points correspond to interior points of the underlying surface, while reflector curves and corner reflectors correspond to its boundary points.

4.2. Orbifold coverings and the orbifold fundamental group

We describe the notions of orbifold coverings and orbifold fundamental groups for orbifolds with only cone point singularities, as that is all that will be needed in what follows.

Definition 57. An orbifold covering is a continuous map $p : Y \to X$, where X and Y are orbifolds, such that every point x of X has a neighbourhood U satisfying the following conditions:

- U is homeomorphic to the quotient of an open set $\tilde{U}$ of $\mathbb{R}^2$ by the action of a finite group Γ;
- $p^{-1}(U)$ is a disjoint union of sets V_i, each of which is homeomorphic to $\tilde{U}/\Gamma_i$ for some subgroup Γ_i of Γ;
- the following diagram commutes for each i:

$$\begin{tikzcd} V_i \arrow[r, "\cong"] \arrow[d, "p"'] & \tilde{U}/\Gamma_i \arrow[d] \\ U \arrow[r, "\cong"] & \tilde{U}/\Gamma \end{tikzcd}$$

The notion of orbifold coverings allows us to define the orbifold fundamental group by analogy with the usual fundamental group: given an orbifold X, it has a unique maximal connected orbifold covering $\tilde{X}$, and we define the orbifold fundamental group $\pi_1^{orb}(X)$ to be the group of automorphisms $\tilde{X}$ respecting the covering. If $\tilde{X}$ is a surface (i.e. each Γ_i is trivial), X is said to be *good*, and otherwise it is said to be *bad*. There are relatively few bad orbifolds (in dimension two); see [**50**] for a classification.

We will give an alternate description of the orbifold fundamental group, following [**45**]. For convenience, we shall do this in the case of an orbifold with only cone point singularities, as we will not need the more general situation in what follows. Note that in this case, the underlying surface is closed (see the final remark in the preceding section), and as such, is classified by its genus. We will use the following convention for the genus of a non-orientable closed surface Σ :

$$g(\Sigma) = 2 - \chi(\Sigma)$$

while the genus of an orientable closed surface will be defined as usual by

$$g(\Sigma) = \frac{2 - \chi(\Sigma)}{2},$$

so that non-orientable closed surfaces are those with negative genus.

Let X be a compact 2 dimensional orbifold with only cone point singularities and underlying surface Σ. By compactness, there are only finitely many cone points $x_1, \ldots, x_n$. Letting α_i denote the order of x_i, we will use the following notation for such an X:

$$X = \Sigma(\alpha_1, \ldots, \alpha_n).$$

PROPOSITION 58. *Let* $X = \Sigma(\alpha_1, \ldots, \alpha_n)$*, and let* Σ_0 *denote* Σ *with regular neighbourhoods* U_i *of the cone points removed. Letting* γ_i *denote an element of* $\pi_1(\Sigma_0)$ *represented by the boundary of* U_i*, the orbifold fundamental group* $\pi_1^{orb}(X)$ *of is given by* $\pi_1(\Sigma_0)$ *quotiented by the relations* $\gamma_i^{\alpha_i} = 1$.

We will not prove this here; see page 424 of [**45**] for details. Note that we recover the fundamental group of the surface by further quotienting by the relations $\gamma_i = 1$, so that $\pi_1^{orb}(X)$ surjects onto $\pi_1(\Sigma)$.

4.3. Geometric structures on orbifolds

DEFINITION 59. Let $X = \Sigma(\alpha_1, \ldots, \alpha_n)$. The orbifold Euler characteristic of X is defined to be

$$\chi^{orb}(X) = \chi(\Sigma) - \sum_{i=1}^{n}(1 - \frac{1}{\alpha_i}).$$

As in the previous section, this definition arises by analogy with the situation for closed surfaces. We refer the reader to page 426 of [**45**], where this is explained in detail. We record here the following fact:

LEMMA 60. *Let* $X \to Y$ *be an n-fold orbifold covering map. Then*

$$\chi^{orb}(X) = n\chi^{orb}(Y).$$

Note that in [**45**], the orbifold Euler characteristic is defined by the above relation for good orbifolds, and the expression we have used as the definition is then derived as a consequence of this relation, and is used to extend the definition to bad orbifolds.

The orbifold Euler characteristic will be useful in what follows because it determines when X admits a geometric structure.

DEFINITION 61. An orbifold X is said to *admit a geometric structure* if it admits an orbifold atlas $\{(U_i, \Gamma_i, \tilde{U}_i, \phi_i)\}_{i \in I}$ subject to one of the following restrictions:

- Each $\tilde{U}_i$ is an open set of $\mathbb{E}^2$ for which Γ_i is a group of Euclidean isometries, and the transition maps act as Euclidean isometries;
- Each $\tilde{U}_i$ is an open set of S^2 for which Γ_i is a group of isometries of the sphere, and the transition maps act as isometries of the sphere;

- Each $\tilde{U}_i$ is an open set of $\mathbb{H}^2$ for which Γ_i is a group of hyperbolic isometries, and the transition maps act as hyperbolic isometries.

The geometric structures corresponding to the three above situations are respectively called *parabolic, elliptic,* and *hyperbolic.*

The fundamental theorem relating the orbifold Euler characteristic to geometric structures is the following:

THEOREM 62. *Let X be a closed orbifold. Then X admits a geometric structure if and only if it is good, and more precisely,*

- *X admits a parabolic structure if and only if $\chi^{orb}(X) = 0$;*
- *X is bad or admits an elliptic structure if and only if $\chi^{orb}(X) > 0$;*
- *X admits a hyperbolic structure if and only if $\chi^{orb}(X) < 0$.*

PROOF. See theorem 13.3.6 in [**50**]. □

CHAPTER 5

SEIFERT SPACES

5.1. Basic definitions and facts

The *trivial fibred solid torus* is the space

$$D^2 \times S^1 = \{(re^{i\theta}, e^{i\phi}) | 0 \leq r \leq 1, 0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi\}$$

equipped with the product foliation by circles, that is, the foliation whose fibres are the $\{x\} \times S^1$ for x in D^2.

More generally, given coprime integers p and q with $0 \leq q \leq p/2$, the (p, q) *fibred solid torus* is defined to be the quotient of the space $D^2 \times I$, where I denotes the unit interval, under the identification

$$(re^{i\theta}, 0) \sim (re^{i(\theta + 2\pi\frac{q}{p})}, 1),$$

equipped with the induced foliation by circles. This space will be denoted $\mathbb{T}(p, q)$. It is a fact that $\mathbb{T}(p, q)$ is fibre-preservingly homemorphic to $\mathbb{T}(p', q')$ if and only if $p = p'$ and $q = q'$.

Similarly, the *fibred solid Klein bottle* is defined to be the quotient of the space $D^2 \times I$ under the identification

$$(re^{i\theta}, 0) \sim (re^{-i\theta}, 1),$$

equipped with the induced foliation by circles.

Note that $\mathbb{T}(p, q)$ is p-fold covered by the trivial fibred solid torus via the map

$$(re^{i\theta}, e^{i\phi}) \mapsto (re^{i\frac{q}{p}\theta}, e^{ip\phi}),$$

and that this covering induces an action of $\mathbb{Z}_p$ on $D^2 \times S^1$ by deck transformations.

Similarly, the fibred solid Klein bottle is twofold covered by the trivial fibred solid torus via the map

$$(re^{i\theta}, e^{i\phi}) \mapsto (re^{-i\theta}, -e^{i\phi}),$$

covering that induces an action of $\mathbb{Z}_2$ on $D^2 \times S^1$.

A three-manifold M is said to be *Seifert fibred* if can be expressed as a disjoint union of simple closed curves, called *fibres*, such that every fibre has a closed fibred neighbourhood (that is, a closed neighbourhood which is a union of fibres) which is fibre-homeomorphic to a fibred solid torus or to a fibred solid Klein bottle. A fibre is called *regular* if it has a fibred neighbourhood homeomorphic to a trivial fibred solid torus, and is called *exceptional* otherwise. Note that if an exceptional fibre has a fibred neighbourhood which is fibre-homeomorphic to a fibred solid Klein bottle, said fibre must be orientation-reversing in M.

The space obtained by identifying each fibre to a point is called the *base space* of M. It has a natural orbifold structure: topologically, it is a surface, as each fibred neighbourhood is sent onto a disk by the natural projection. Furthermore, each exceptional fibre corresponds to a cone point or to a reflector point: for example, if f is an exceptional fibre with fibred neighbourhood isomorphic to $\mathbb{T}(p, q)$, we can express $\mathbb{T}(p, q)$ as the quotient of $\mathbb{T}(1, 0)$ by the action of $\mathbb{Z}_p$ described above. This action factors through the natural projection map from $\mathbb{T}(1, 0)$ (viewed as a Seifert fibred space) onto its base space D^2 and expresses the base space of $\mathbb{T}(p, q)$ as the quotient $D^2/\mathbb{Z}_p$.

We end this section with a fundamental result about Seifert fibred spaces which will be useful later. We denote the non-orientable S^1-bundle over S^2 by $S^1 \tilde{\times} S^2$.

THEOREM 63. *Let M be a compact, connected Seifert fibred space.*

- *M is either $S^1 \times S^2$, $S^1 \tilde{\times} S^2$, $P^3 \# P^3$, or is irreducible.*
- *If M is irreducible, then M is $P^2 \times S^1$ or is P^2-irreducible.*

PROOF. See proposition 4.1 in [**9**]. □

5.2. The Seifert invariants

The following two sections are based on the exposition given in [**26**]. Let M be a closed, oriented, Seifert fibred manifold. Note that because M is oriented, it has no orientation-reversing singular fibres. Therefore B is an orbifold whose singularities are all cone points, and its underlying surface S is closed and connected (but possibly non-orientable). Furthermore, by compactness of M, there are only finitely many singular fibres, as M can be covered by interiors of fibred neighbourhoods. We will describe a set of invariants of M that, when suitably normalized, characterize it completely. We first treat the case where the base orbifold B of M is oriented.

Given such an M, we let $f_1, \ldots f_n$ denote a nonempty set of fibres of M containing the exceptional fibres of M. For each i, let F_i denote a neighbourhood of f_i which is homeomorphic to a fibred solid torus, and let D_i denote the image of F_i under the canonical projection to the base orbifold. Set $M_0 = M - (\mathring{F}_1 \cup \ldots \cup \mathring{F}_n)$ and $B_0 = B - (\mathring{D}_1 \cup \ldots \cup \mathring{D}_n)$. Then the restricted projection $M_0 \to B_0$ is an S^1-bundle over a compact oriented surface with nonempty boundary, and must therefore be trivial. In particular, it admits a section s. Let h_i denote the homology class in $H_1(\partial F_i)$ represented by a regular fibre, and let q_i denote the homology class in $H_1(\partial F_i)$ represented by $s(\partial B_i)$. Letting l denote the positive generator of $H_1(F_i)$, the *Seifert invariants* of f_i are defined by the following relations:

- $h_i := \alpha_i l_i$;
- $q_i := -\beta_i l_i$.

The Seifert invariants of M with respect to the section s are defined to be

$$(g, (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)),$$

where g is the genus of the surface S.

As can be seen from the construction, these invariants depend on the choice of section s. We now describe a normalization of the Seifert invariants which will allow us to assign a unique invariant to each oriented Seifert manifold. Given a trivialization of the S^1-bundle $M_0 \to B_0$, we can assume that the section s chosen above has the form $x \mapsto (x, 1)$. Another section $\tilde{s}$ of the bundle would then take the form $x \mapsto (x, f(x))$ where f is a map from B_0 into S^1. As varying f within its homotopy class does not affect the α_i and the β_i, we are interested in determining which homotopy classes can arise for f. Recall that the homotopy classes of maps of a manifold X into S^1 are represented by the elements of $H^1(X, \mathbb{Z})$.

LEMMA 64. *Writing $f_i = f|_{\partial D_i}$, let $(\deg f_1, \ldots \deg f_n)$ denote the homotopy class of $f|_{\partial B_0}$ in $H^1(\partial B_0, \mathbb{Z}) \cong \mathbb{Z}^n$. Then f comes from a section of the bundle $M_0 \to B_0$ if and only if $\sum_{i=1}^n \deg f_i = 0$.*

PROOF. We consider the following diagram, in which both rows are exact, and where g is the map $(m_1, \dots m_n) \mapsto \sum_{i=1}^n m_i$:

$$\begin{tikzcd}
H^1(B_0) \arrow[r] \arrow[d, "\cong"] & H^1(\partial B_0) \arrow[r, "g"] \arrow[d, "\cong"] & H^2(B_0, \partial B_0) \arrow[d, "\cong\text{(Lefschetz duality)}"] \\
H_1(B_0, \partial B_0) \arrow[r] & H_0(\partial B_0) \arrow[r] \arrow[d, "\cong"] & H_0(B_0) \arrow[d, "\cong\text{(by connectedness of } B_0)"] \\
 & \mathbb{Z}^n & \mathbb{Z}
\end{tikzcd}$$

By exactness, an element of $H^1(\partial B_0, \mathbb{Z})$ comes from an element of $H^1(B_0, \mathbb{Z})$ if and only if $\sum_{i=1}^n \deg f_i = 0$. □

Thus when passing from s to s', we replace each q_i by $q_i' = q_i + m_i h_i$. Expressing this in terms of l_i, we get $q_i' = (\alpha_i m_i - \beta_i) l_i$ so that the Seifert invariants of f_i with respect to s' are $(\alpha_i, \beta_i - \alpha_i m_i)$. This allows us to replace each β_i by $\beta_i - \alpha_i m_i$, subject only to the condition that the m_i must sum to zero.

In the case where the base orbifold B is non-orientable, we proceed similarly: expressing the underlying surface S as a connected sum $S = T\#R$ where T is orientable and R is P^2 or a Klein bottle, we may assume that the singular fibres of M all lie over T. Using the same notation for M_0 and for D_i as above, and setting $T_0 = T - (\mathring{D}_1 \cup \dots \cup \mathring{D}_n)$, we get an orientable S^1-bundle over $T_0\#R$, which admits a section s with respect to which we have Seifert invariants $(g, (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$. Note that in this case the genus g of S is negative. The Seifert invariants can then be normalized exactly as above. We summarize this discussion in the following proposition:

PROPOSITION 65. *Let M be a closed, oriented Seifert fibred manifold with orientable base orbifold. Then M admits a unique normalized Seifert invariant, given by*

$$(g; (1, \beta_0), (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)),$$

where g denotes the genus of the base orbifold of M and we have $0 < \beta_i < \alpha_i$ for $i = 1, \dots, n$.

Given a Seifert space with Seifert invariants $(g; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$, we will write $M = M(g; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$. If the invariants are normalized, so that they take the form $(g;(1,\beta_0),(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ with $0<\beta_i<\alpha_i$ for $i=1,\ldots,n$, we will set $b:=\beta_0$ and write $M=M(g,b;\frac{\beta_1}{\alpha_1},\ldots,\frac{\beta_n}{\alpha_n})$.

We have thus established a one-to-one correspondence between closed, oriented Seifert manifolds and their normalized Seifert invariants. It is natural to ask whether different Seifert invariants yield non-homeomorphic manifolds. This is not true in general: perhaps unsurprisingly at this point, lens spaces offer counterexamples. Before proceeding, we introduce the notation $L(0,1)$ for the space $S^2\times S^1$ (recall that we defined the lens space $L(p,q)$ only for $p>0$). Despite this notation, we will *not* be considering $S^2\times S^1$ as a lens space.

Proposition 66. *$L(p,q)$ is homeomorphic to $M(0;\beta_1/\alpha_1,\beta_2/\alpha_2)$, for any choice of α_i,β_i satisfying*

$$p=\det\begin{bmatrix}\alpha_1 & \alpha_2\\ -\beta_1 & \beta_2\end{bmatrix}\ \textit{and}\ q=\det\begin{bmatrix}\alpha_1 & \alpha_2'\\ -\beta_1 & \beta_2'\end{bmatrix},$$

where α_2' and β_2' are given by the relation $\det\begin{bmatrix}\alpha_2 & \alpha_2'\\ \beta_2 & \beta_2'\end{bmatrix}=1.$

Proof. We refer the reader to page 30 of **[26]** for a proof. □

Remark 67. By this result, any orientable Seifert manifold with base orbifold over S^2 with zero, one or two exceptional fibres which is not $S^2\times S^1$ is a lens space.

The above result can be used to show that lens spaces admit many different Seifert structures which are not related by fibre-preserving homeomorphisms. This situation is atypical among closed, orientable Seifert manifolds. The Seifert invariants allow us to state this precisely:

Theorem 68. *Let M be a closed, orientable Seifert manifold. Then M admits a unique Seifert structure up to isotopy, unless M is one of the following:*

- $M(0,1;\beta/\alpha)$;
- $M(0,1;1/2,1/2)$;
- $M(0,0;\alpha_1/\beta_1,\alpha_2/\beta_2)$;
- $M(-1,0;\beta/\alpha)$ *with α,β both nonzero;*
- $M(-2,0;)$

In particular, omitting the above cases, two closed, orientable Seifert manifolds with different normalized Seifert invariants cannot be homeomorphic.

Proof. This is almost entirely proven in **[22]**, with all gaps filled by appropriate references. □

5.3. The fundamental group of an orientable Seifert space

Theorem 69. *Let M be closed, oriented, and Seifert fibred, and write $M = M(g, b; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$. The fundamental group of M admits the following presentation:*

If $g \geq 0$, it is generated by a set $\{a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_n, h\}$ with the following relations:

- *h is central;*
- *$c_j^{\alpha_j} = h^{-\beta_j}$ for each j in $\{1, \ldots, n\}$;*
- *$[a_1, b_1] \ldots [a_g, b_g] c_1 \ldots c_n = h^b$.*

If $g < 0$, it is generated by a set $\{a_1, \ldots, a_{|g|}, c_1, \ldots, c_n, h\}$ with the following relations:

- *$a_j h a_j^{-1} = h^{-1}$ for each j in $\{1, \ldots, |g|\}$;*
- *$c_j h c_j^{-1} = h$ for each j in $\{1, \ldots, n\}$;*
- *$c_j^{\alpha_j} = h^{-\beta_j}$ for each j in $\{1, \ldots, n\}$;*
- *$a_1^2 \ldots a_{|g|}^2 c_1 \ldots c_n = h^b$.*

Proof. For $g \geq 0$, this follows by applying Van Kampen's theorem to the decomposition of M as

$$M = M_0 \cup F_0 \cup \ldots \cup F_n,$$

where the notation is the same as in the last section (with the exception that our indices for the F_i begin at zero because we are using normalized invariants). We may view M_0 as $B_0 \times S^1$, yielding the following presentation for $\pi_1(M_0)$:

$$\langle a_1, \ldots, a_g, b_1, \ldots, b_g, c_0, \ldots, c_n, h | [a_1, b_1] \ldots [a_g, b_g] c_0 \ldots c_n = 1, h \text{ is central} \rangle,$$

where the a_i and b_i are standard generators of $H_1(B)$, the c_i represent the boundary components of B_0, and h represents the S^1 fibre. We draw attention to the presence of an element c_0 in this presentation which does not appear in the theorem statement. By Van Kampen's theorem, gluing in each F_j adds a generator t_j as well as the relations

- $c_j^{\alpha_j} = h^{-\beta_j}$;
- $c_j^{\alpha'_j} = h^{-\beta'_j} t_j$.

The second of these relations expresses t_j in terms of the other generators, so that it can be omitted in the presentation. Note that for $(\alpha_0, \beta_0) = (1, b)$, we get

- $c_0 = h^{-b}$,

so that we may also get rid of c_0, writing

$$1 = [a_1, b_1] \dots [a_g, b_g] c_0 \dots c_n = [a_1, b_1] \dots [a_g, b_g] h^{-b} c_1 \dots c_n.$$

As h is central, this reduces to

$$1 = [a_1, b_1] \dots [a_g, b_g] c_1 \dots c_n h^{-b},$$

yielding the claimed presentation. The situation is analogous for $g < 0$. □

Corollary 70. *If $M = M(g, b; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$ with $g \geq 0$, then we have*

$$H_1(M, \mathbb{Z}) = \langle a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n, h | \alpha_j c_j + \beta_j h = 0, c_1 + \dots + c_n = -bh \rangle.$$

In particular:

- $H_1(M, \mathbb{Q}) = \mathbb{Q}^{2g+1}$ *if* $b = \sum_{i=1}^n \beta_i / \alpha_i$.
- $H_1(M, \mathbb{Z}) = \mathbb{Z}^{2g} \oplus H$ *with* $|H| = |b - \sum_{i=1}^n \beta_i / \alpha_i| \alpha_1 \dots \alpha_n$ *if* $b \neq \sum_{i=1}^n \beta_i / \alpha_i$.

Proof. The presentation given for $H_1(M, \mathbb{Z})$ is a direct consequence of the above theorem. Upon passing to coefficients in $\mathbb{Q}$, we may express each of the c_i in terms of h. If in addition we have $b = \sum_{i=1}^n \beta_i / \alpha_i$, the last relation in the above presentation becomes vacuous, so that $H_1(M, \mathbb{Q})$ is free over $(a_1, \dots, a_g, b_1, \dots, b_g, h)$.

If $b \neq \sum_{i=1}^{n} \beta_i/\alpha_i$, we view $\mathbb{Z}^{n+1}$ as a free group over a basis $(C_1, \ldots, C_n, H)$ and consider the map represented by the matrix

$$M = \begin{bmatrix} b & \beta_1 & \ldots & \beta_n \\ 1 & \alpha_1 & & \\ \vdots & & \ddots & \\ 1 & & & \alpha_n \end{bmatrix}$$

with respect to this basis. Then $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus H$ with $H \cong \mathrm{coker}(M)$, and the order of H is given by $|\det M\,|$. We can view M as a matrix with coefficients in $\mathbb{Q}$ to compute this determinant, obtaining

$$|\det M\,| = \begin{vmatrix} (b - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}) & \beta_1 & \ldots & \beta_n \\ 0 & \alpha_1 & & \\ \vdots & & \ddots & \\ 0 & & & \alpha_n \end{vmatrix} = |b - \sum_{i=1}^{n} \beta_i/\alpha_i| \alpha_1 \ldots \alpha_n.$$

□

As a second corollary of theorem 69, we obtain the following lemma relating the fundamental group of a Seifert fibred space to the orbifold fundamental group of its base space:

Lemma 71. *Let M be an orientable Seifert fibred with base orbifold B, and let h be an element of the fundamental group of M represented by a regular fibre. Then there is an exact sequence*

$$1 \to \langle h \rangle \to \pi_1(M) \to \pi_1^{orb}(B) \to 1.$$

Proof. This is a direct consequence of the presentations we have given for $\pi_1(M)$ and $\pi_1^{orb}(B)$: note that if $M = M(g, b; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$, quotienting $\pi_1(M)$ by $\langle h \rangle$ yields exactly $\pi_1^{orb}(B)$. □

Remark 72. For convenience, we have not defined the orbifold fundamental group for orbifolds with reflector curves, as we will not need such generality in what follows. Modulo such a definition, the above proposition remains true even if M is not assumed to be orientable.

Theorem 73. *Let M be a closed, orientable Seifert fibred space with base orbifold B and projection map p, and let $g : B' \to B$ be an orbifold covering. Then there* *exists an orientable Seifert fibred space M' with base orbifold B' and projection map p' as well as a covering map $\hat{g} : M' \longrightarrow M$ such that*

(1) $p \circ \hat{g} = g \circ p'$
(2) *for every regular fibre ϕ of M and for every connected component ϕ' of $\hat{g}^{-1}(f)$, the map $\hat{g}$ restricts to a homeomorphism from ϕ' to ϕ.*

Proof. Let $f_1, \ldots, f_n$ be the singular fibres of M and let $F_1, \ldots F_n$ and $D_1, \ldots, D_n$ be neighbourhoods as described in the beginning of section 5.2. Let

$$x_1 = p(f_1), \ldots, x_n = p(f_n)$$

denote the cone points of B. Set

- $B_0 = B - (\mathring{D}_1 \cup \ldots \cup \mathring{D}_n)$,
- $B_0' = g^{-1}(B_0)$,
- $g_0 = g|_{B_0'}$
- $M_0 = M - (\mathring{F}_1 \cup \ldots \cup \mathring{F}_n)$,
- $p_0 = p|_{M_0}$,

so that g_0 is a covering map between connected surfaces and p_0 is a locally trivial circle bundle.

We now set $M_0' = \{(f, x) : p_0(f) = g_0(x)\} \subseteq M_0 \times B_0'$ and define $\hat{g}_0$ and p_0' to be the projections of M_0' onto M_0 and onto B_0' respectively. Note that this is simply the standard pullback construction, so that by general fibre space theory, $\hat{g}_0$ is a covering map between 3-manifolds and p_0' is a locally trivial circle bundle. It is clear that if f is a fibre of M_0 and f' is a connected component of $\hat{g}_0{}^{-1}(f)$, then the map $\hat{g}_0$ restricts to a homeomorphism from f' to f. Note also that by construction, we have $p_0 \circ \hat{g}_0 = g_0 \circ p_0'$:

$$\begin{tikzcd} M_0' \arrow[r, "\hat{g}_0"] \arrow[d, "p_0'"'] & M_0 \arrow[d, "p_0"] \\ B_0' \arrow[r, "g_0"] & B_0 \end{tikzcd}$$

It remains to extend $\hat{g}_0$ and p_0' to obtain our desired covering and projection maps.

This can be done stepwise as follows: given x_i, fix a component D'_i of $g^{-1}(D_i)$. Set $T_i = \partial F_i$ and $T'_i = p_0^{-1}(\partial D'_i) \cong S^1 \times S^1$. Then T'_i is a connected component of $\hat{g}_0^{-1}(T_i)$ and $\hat{g}_0|_{T'_i}$ is a covering map with image T_i. As the x_j for $j \neq i$ and their associated neighbourhoods will not intervene in the rest of the proof, we will drop the index i. Summarizing, we have the following commutative diagram:

$$\begin{tikzcd} T' \arrow[r, "\hat{f}"] \arrow[d, "\rho'"'] & T \arrow[d, "\rho"] \\ D' \arrow[r, "h"] & D \end{tikzcd}$$

where $\hat{f} = \hat{g}_0|_{T'}$, $h = g_0|_{D'}$, $\rho' = p'_0|_{T'}$, and $\rho = p_0|_T$ (this choice of notation will be made clear later on).

By hypothesis, h is an orbifold covering between the orbifolds D' with one cone point at $y := h^{-1}(x)$, which we shall denote $D'(y)$, and D with one cone point at x, which we shall denote $D(x)$. Let β and α denote the respective orders of y and x as cone points. By the general theory of orbifold coverings, β divides α and the map h restricts to a $\delta := \alpha/\beta$-fold covering map outside of y and x. In particular, the map

$$f := h|_{C'} : C' \to C,$$

where $C' = \partial D'$ and $C = \partial D$, is a δ-fold covering map. Thus we have determined the order of the covering map $\hat{f} : T' \to T$: it is also a δ-fold covering map by commutativity of the above diagram.

Our objective is to fill T' with a solid torus F' and construct a covering map $\hat{h} : F' \to F$ which restricts to $\hat{f}$ on T'. We do this as follows: let μ be a meridian of F. The connected components of $\hat{f}^{-1}(\mu)$ are disjoint essential simple closed curves in $\tilde{T}$, hence are isotopic. Letting $\tilde{\mu}$ denote any one of these components, we glue a solid torus F' into T' by identifying its meridian to $\tilde{\mu}$ (this operation is not affected by varying $\tilde{\mu}$ in its isotopy class). Note that this also identifies $D(y)$ with a meridian disk in F'. In order to extend the domain of $\hat{f}$ from T' to F', we must show that that $\hat{f}^{-1}(\mu)$ has exactly δ connected components in T', so that each of them maps homeomorphically onto μ.

Let c be the number of connected components of $\hat{f}^{-1}(\mu)$, and let ϕ be some regular fibre of T. When placed in minimal intersection position, μ and ϕ intersect in precisely α points of same sign (as x is of order α). Hence $\hat{f}^{-1}(\mu)$ and $\hat{f}^{-1}(\phi)$

intersect in precisely $\delta\alpha$ points of same sign. Now $\hat{f}^{-1}(\phi)$ has precisely δ connected components, as $\hat{f}$ is a δ-fold covering map. Letting $\tilde{\phi}$ denote a connected component of $\hat{f}^{-1}(\phi)$, we see that $\hat{f}^{-1}(\phi)$ represents δ times ϕ on the level of homology. Similarly, $\hat{f}^{-1}(\mu)$ represents c times $\tilde{\mu}$. Finally, because the projection map from T' to C has degree β (as y is a cone point of order β), $\tilde{\mu}$ and $\tilde{\phi}$ intersect in precisely β points. Thus, denoting the intersection number of two curves γ and γ' by $\#(\gamma, \gamma')$, we have

$$\delta\alpha = \#(\hat{f}^{-1}(\mu), \hat{f}^{-1}(\phi)) = c\delta\#(\tilde{\mu}, \tilde{\phi}) = c\delta\beta,$$

so that $c = \alpha/\beta = \delta$. We may therefore extend $\hat{f}$ to a δ-fold covering map $\hat{h} : F' \to F$.

We then equip F' with a Seifert fibre structure by pulling back the fibres of F via $\hat{h}$. As $D(y)$ is identified with a meridian disk of F', each fibre of F' can be collapsed to a point in $D(y)$, yielding a projection $F' \to D(y)$. By construction, the diagram

$$\begin{tikzcd} F' \arrow[r, "\hat{h}"] \arrow[d, "\rho'"'] & F \arrow[d, "\rho"] \\ D' \arrow[r, "h"] & D \end{tikzcd}$$

is commutative and restricts to

$$\begin{tikzcd} T' \arrow[r, "\hat{f}"] \arrow[d, "\rho'"'] & T \arrow[d, "\rho"] \\ C' \arrow[r, "f"] & C \end{tikzcd}$$

on the boundaries, so that we have effectively extended the domain of $\hat{g}_0$ and p'_0 to $M'_0 \cup p^{-1}(\mathring{F})$.

Proceeding as above with each of the x_i, after n steps we will have extended $\hat{g}_0$ and p'_0 to M', obtaining maps $\hat{g}$ and p' as well as a commutative diagram of the form

$$\begin{tikzcd}
M' \arrow[rrr, "\hat{g}"] \arrow[dr] \arrow[ddd, "p'"'] & & & M \arrow[dl] \arrow[ddd, "p"] \\
& M'_0 \arrow[r, "\hat{g}_0"] \arrow[d, "p'_0"'] & M_0 \arrow[d, "p_0"] & \\
& B'_0 \arrow[r, "g_0"] & B_0 & \\
B' \arrow[rrr, "g"] \arrow[ur] & & & B \arrow[ul]
\end{tikzcd}$$

as desired. Property (2) of the theorem is clear from the construction. □

THEOREM 74. *Let M be a closed, orientable Seifert fibred space with base orbifold $S^2(\alpha_1, \alpha_2, \alpha_3)$ which is not $S^2 \times S^1$. Then $\pi_1(M)$ is finite if and only if $(\alpha_1, \alpha_2, \alpha_3)$ satisfies*

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1.$$

PROOF. Examining the presentation given above for the case $g = 0$ and $n = 3$, we get

$$\pi_1^{orb}(B) = \langle c_1, c_2, c_3 | c_1^{\alpha_1} = c_2^{\alpha_2} = c_3^{\alpha_3} = c_1c_2c_3 = 1\rangle = \Delta(\alpha_1, \alpha_2, \alpha_3).$$

By proposition 14 this group is infinite if and only if $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \leq 1$. Thus if $\pi_1(M)$ is finite, $\pi_1^{orb}(B)$ must be finite by lemma 71, yielding $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$.

It remains to see that when $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$, $\pi_1(M)$ is finite. When this is the case, $\pi_1^{orb}(B)$ can be identified with a finite isometry group of S^2, realizing B as a quotient of S^2, thus yielding an orbifold covering map $S^2 \to B$ (see pages 413 and 425 of [**45**] for details). Theorem 73 then yields an orientable Seifert space M' with base space S^2 and no exceptional fibres, as well as a covering map from M' to M. Thus M' must be a lens space by the exclusion of $S^1 \times S^2$ and by remark 67, and as such is universally covered by S^3. It then follows from the compactness of S^3 that M has finite fundamental group. Indeed, M is universally covered by S^3, so that by fixing a point p in M, there exists a bijection between $\pi_1(M)$ and the preimage γ of p via the covering map from S^3 to M. Because γ is discrete and closed in S^3, it is finite. □

Theorem 75. *Let M be a closed oriented Seifert manifold which is not $S^2 \times S^1$. Then M has finite fundamental group if and only if M has base orbifold one of the following:*

(1) $S^2(\alpha_1)$;
(2) $S^2(\alpha_1, \alpha_2)$;
(3) $S^2(\alpha_1, \alpha_2, \alpha_3)$ *with* $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$;
(4) $P^2(\alpha_1)$.

Proof. By the presentations given above, M has infinite fundamental group if the surface S underlying its base orbifold is not S^2 or P^2. Indeed, we have the following string of surjective group homomorphisms:

$$\pi_1(M) \longrightarrow \pi_1^{orb}(B) \longrightarrow \pi_1(S) \longrightarrow H_1(S),$$

so that we may restrict to the cases where $S = S^2$ or $S = P^2$.

If $S = S^2$, by remark 67, the Seifert spaces corresponding to cases 1 and 2 in the theorem statement are S^3 or are lens spaces. Case 3 is covered by theorem 74. If M has more that four exceptional fibres, the group $\pi_1^{orb}(B)$ has presentation

$$\langle c_1, \ldots, c_n | c_1^{\alpha_1} = \ldots = c_n^{\alpha_n} = c_1 \ldots c_n = 1 \rangle,$$

with $n \geq 4$, and this group can be realized as the conformal subgroup Γ^* of the group Γ generated by the reflections through the sides of a hyperbolic (resp. euclidean for $n = 4$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2$) n-gon Δ with angles $\frac{\pi}{\alpha_1}, \ldots, \frac{\pi}{\alpha_n}$. The group Γ is infinite: in fact, it can be shown that it acts on $\mathbb{H}^2$ (resp $\mathbb{R}^2$) with fundamental domain Δ, yielding a tiling whose tiles are in one-to-one correspondence with the elements of Γ. See **[31]** for details in the hyperbolic case. As Γ^* is of index two in Γ, it is also infinite, so that ultimately $\pi_1(M)$ is as well.

For the case where $S = P^2$, we refer the reader to chapters 9 and 10 of the final section of **[47]**, where case 4 is proved by studying covering maps between Seifert spaces. □

We end this section with a lemma about Seifert spaces with base orbifolds over S^2 or P^2, which will be useful later.

LEMMA 76. *Suppose that* $M = M(g; b, \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$ *where* $g = 0$ *or* $g = -1$. *Let* $\phi : \pi_1(M) \to Homeo_+(\mathbb{R})$ *be a group homomorphism. Then* $\phi(h)$ *is conjugate to translation by* ± 1 *if and only if the action induced by* ϕ *has no global fixed points.*

PROOF. If $\phi(h)$ is conjugate to translation by ± 1, then the action induced by ϕ cannot have any global fixed points. Conversely, suppose that the action induced by ϕ has no global fixed points. By lemma 38, it suffices to show that $\phi(h)$ has no fixed points. We proceed by contradiction: suppose there is a $t \in \mathbb{R}$ such that $\phi(h)(t) = t$. By theorem 69, we have

$$\phi(\gamma_i)^{\alpha_i}(t) = \phi(h)^{-\beta_i}(t) = t$$

for all i, $1 \leq i \leq n$. Because $\phi(\gamma_i)$ is orientation-preserving, it must also fix t. If $g = -1$, we also have

$$\phi(a_1)^2(t) = \phi(a_1)^2\phi(\gamma_1)\ldots\phi(\gamma_n)(t) = \phi(h)^b(t) = t,$$

so that $\phi(a_1)$ also fixes t. But this means that t is a global fixed point for the action of M, which is a contradiction. Thus $\phi(h)$ cannot have any fixed points. □

5.4. Seifert spaces, homology spheres, and torus knots

Corollary 70 allows us to determine exactly when a Seifert fibred space is an integer homology sphere:

THEOREM 77. *Let* $M = M(g, b; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$ *be a Seifert-fibred integer homology sphere. Then* M *has base orbifold over* S^2 *with exceptional fibres* $f_1, \ldots, f_n$. *Writing* (α_i, β_i) *for the Seifert invariant of* f_i, *we have* $\gcd(\alpha_i, \alpha_j) = 1$ *whenever* $i \neq j$. *Conversely, given any list* $\alpha_1, \ldots, \alpha_n$ *of integers satisfying* $\gcd(\alpha_i, \alpha_j) = 1$ *whenever* $i \neq j$, *there exists a unique integer homology sphere with base orbifold* $S^2(\alpha_1, \ldots, \alpha_n)$.

PROOF. Let M be an integer homology sphere and write $M = M(g, b; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$. First note that if we had $g < 0$, the underlying surface of the base orbifold of M would admit a double cover, which would lift to a double cover of M, which would imply that $H^1(M; \mathbb{Z}_2)$ is nontrivial as it surjects onto $\mathbb{Z}_2$. Combining this with corollary 70, we see that we must have $b \neq \sum_{i=1}^{n} \beta_i/\alpha_i$ and $g = 0$ for otherwise, M would not even be a rational homology sphere. We have

$$1 = |H_1(M)| = |b - \sum_{i=1}^{n} \beta_i/\alpha_i|\alpha_1 \ldots \alpha_n = |b\alpha_1 \ldots \alpha_n - \sum_{i=1}^{n} \beta_i\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n|,$$

where $\hat{\alpha}_i$ indicates that α_i is to be omitted from the expression. This gives a Bezout relation for the set of $n+1$ integers $\{b\alpha_1 \ldots \alpha_n\} \cup \{\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n\}_{1\le i\le n}$, which implies that $\gcd(\alpha_i, \alpha_j) = 1$ whenever $i \neq j$. Indeed, if d divides α_i and α_j with $i \neq j$, then d must divide $b\alpha_1 \ldots \alpha_n - \sum_{i=1}^{n} \beta_i\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n = \pm 1$, so that $d = \pm 1$.

Conversely, given a list $\alpha_1, \ldots, \alpha_n$ of integers satisfying $\gcd(\alpha_i, \alpha_j) = 1$ whenever $i \neq j$, we may find integers B and γ_i $(1 \le i \le n)$ such that $B\alpha_1 \ldots \alpha_n - \sum_{i=1}^{n} \gamma_i\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n = 1$, and by the forward direction of the proof, setting $M = M(0, \frac{B}{1}; \frac{\gamma_1}{\alpha_1}, \ldots, \frac{\gamma_n}{\alpha_n}) = M(0, b; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$, we obtain an integer homology sphere. This solution is unique: indeed, given another sequence of integers B' and γ_i' $(1 \le i \le n)$ such that

$$B'\alpha_1 \ldots \alpha_n - \sum_{i=1}^{n} \gamma_i'\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n = 1,$$

we write $M' = M'(0, \frac{B'}{1}; \frac{\gamma_1'}{\alpha_1}, \ldots, \frac{\gamma_n'}{\alpha_n}) = M'(0, b'; \frac{\beta_1'}{\alpha_1}, \ldots, \frac{\beta_n'}{\alpha_n})$. By the forward direction of the proof, we have

$$b\alpha_1 \ldots \alpha_n - \sum_{i=1}^{n} \beta_i\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n = 1$$

as well as

$$b'\alpha_1 \ldots \alpha_n - \sum_{i=1}^{n} \beta_i'\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n = 1.$$

Subtracting the second of these expressions from the first and reducing modulo α_i yields $\beta_i - \beta_i' \equiv 0 \mod \alpha_i$ for each i. As these are normalized Seifert invariants, they satisfy $0 < \beta < \alpha_i$ and $0 < \beta_i' < \alpha_i$ for each i, so that $\beta_i = \beta_i'$ for each i and thus $b = b'$ as well.

We note that for $n \le 2$, the integer homology sphere obtained is S^3, as otherwise we would obtain a lens space, and for $n \ge 3$, we obtain distinct Seifert manifolds following our choices of α_i by theorem 68. □

The above result implies that Seifert fibred integer homology spheres generically have hyperbolic base orbifolds. Specifically, we have the following:

PROPOSITION 78. *Let M be a Seifert fibred integer homology sphere. If M is neither S^3 nor the Poincaré homology sphere, then its base orbifold is hyperbolic.*

PROOF. By theorem 77, M has base orbifold S^2 with n exceptional fibres, and if $n \leq 2$, then M is S^3. If $n \geq 4$, then either its base orbifold is hyperbolic by theorem 62, or $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2$, in which case it cannot be an integer homology sphere because the α_i are not coprime. If $n = 3$, then its base orbifold is hyperbolic unless $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \geq 1$ (again by by theorem 62). The only triple $(\alpha_1, \alpha_2, \alpha_3)$ satisfying this condition with the α_i pairwise coprime is $(2, 3, 5)$, and in this case M is the Poincaré homology sphere (see **[29]** for more details on this manifold). □

COROLLARY 79. *Let M be a Seifert-fibred integer homology sphere. If M is neither the Poincaré homology sphere nor S^3, then its fundamental group is left-orderable.*

PROOF. Let M be as stated. By the above proposition, the base orbifold B of M is hyperbolic, so that $\pi_1^{orb}(B) = \pi_1(M)/\langle h\rangle$ is a nontrivial group of hyperbolic isometries, i.e. a nontrivial subgroup of $PSL_2(\mathbb{R}) \subset Homeo_+(S^1)$. By proposition 43, $\pi_1(M)$ is left-orderable. □

The branched cyclic covers of torus knots are always Seifert fibred, and a complete description of their Seifert invariants is known. We will not state the full result here; we refer the reader to **[37]**. We state only the cases that will be necessary for our purposes later:

THEOREM 80. *Let n, p, and q be integers with p and q coprime such that $0 \leq q < p$. Let d, p be such that $dp + qb = -1$. Then $\Sigma_n T_{p,q}$ is Seifert fibred. Suppose that $\gcd(pq, n) = a \neq 1, n$, and take m such that $\frac{pq}{a}m \equiv 1 \mod n/a$. Let k be defined by $\frac{pq}{a}m = 1 - k\frac{n}{a}$.*

(1) *If $\gcd(p, n) = 1$, then*
$$\Sigma_n T_{p,q} = M\left(0; \frac{-m}{n/a}, \frac{dk}{q/a}, \frac{bk}{p}, \dots, \frac{bk}{p}\right),$$
where $\frac{bk}{p}$ is repeated a times.

(2) *If* $\gcd(p,n) = a$, *then*

$$\Sigma_n T_{p,q} = M\left(0; \frac{-m}{n/a}, \frac{dk}{q}, \dots, \frac{dk}{q}, \frac{bk}{p/a}, \right),$$

where where $\frac{dk}{q}$ *is repeated a times.*

As a corollary of the main result in **[37]**, we have the following:

COROLLARY 81. $\Sigma_n T_{p,q}$ *is an integer homology sphere if and only if* $\gcd(pq, n) = 1$.

CHAPTER 6

FOLIATIONS

6.1. Basic definitions and general results

Let M be a 3-dimensional manifold. A *foliated chart of codimension* k (with $k = 1$ or $k = 2$) is an open set U in M together with a diffeomorphism

$$\varphi : U \to B_\tau \times B_\pitchfork \subset \mathbb{R}^{3-k} \times \mathbb{R}^k,$$

where

- If $k = 1$, $B_\pitchfork$ is an interval in $\mathbb{R}$, and B_τ is of the form $J_1 \times J_2$, where the J_i are intervals in $\mathbb{R}$;
- If $k = 2$, B_τ is an interval in $\mathbb{R}$, and $B_\pitchfork$ is of the form $J_1 \times J_2$, where the J_i are intervals in $\mathbb{R}$.

Given a foliated chart (U, φ), the sets

$$P_s = \varphi^{-1}(B_\tau \times \{s\})$$

and

$$S_t = \varphi^{-1}(\{t\} \times B_\pitchfork)$$

are called, respectively, *plaques* and *transversals* of the foliated chart.

Let $\mathcal{F} = \{F_\lambda\}_{\lambda \in \Lambda}$ be a decomposition of M into connected, immersed surfaces, and suppose that M admits an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ of foliated charts of codimension k such that for each $\alpha \in \mathcal{A}$ and for each $\lambda \in \Lambda$, the set $L_\lambda \cap U_\alpha$ is a union of plaques. Then $\mathcal{F}$ is said to be a *foliation of codimension* 1, and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is called a foliated atlas associated to $\mathcal{F}$.

We call each L_λ a *leaf* of the foliation. Given two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), we can write the transition maps $\phi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}$ as

$$\phi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

$$(x, y, z) \mapsto (\psi_{\alpha\beta}(x, y, z), z_{\alpha\beta}(x, y, z)).$$

We call the maps $z_{\alpha\beta}$ the *transverse coordinate changes* of the two given charts.

As is standard for objects defined on manifolds, an equivalence relation is defined on foliated atlases in such a way that equivalent foliated atlases correspond to the same foliation. We refer the interested reader to **[11]** for a careful treatment of foliations. The important fact for our purposes is that for a given foliation $\mathcal{F}$, we can choose a foliated atlas $\mathcal{U}$ such that it satisfies the following conditions:

- if P and Q are two plaques in distinct charts of $\mathcal{U}$, then $P \cap Q$ is open in P and in Q;
- the transverse coordinate changes $z_{\alpha\beta}(x, y, z)$ are locally independent of (x, y) (for the case $k = 1$) or of x (for the case $k = 2$);
- the transverse coordinate changes $z_{\alpha\beta}$ satisfy the cocyle conditions:

$$z_{\alpha\beta} \circ z_{\beta\gamma} = z_{\alpha\gamma}$$

Definition 82. A foliation $\mathcal{F}$ is called *co-orientable* if $\det z_{\alpha\beta} > 0$ for all $\alpha, \beta \in \mathcal{A}$, that is, if the transverse foliation to $\mathcal{F}$ can be coherently oriented.

The space obtained by identifying each leaf of a foliation to a point is called the *leaf space* of the foliation. Its topology can be quite complicated in general. We will not need to know much about the leaf space of a foliation, but we will need the following definition:

Definition 83. Let $\mathcal{F}$ be a codimension one foliation on a three-dimensional manifold M. Let $\tilde{\mathcal{F}}$ denote the pullback foliation of $\mathcal{F}$ to the universal cover of M (whose leaves are the connected components of the preimages of leaves of $\mathcal{F}$ via the projection map). Then $\mathcal{F}$ is said to be $\mathbb{R}$*-covered* if the leaf space of $\tilde{\mathcal{F}}$ is homeomorphic to $\mathbb{R}$. Given such a foliation, the action of $\pi_1(M)$ on $\tilde{\mathcal{F}}$ induces a group homomorphism $\phi : \pi_1(M) \to Homeo(\mathbb{R})$.

Lemma 84. *Let M be a compact, connected three manifold, and let $\mathcal{F}$ be a co-oriented foliation of M which is $\mathbb{R}$-covered. Then, with the notation of the previous definition, the image of ϕ lies in $Homeo_+(\mathbb{R})$.*

Proposition 85. *Let M be a compact, connected, P^2-irreducible manifold which admits a co-oriented $\mathbb{R}$-covered foliation. Then M has left-orderable fundamental group.*

PROOF. See proposition 5.3 of [**9**]. □

DEFINITION 86. A foliation is called *taut* if there exists a single simple closed curve which is everywhere transverse to the leaves of the foliation.

LEMMA 87. *A transverse loop to a taut foliation has infinite order in the fundamental group.*

PROOF. See section 3.1 of [**34**]. □

THEOREM 88. *Suppose $b_1(M) > 0$. Then M admits a taut foliation.*

PROOF. This is theorem 5.5 of [**18**]. □

We end this section with a theorem relating L-spaces to taut foliations, first proved in [**42**]. The proof was reliant on a result due to Eliashberg and Thurston [**16**] concerning approximations of on C^2 taut foliations by contact structures. This was recently extended to C^0 foliations in [**28**] and in [**5**].

THEOREM 89. *Suppose M admits a co-orientable taut foliation. Then M is not an L-space.*

6.2. Foliations and Seifert manifolds

DEFINITION 90. Let M be compact and Seifert fibred. A *horizontal foliation* is a codimension 1 foliation that is everywhere transverse to the Seifert fibres.

LEMMA 91. *Horizontal foliations are taut. In particular, if M admits a horizontal foliation, then $\pi_1(M)$ has infinite fundamental group.*

PROOF. This is by definition, because a Seifert fibre is a circle. □

Note that this immediately shows that Seifert fibred spaces need not admit horizontal foliations. In fact, we know exactly when a Seifert manifold admits a horizontal foliation; we consider only the orientable case as this is all that we will need later.

THEOREM 92. *Let $M = M(0, b; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$ with $n \geq 3$. Then M admits a horizontal foliation if and only if one of the following holds:*

(1) $-(n-2) \leq b \leq -2$;

(2) $b = -1$ *and there exist integers* x, y *with* $0 < x < y$ *such that after some permutation of the* $\frac{\beta_i}{\alpha_i}$, *we have*
- $\frac{\beta_1}{\alpha_1} < \frac{x}{y}$;
- $\frac{\beta_2}{\alpha_2} < \frac{y-x}{y}$;
- $\frac{\beta_i}{\alpha_i} < \frac{1}{y}$ *for* $3 \leq i \leq n$;

(3) $b = -(n-1)$ *and condition* 2 *holds for* $M(0, -1; \frac{\alpha_1-\beta_1}{\alpha_1}, \ldots, \frac{\alpha_n-\beta_n}{\alpha_n})$.

Proof. This is due to a combination of papers: see **[15, 27, 36]** as well as **[9]** and **[20]** for details. □

The following results will be useful later

Proposition 93. *Let M be a compact, connected, orientable Seifert manifold, and let* $\mathcal{F}$ *be a horizontal foliation on M. Then* $\mathcal{F}$ *is co-orientable if and only if the underlying surface S to the base orbifold of M is orientable.*

Proof. Let M and $\mathcal{F}$ be as in the theorem statement, and let h be a regular fibre of M. Then the tangent field to $\mathcal{F}$ restricted to a fibred neighbourhood F of h is just the pullback of the tangent bundle of S restricted to the image of F under the Seifert projection. Thus the orientability of $\mathcal{F}$ is equivalent to the orientability of S. On the other hand, since M is orientable, the orientability of $\mathcal{F}$ is also equivalent to the co-orientability of the foliation transverse to $\mathcal{F}$. □

Proposition 94. *Let M be a closed, connected,* P^2*-irreducible Seifert fibred manifold with infinite fundamental group. Then every horizontal foliation on M is* $\mathbb{R}$*-covered.*

Proof. See lemma 5.6 of **[9]**. □

CHAPTER 7

THE L-SPACE CONJECTURE

7.1. When Seifert manifolds have left-orderable fundamental groups

The goal of this section is to prove the following theorem:

THEOREM 95. *Let M be a compact, connected, Seifert fibred three-manifold. Then M has a left-orderable fundamental group if and only if one of the following holds:*

- $b_1(M) > 0$ and M is not $P^2 \times S^1$
- $b_1(M) = 0$, M is orientable, has base orbifold over S^2, and admits a horizontal foliation.

The proof will be split up into several pieces. First we deal with the case where M is not P^2-irreducible:

PROPOSITION 96. *Let M be a compact, connected, Seifert fibred three-manifold which is not P^2-irreducible. Then either*

- *M is $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$ so has nonzero first betti number and left-orderable fundamental group (equal to $\mathbb{Z}$); or*
- *M is $P^2 \times S^1$, has nonzero first betti number, base orbifold over P^2, and has non left-orderable fundamental group (equal to $\mathbb{Z}_2 \times \mathbb{Z}$); or*
- *M is $P^3 \# P^3$, has zero first betti number, is orientable, and has non left-orderable fundamental group (equal to $\mathbb{Z}_2 * \mathbb{Z}_2$).*

PROOF. This essentially follows from theorem 63: if M is as in the statement, then either

- M is reducible, in which case it is $P^3 \# P^3$, $S^2 \times S^1$, or $S^2 \tilde{\times} S^1$, or

- M is irreducible, in which case it is not P^2-irreducible if and only if it is nonorientable and its fundamental group contains an element of order two, by theorem 13. The only such Seifert manifold is $P^2 \times S^1$ by theorem 63.

The statements about the fundamental groups of these spaces are standard and will not be proved here. □

PROPOSITION 97. *Let M be a compact, connected, Seifert-fibred three-manifold with $b_1(M) > 0$. Then M has a left-orderable fundamental group if and only if M is not $P^2 \times S^1$.*

PROOF. If $\pi_1(M)$ is left-orderable, then M cannot be $P^2 \times S^1$. Conversely, suppose M is not $P^2 \times S^1$. Then M either has fundamental group equal to $\mathbb{Z}$ (which is left-orderable) or is P^2-irreducible by the previous proposition. In the latter case, M also has left-orderable fundamental group by corollary 42. □

We now turn to the case when $b_1(M) = 0$.

THEOREM 98. *Let M be a compact, connected, P^2-irreducible Seifert-fibred three-manifold with $b_1(M) = 0$. Then the fundamental group of M is left-orderable if and only if M is orientable, has base orbifold over S^2, and admits a horizontal foliation.*

PROOF. Suppose M is orientable, has base orbifold over S^2, and admits a horizontal foliation $\mathcal{F}$. By proposition 93, $\mathcal{F}$ is co-orientable because the base orbifold is orientable. By proposition 94, $\mathcal{F}$ is $\mathbb{R}$-covered. Thus by proposition 85, M has left-orderable fundamental group.

Conversely, suppose M has left-orderable fundamental group. By lemma 12, M is closed and orientable. To see that the base orbifold B must be S^2 or P^2, we consider the following composition of group homomorphisms:

$$\pi_1(M) \to \pi_1(M)/\langle h \rangle = \pi_1^{orb}(B) \to \pi_1(B) \to H_1(B),$$

where every map is the canonical surjection, and h is an element of $\pi_1(M)$ represented by a Seifert fibre. This gives a surjective map from $\pi_1(M)$ onto the abelian group $H_1(B)$, which yields a surjective map of $H_1(M)$ onto $H_1(B)$ by the universal property of the abelianization map. Because $H_1(M)$ is finite, $H_1(B)$ must be as well, so that B has genus 0 or -1.

By our conventions for the Seifert invariants, we have $\alpha_i \geq 2$ for each i, so that in general we have

$$\sum_{i=1}^{n}(1-\frac{1}{\alpha_i}) \geq \sum_{i=1}^{n}(1-\frac{1}{2}) = \frac{n}{2}.$$

we now split according to several cases. We first note that $\pi_1(M)$ is infinite because it is nontrivial and torsion-free. Therefore,

- if $B = P^2(\alpha_1, \dots, \alpha_n)$, then $n \geq 2$, by theorem 75. In this case, the above inequality yields $\sum_{i=1}^{n}(1-\frac{1}{\alpha_i}) \geq 1$.
- if $B = S^2(\alpha_1, \dots, \alpha_n)$, then either $n > 3$ or $n = 3$ with $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \leq 1$, also by theorem 75.
  - in the first case, the above inequality yields $\sum_{i=1}^{n}(1-\frac{1}{\alpha_i}) \geq 2$;
  - in the second case, we compute directly, obtaining $\sum_{i=1}^{n}(1-\frac{1}{\alpha_i}) = 3 - (\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}) \geq 2$

In any case, we get $\chi^{orb}(B) \leq 0$, so that B admits an Euclidean or hyperbolic structure by theorem 62, and $\pi_1^{orb}(B)$ acts properly discontinuously on $\mathbb{E}^2$ or $\mathbb{H}^2$ with quotient B. Note also that M admits a unique Seifert structure up to isotopy by theorem 68.

Now because $\pi_1(M)$ is left-orderable, it may be viewed as a subgroup of $Homeo_+(\mathbb{R})$ by theorem 36 and because $\pi_1(M)$ is finitely presented (hence countable). We may suppose that the induced action on $\mathbb{R}$ is fixed point free by lemma 37 (lemma 5.1 in **[9]**). By lemma 76, $\phi(h)$ is conjugate to translation by ± 1. Replacing h by h^{-1} if necessary, we can therefore assume that we have a group homomorphism $\phi : \pi_1(M) \longrightarrow Homeo_+(\mathbb{R})$ such that $\phi(h)$ is translation by 1.

Let $\tilde{B}$ denote the universal orbifold cover of B. As $\tilde{B}$ is $\mathbb{E}^2$ or $\mathbb{H}^2$, it is homeomorphic to $\mathbb{R}^2$. We let $\pi_1(M)$ act on $\tilde{B} \times \mathbb{R}$ as follows:

$$\gamma(x,t) = (p(\gamma)x, \phi(\gamma)t),$$

where $p : \pi_1(M) \longrightarrow \pi_1^{orb}(B) = \pi_1(M)/\langle h \rangle$ is the natural projection. We claim that this action is free and properly discontinuous.

To see that it is free, suppose that $\gamma(x,t) = (x,t)$. Although $\pi_1^{orb}(B)$ does not act freely on the universal cover $\tilde{B}$ (due to the presence of cone points), its action is properly discontinuous. Combining this with the fact that $p(\gamma)x = x$, we see that $p(\gamma)$ has finite order $k > 0$ in $\pi_1^{orb}(B)$. Thus $\gamma^k = h^n$ for some $n \in \mathbb{N}$, so that if n is nonzero, the action of $\phi(\gamma^k)$ on $\mathbb{R}$ has no fixed points. This in turn implies that $\phi(\gamma)$ acts on $\mathbb{R}$ without fixed points, contradicting the fact that $\gamma(x,t) = (x,t)$. Thus n must be zero, so that γ^k is trivial in $\pi_1(M)$. As $\pi_1(M)$ is torsion-free, γ must be the identity element of $\pi_1(M)$.

To see that the action is properly discontinuous, recall that $\pi_1^{orb}(B)$ acts properly discontinuously on $\tilde{B}$. Let (x,t) be a point in $\tilde{B}\times\mathbb{R}$, and let U be a neighbourhood of x in $\tilde{B}$ such that the set

$$\{\gamma \in \pi_1(M) | p(\gamma)U \cap U \neq \emptyset\}$$

is finite. Let I denote the interval $(t-\frac{1}{2}, t+\frac{1}{2})$. For any γ in $\pi_1(M)$ which is not the identity, the condition that $p(\gamma)U \cap U$ be empty implies that $\gamma(U\times I) \cap U \times I$ is also empty, unless $p(\gamma)$ is the identity element of $\pi_1^{orb}(B)$. However, if this is the case, γ must be some nontrivial power of h, so that $\phi(\gamma)$ acts as a nonzero integer translation on I. Thus in this case as well, the set $\gamma(U\times I) \cap U\times I$ is empty, so that the set

$$\{\gamma \in \pi_1(M) | \gamma(U\times I) \cap U \times I \neq \emptyset\}$$

is reduced to the identity element of $\pi_1(M)$.

The quotient of $\tilde{B}\times\mathbb{R}$ by this action is therefore a manifold N, and we have

$$\pi_1(M) \cong \pi_1(N)/p'_*\pi_1(\tilde{B}\times\mathbb{R}),$$

where $p' : \tilde{B}\times\mathbb{R} \to N$ is the natural projection. As $\tilde{B}\times\mathbb{R}$ is homeomorphic to $\mathbb{R}^3$, N and M have isomorphic fundamental groups.

The lines $\{x\}\times\mathbb{R}$ in $\tilde{B}\times\mathbb{R}$ descend to a Seifert structure on N, so that N is Seifert fibred with infinite fundamental group isomorphic to $\pi_1(M)$. By theorem 7, M and N are homeomorphic, and by theorem 68, their Seifert structures are isomorphic.

Furthermore, the planes $\tilde{B}\times\{t\}$ descend to a horizontal foliation of M, which is co-orientable because the $\mathbb{R}$-action upstairs is by orientation-preserving homeomorphisms. By proposition 93, the base orbifold is orientable, so that $B = S^2(\alpha_1, \ldots, \alpha_n)$. □

7.2. Relation to L-spaces

As a corollary of theorem 98, we get the following:

PROPOSITION 99. *Let M be a compact, connected, orientable Seifert fibred rational homology sphere with base orbifold $S^2(\alpha_1, \dots, \alpha_n)$. Then $\pi_1(M)$ is left-orderable if and only if M admits a horizontal foliation.*

The following theorem is proved in [**30**]:

PROPOSITION 100. *Let M be a compact, connected, orientable Seifert fibred rational homology sphere with base orbifold $S^2(\alpha_1, \dots, \alpha_n)$. Then the following are equivalent:*

- *M is not an L-space;*
- *M admits a horizontal foliation;*
- *M admits a taut foliation.*

It is natural to ask whether the above theorem remains true if we drop the assumption that the base orbifold be orientable. Note that by theorem 95, we have the following facts:

- the only compact, connected, Seifert fibred three-manifold with positive first betti number whose fundamental group is not left-orderable is $P^2 \times S^1$, with base orbifold P^2;
- no compact, connected, Seifert fibred rational homology sphere with base orbifold P^2 admits a left-orderable fundamental group.

Drawing intuition from proposition 100, and hoping that having a left-orderable fundamental group might be an obstruction to being an L-space, one could conjecture that every compact, Seifert fibred space with base orbifold over P^2 is an L-space. This turns out to be true:

PROPOSITION 101. *Let M be a Seifert fibred space with base orbifold $P^2(\alpha_1, \dots, \alpha_n)$, where $\alpha_1 \geq 1$ if $n = 1$ and $\alpha_i \geq 2$ for all $1 \leq i \leq n$ if $n > 1$. Then M is an L-space.*

PROOF. See proposition 18 of [**8**]. □

These results combine to give us the following:

THEOREM 102. *Let M be a compact, connected, Seifert fibred space. Then $\pi_1(M)$ is left-orderable if and only if M is not an L-space.*

7.3. Application to branched cyclic covers of torus knots

We define a manifold to be *excellent* if it has left-orderable fundamental group and admits a co-orientable taut foliation (hence is also not an L-space by Theorem 89), and we call a manifold a *total L-space* if it is an L-space, admits no co-orientable taut foliation, and has non-left-orderable fundamental group. Note that by theorem 102, if M is a compact, connected, Seifert fibred manifold which is not excellent, then either

- $\pi_1(M)$ is not left-orderable, in which case it is an L-space, and thus cannot admit a co-orientable taut foliation by theorem 89; or
- M does not admit a co-orientable taut foliation, in which case $b_1(M) = 0$ by theorem 88. Now M cannot both admit a horizontal foliation and have base orbifold S^2, as this would mean that there exists a co-orientable taut foliation on M. Thus $\pi_1(M)$ is not left-orderable by theorem 95, so that M is an L-space.

In any case, we see that M is a total L-space. We summarize this with the following proposition:

PROPOSITION 103. *Let M be a compact, connected, Seifert fibred manifold. Then M is excellent if and only if it is not a total L-space.*

We have seen earlier that cyclic branched covers of S^3 branched over torus knots are Seifert fibred. Thus by the above proposition, they are either total L-spaces or are excellent. Certainly if their fundamental group is finite, they must be total L-spaces, as their fundamental group cannot be left-orderable. As it turns out, this is the only restriction, i.e. such manifolds are excellent if and only if their fundamental group is infinite. To see this, we begin with a few preliminary results.

LEMMA 104. *Let d divide n and suppose $\Sigma_d T_{p,q}$ is excellent. Then $\Sigma_n T_{p,q}$ is also excellent.*

PROOF. Because d divides n, there is a natural map $\rho_{n,d}$ such that the following diagram commutes (where ρ_n and ρ_d are the branched covering projections):

$$\begin{tikzcd}
\Sigma_n T_{p,q} \arrow[r, "\rho_{n,d}"] \arrow[dr, "\rho_n"'] & \Sigma_d T_{p,q} \arrow[d, "\rho_d"] \\
 & S^3
\end{tikzcd}$$

As ρ_n and ρ_d are nonzero degree maps, so is $\rho_{n,d}$. By hypothesis, $\Sigma_d T_{p,q}$ is excellent, and in particular, has left-orderable fundamental group. Because torus knots are prime knots (see page 95 of **[10]**), their branched cyclic covers are prime manifolds (see **[43]**), so that $\Sigma_n T_{p,q}$ must also have left-orderable fundamental group by corollary 41. Thus by theorem 95, $\Sigma_n T_{p,q}$ is excellent. □

We will repeatedly use the fact that $T_{p,q} = T_{q,p}$ in the following (see section 3.E of **[10]** for basic properties of torus knots).

PROPOSITION 105. *If $gcd(n, pq) = 1$, then $\Sigma_n T_{p,q}$ is excellent unless $\{p, q, n\} = \{2, 3, 5\}$.*

PROOF. Suppose $gcd(n, pq) = 1$. By corollary 81, we can view $\Sigma_n T_{p,q}$ as a Seifert-fibred integer homology sphere with three exceptional fibres. By corollary 79, $\pi_1(\Sigma_n T_{p,q})$ is left-orderable unless $\{p, q, n\} = \{2, 3, 5\}$. By theorem 95, $\Sigma_n T_{p,q}$ is therefore excellent unless $\{p, q, n\} = \{2, 3, 5\}$. □

PROPOSITION 106. *Suppose that r divides either p or q. Then $\Sigma_r T_{p,q}$ is excellent except in the following cases:*

- $\Sigma_2 T_{4,3}$;
- $\Sigma_3 T_{3,2}$;
- $\Sigma_2 T_{k,2}$ *for some integer* $k \geq 2$.

PROOF. The proof of this proposition makes extensive use of theorem 80 and of 92 and is quite lengthy. We refer the interested reader to **[20]**, where it is done in full detail. □

We are now ready to prove the main theorem of this section:

THEOREM 107. *Let $n, p,$ and q be integers ≥ 2 and such that p and q are relatively prime with $q < p$. Then $\Sigma_n T_{p,q}$ is excellent if and only if its fundamental group is not finite, that is, if and only if $\Sigma_n T_{p,q}$ is not one of the following:*

- $\Sigma_2 T_{5,3}$;
- $\Sigma_2 T_{4,3}$;
- $\Sigma_n T_{3,2}$ *with* $2 \leq n \leq 5$;
- $\Sigma_2 T_{p,2}$ *with* $3 \leq p$; *(note that strictly speaking, we need only write* $7 \leq p$ *here)*
- $\Sigma_n T_{5,2}$ *with* $2 \leq n \leq 3$.

If $\Sigma_n T_{p,q}$ is not excellent, then it is a total L-space.

PROOF. By proposition 105, $\Sigma_2 T_{5,3}$ is a total L-space and we may suppose that we are dealing with $\Sigma_n T_{p,q}$ with $\gcd(n, pq) \neq 1$. Note that if $\gcd(n, p)$ and $\gcd(n, q)$ were both equal to one, we would have $\gcd(n, pq) = 1$, so that n must have some divisors in common with either p or q. Let d be such an integer. Thus $\Sigma_d T_{p,q}$ is excellent exactly when it does not take one of the forms listed in proposition 106. As d divides n, we therefore have by lemma 104 that $\Sigma_n T_{p,q}$ is excellent except maybe in the cases

- $(n, p, q) = (n, 4, 3)$;
- $(n, p, q) = (n, 3, 2)$;
- $(n, p, q) = (n, k, 2)$ for some integer $k > 3$,

which we have split up in this way for convenience of proof. Writing $n = 2^a 3^b 5^c m$, where m is some integer such that $\gcd(m, 2 \cdot 3 \cdot 5) = 1$, we first deal with the case where $m \neq 1$, so that $m \geq 7$. Note also that in any case, $\gcd(m, q) = 1$.

- If we also have $\gcd(m, p) = 1$, then $\gcd(m, pq) = 1$ so that by proposition 105, $\Sigma_m T_{p,q}$ is excellent, and by lemma 104, $\Sigma_n T_{p,q}$ is as well.
- If $\gcd(m, p) \neq 1$, let s be a prime factor of m and of p. Then $s \geq 7$, so that $\Sigma_s T_{p,q}$ is excellent by proposition 106, and so is $\Sigma_n T_{p,q}$ by lemma 104.

In any case, we see that if $m \neq 1$, $\Sigma_n T_{p,q}$ is excellent. We may thus suppose that n is of the form $n = 2^a 3^b 5^c$. We now split into three cases following the values of p and q:

Case 1: $(p,q) = (4,3)$. Note that $\Sigma_5 T_{4,3}$ is excellent by proposition 105. Thus if n is divisible by 5, $\Sigma_n T_{4,3}$ is excellent by lemma 104. We may therefore restrict to the case where $c = 0$. Noting that $\Sigma_3 T_{4,3}$ is excellent by proposition 106, we again have by lemma 104 $\Sigma_n T_{4,3}$ is excellent if n is divisible by 3. We may thus restrict to the case where n is of the form $n = 2^a$. Note that $\pi_1(\Sigma_2 T_{4,3})$ is finite by corollary 20, so that $\Sigma_2 T_{4,3}$ is a total L-space. By proposition 106, $\Sigma_4 T_{4,3}$ is excellent, so that by lemma 104, $\Sigma_n T_{4,3}$ is excellent whenever n is of the form $n = 2^a$ with $a \geq 2$.

Case 2: $(p,q) = (3,2)$. We first note that $\Sigma_2 T_{3,2}$, $\Sigma_3 T_{3,2}$, $\Sigma_4 T_{3,2}$, and $\Sigma_5 T_{3,2}$ have finite fundamental group by corollary 20, so that they are total L-spaces. We also note that $\Sigma_{25} T_{3,2}$ is excellent by 105, so that by lemma 104, $\Sigma_n T_{3,2}$ is excellent if $c \geq 2$. Furthermore, if we can show that $\Sigma_8 T_{3,2}$ and $\Sigma_9 T_{3,2}$ are excellent, we will have shown that $\Sigma_n T_{3,2}$ is excellent if $a \geq 3$, $b \geq 2$, again by lemma 104. It would then only be necessary to verify that $\Sigma_6 T_{3,2}$, $\Sigma_{10} T_{3,2}$, and $\Sigma_{15} T_{3,2}$ are excellent, which would give us that $\Sigma_{12} T_{3,2}$ and $\Sigma_{20} T_{3,2}$ are excellent by lemma 104, completing the proof for the case $(p,q) = (3,2)$.

Case 3: $(p,q) = (k,2)$ for some integer $k > 3$. Then the spaces $\Sigma_2 T_{p,2}$ are total L-spaces because they have finite fundamental group by corollary 20. Note also that because p and q were taken to be coprime, we in fact have $p \geq 5$. If $\gcd(p,5) = 1$, then $\Sigma_5 T_{p,2}$ is excellent by proposition 105. If $\gcd(p,5) \neq 1$, then p must be divisible by 5, so that we also have that $\Sigma_5 T_{p,2}$ is excellent by proposition 106. We therefore have that $\Sigma_n T_{p,2}$ is excellent if $c \geq 1$ by lemma 104, so that we may suppose n to be of the form $2^a 3^b$.

If $p \geq 7$, then $\Sigma_3 T_{p,2}$ is excellent: indeed, if $\gcd(3,p) = 1$, this follows from proposition 105, while if $\gcd(3,p) \neq 1$, it follows from proposition 106. We therefore have that $\Sigma_n T_{p,2}$ is excellent when $p \geq 7$ and $b \geq 1$ by lemma 104. If we can show that $\Sigma_4 T_{p,2}$ is excellent, we will have dealt with the case $(p,q) = (k,2)$ with $k \geq 7$.

If $p = 5$, then $\Sigma_2 T_{5,2}$ and $\Sigma_3 T_{5,2}$ are total L-spaces because they have finite fundamental group, while $\Sigma_9 T_{5,2}$ is excellent by proposition 105, so that $\Sigma_n T_{5,2}$ is excellent when $b \geq 2$ by lemma 104. Thus if we can show that $\Sigma_4 T_{5,2}$ and $\Sigma_6 T_{5,2}$ are excellent, the theorem will be proved.

We list the specific cases still requiring proof, in the order in which they appeared above:

(1) $\Sigma_8 T_{3,2}$;
(2) $\Sigma_9 T_{3,2}$;
(3) $\Sigma_6 T_{3,2}$;
(4) $\Sigma_{10} T_{3,2}$;
(5) $\Sigma_{15} T_{3,2}$;
(6) $\Sigma_4 T_{p,2}$ for $p \geq 5$;
(7) $\Sigma_6 T_{5,2}$.

Cases 1 through 3 and case 6 are dealt with in **[20]**, who leave cases 4, 5, and 7 to the reader. We will therefore cover these cases here, using theorem 80 to explicitly compute the Seifert invariants of our manifolds.

Case 4: Using the notations of theorem 80, we can take $b = 1, d = -1$, and $m = 7$, yielding $k = -4$, so that

$$\Sigma_{10} T_{3,2} = M\left(0; \frac{-7}{5}, 4, \frac{-4}{3}, \frac{-4}{3}\right) = M\left(0, -2; \frac{2}{3}, \frac{2}{3}, \frac{3}{5}\right).$$

We see that $M\left(0, -1; \frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right)$ satisfies condition 2 of theorem 92 with $x = 1$ and $y = 2$, so that $\Sigma_{10} T_{3,2}$ admits a horizontal foliation, and thus is excellent by theorem 95.

Case 5: We can take $b = 1, d = -1$, and $m = 8$, yielding $k = -3$, so that

$$\Sigma_{15} T_{3,2} = M\left(0; \frac{-8}{5}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -3\right) = M\left(0, -2; \frac{2}{5}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

We see that $M\left(0, -2; \frac{2}{5}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ satisfies condition 1 of 92.

Case 7: We can take $b = 2, d = -1$, and $m = -1$, yielding $k = 2$, so that $\Sigma_{10} T_{3,2}$ admits a horizontal foliation, and thus is excellent by theorem 95.

$$\Sigma_6 T_{5,2} = M\left(0; \frac{1}{3}, -2, \frac{4}{5}, \frac{4}{5}\right) = M\left(0, -2; \frac{1}{3}, \frac{4}{5}, \frac{4}{5}\right).$$

We see that $M\left(0, -1; \frac{2}{3}, \frac{1}{5}, \frac{1}{5}\right)$ satisfies condition 2 of 92 with $x = 3$ and $y = 4$, so that $\Sigma_6 T_{5,2}$ admits a horizontal foliation, and thus is excellent by theorem 95. $\square$

7.4. The L-space conjecture

We end by discussing the main conjecture connecting L-spaces to taut foliations and left-orderability. We begin with a definition: a *graph manifold* is a 3 manifold X containing a family of disjoint embedded tori $(T_i)_{i\in I}$ such that the components of $X - \bigcup T_i$ are Seifert-fibred. Equivalently, graph manifolds are manifolds whose JSJ components are all Seifert-fibred. Graph manifolds will be important in what follows because many of the conjectures we will introduce are verified for such manifolds.

Let M be a closed, connected, orientable, irreducible 3-manifold (we draw attention to the fact that we are now imposing an orientability condition on M). We have seen that when $b_1(M) > 0$,

- M has a left-orderable fundamental group (Corollary 42);
- M is not an L-space (Proposition 55);
- M admits a co-oriented taut foliation (Theorem 88).

In other words, M is excellent.

Further, we have seen that when $b_1(M) = 0$ and M is Seifert-fibred, then M is excellent if and only if it is not a total L-space. The L-space conjecture posits that the assumption that M be Seifert-fibred is superfluous:

Conjecture 108. *Let M be a closed, connected, orientable, irreducible three-manifold. Then M is excellent if and only if it is not a total L-space.*

To summarize what is known so far, we decompose the above statement as follows:

- **If M admits a co-oriented taut foliation, then M is not an L-space**: This is known to be true in general: see the references preceding theorem 89.
- **If M is not an L-space, then M admits a co-oriented taut foliation**: Not known in general; proven for graph manifolds in **[21]**.
- **$\pi_1(M)$ is left-orderable if and only if M admits a co-oriented taut foliation:** Not known in general; proven for graph manifolds in **[7]**.

- **If $\pi_1(M)$ is left-orderable, then M is not an L-space** : Not known in general; proven for graph manifolds in **[6]**.
- **If M is not an L-space, then $\pi_1(M)$ is left-orderable:** It follows from the above results that this is true for graph manifolds.

Thus the L-space conjecture is verified for graph manifolds. In particular, the twofold branched covers of arborescent knots verify the L-space conjecture (see appendix A of **[4]**). In light of this, it would be interesting to see evidence for non graph manifolds. The following theorem is proved in **[8]**:

THEOREM 109. *(**[8]**, corollary 6) Let M be a closed, connected, Sol manifold. Then M verifies the L-space conjecture.*

Because Seifert spaces account for all geometric manifolds except for Sol manifolds and hyperbolic manifolds, the above theorem combines with theorem 102 to give us the following:

THEOREM 110. *Let M be a closed, connected, non-hyperbolic geometric manifold. Then $\pi_1(M)$ is left-orderable if and only if M is not an L-space.*

Hyperbolic manifolds can also arise as cyclic branched covers of knots and links. The conjecture has been verified for certain such families of manifolds, for example:

THEOREM 111. *(**[8]**, theorem 8) Let L be a non-split alternating link. Then $\Sigma_2(L)$ verifies the L-space conjecture.*

In addition to this, Nathan Dunfield has verified the conjecture by hand for many hyperbolic manifolds. Based on the census of 11,031 closed hyperbolic three-manifolds elaborated by Hodgson and Weeks **[13]**, Dunfield has been unable to find any counterexamples to the conjecture thus far. Dunfield then studied two-fold branched covers of non-alternating links with fewer than 16 crossings, generating 265,503 hyperbolic rational homology spheres. Again, no counterexample was found. For information on how this was done, see **[14]**.

We end with another important conjecture, first formulated by Ozsváth and Szabó in the context of L-spaces:

CONJECTURE 112. *(Heegaard-Floer Poincaré conjecture) The only irreducible integer homology spheres which are L-spaces are S^3 and the Poincaré homology sphere $\Sigma(2,3,5)$.*

In [**3**], Boyer and Boileau proved the following:

THEOREM 113. *([**3**]) Let W be a graph manifold integer homology sphere. Then W admits a co-oriented taut foliation if and only if W is not S^3 or the Poincaré homology sphere $\Sigma(2,3,5)$.*

Combined with the above results on graph manifolds, this verifies the Heegaard-Floer Poincaré conjecture for graph manifolds. The above conjecture, combined with the presumed equivalence between co-oriented taut foliations, left-orderable fundamental groups, and non L-spaces, lead to the following problems:

PROBLEM 114. Show that the only irreducible integer homology spheres with non left-orderable fundamental group are S^3 and the Poincaré homology sphere $\Sigma(2,3,5)$.

PROBLEM 115. Show that the only irreducible integer homology spheres which do not admit a co-orientable taut foliation are S^3 and the Poincaré homology sphere $\Sigma(2,3,5)$.

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